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A REVERSED MEIR'S INEQUALITY AND SOME RELATED RESULTS

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ABSTRACT. Reversed versions are developed for Meir's inequality and some related results.

1. Introduction. In 1981 Meir [**4**] proved the following theorem for nondecreasing sequences.

Theorem A. *Let* $a_0, a_1, \dots, a_{n-1}$ *and* $p_1, p_2, \dots, p_n$ *be nonnegative real numbers satisfying*

$$0 = a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1}, \tag{1.1}$$

$$a_i - a_{i-1} \le p_i, \qquad i = 1, \dots, n-1, \tag{1.2}$$

and

$$p_1 \le p_2 \le \cdots \le p_n. \tag{1.3}$$

If r *and* s *are real numbers with* $r \ge 1$ *and* $s \ge 2r+1$, *then*

$$\left[(s+1)\sum_{i=1}^{n-1} a_i^s (p_i + p_{i+1})/2\right]^{1/(s+1)} \le \left[(r+1)\sum_{i=1}^{n-1} a_i^r (p_i + p_{i+1})/2\right]^{1/(r+1)}. \tag{1.4}$$

Theorem A is sufficiently complicated for its history to be worth noting. This began with Klamkin and Newman [**3**] noting in 1976 that the striking elementary identity

$$\sum_{j=1}^{n} j^3 = \left[\sum_{j=1}^{n} j\right]^2$$

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extends to

$$\sum_{j=1}^{n} a_j^3 \leq \left[\sum_{j=1}^{n} a_j\right]^2,$$

where (a_j) is a nondecreasing sequence with $a_0 = 0$ and $a_j - a_{j-1} \leq 1$. This result turned out to be quite hard to prove in comparison with the integral inequality

$$\int_a^b [f(x)]^3\, dx \leq \left[\int_a^b f(x)\, dx\right]^2$$

for a function $f \in C^1$ with $f(a) = 0$ and $0 \leq f'(x) \leq 1$ on $[a, b]$, which appeared in the 1973 Putnam Competitions. Klamkin and Newman conjectured further the result

$$\left[(s+1)\sum_{i=1}^{n} a_i^s\right]^{1/(s+1)} \leq \left[(r+1)\sum_{i=1}^{n} a_i^r\right]^{1/(r+1)} \tag{1.5}$$

for $s > r > 0$ and the a_i satisfying the same conditions as before, to which their first result reduces for $r = 1$, $s = 3$. We remark parenthetically that even for $a_i = i$, (1.5) is a nontrivial discrete analogue of the 'continuous' identity

$$\left[(s+1)\int_0^x t^s\, dt\right]^{1/(s+1)} = \left[(r+1)\int_0^x t^r\, dt\right]^{1/(r+1)}.$$

Klamkin and Newman were able to establish their conjecture for the case $s = 2r+1$. Thus, for $r \geq 1$, Theorem A gives the second Klamkin and Newman result under the relaxed constraint $s \geq 2r+1$ and further allows weights.

In 1986 G.V. Milovanović and I.Ž. Milovanović [**5**] presented an interesting refinement of Theorem A. Their result is as follows.

Theorem B. *If the numbers a_i, $i = 0, 1, \dots, n-1$, p_i, $i = 1, 2, \dots, n$, r and s satisfy the assumptions of Theorem* A, *then*

$$\begin{aligned}(s+1)\sum_{i=1}^{n-1} a_i^s (p_i + p_{i+1})/2 + \frac{(s+1)(s-r)}{8}\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2)\\ \leq \left[(r+1)\sum_{i=1}^{n-1} a_i^r (p_i + p_{i+1})/2\right]^{(s+1)/(r+1)}.\end{aligned} \tag{1.6}$$

Pečarić [**6**] showed that the conclusion of Theorem A remains valid if assumption (1.3) is replaced by $p_i \le p_n$, $i = 1, \dots, n-1$.

Recently, Alzer [**2**] proved a further result by a different method.

Theorem C. *Let $a_0, a_1, \dots, a_{n-1}$ and $p_1, p_2, \dots, p_n$ be nonnegative real numbers satisfying* (1.1) *and* (1.2) *and*

$$p_i \le p_n, \qquad i = 1, 2, \dots, n-1.$$

If r and s are real numbers with $r \ge 1$ and $s \ge 2r+1$, then

$$\begin{aligned} 0 &\le \frac{(s+1)(s-r)}{8} \sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) \\ &\le \left[(r+1)\sum_{i=1}^{n-1} a_i^r (p_i + p_{i+1})/2\right]^{(s+1)/(r+1)} \\ &\quad - (s+1)\sum_{i=1}^{n-1} a_i^s (p_i + p_{i+1})/2. \end{aligned} \tag{1.7}$$

In the next section we first consolidate these results. Our consolidation leads to some reverse inequalities for suitable domains of the parameters. In particular, these suggest that the constraint $s \ge 2r+1$ cannot always be avoided in Meir-type inequalities.

2. Results. In fact, the proofs from both [**4**] and [**5**] implicitly give the following result.

Theorem 1. *Let $a_0, a_1, \dots, a_{n-1}$ and $p_1, p_2, \dots, p_n$ be nonnegative real numbers satisfying* (1.1) *and* (1.2). *If r and s are real numbers with $r \ge 1$ and $s \ge 2r+1$, then* (1.6) *is valid.*

Further, if

$$\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) \ge 0 \tag{2.1}$$

applies, then (1.7) *and* (1.4) *hold, too.*

Remark. Of course, on using the identity

$$\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) = a_1^{s-1}(p_n^2 - p_1^2) + \sum_{i=2}^{n-1}(a_i^{s-1} - a_{i-1}^{s-1})(p_n^2 - p_i^2), \tag{2.2}$$

we conclude from $p_n^2 \geq p_i^2$ and $a_i^{s-1} \geq a_{i-1}^{s-1}$, $i = 1, \ldots, n-1$, the truth of (2.1).

Similarly, we can prove reversed versions of the previous results. We begin with the following result.

Theorem 2. *Let $a_0, a_1, \ldots, a_{n-1}$ and $p_1, p_2, \ldots, p_n$ be nonnegative real numbers satisfying* (1.1) *and*

$$a_i - a_{i-1} \geq p_i, \qquad i = 1, \ldots, n-1. \tag{2.3}$$

If r and s are real numbers with $0 < r \leq 1$ and $r \leq s \leq 2r+1$, then (1.6) *holds with the inequality reversed.*

Further, if the reverse inequality to (2.1) *holds, then the inequalities reverse to* (1.4) *and* (1.7) *hold too.*

Proof. Without loss of generality, suppose that $a_1 > 0$. Let us consider Stolarsky's means E defined by

$$E(u,v;x,y) = \left[\frac{v}{u}\frac{x^u - y^u}{x^v - y^v}\right]^{1/(u-v)}, \qquad u \neq v, \quad uv \neq 0, \quad x \neq y,$$
$$E(u,v;x,x) = \lim_{y \to x} E(u,v;x,y) = x.$$

It is well known [**7**] that $E(u,v;x,y)$ is increasing in u and v. So we have for $i \in \{1, \ldots, n-1\}$ that

$$E(r+1, 1; a_{i-1}, a_i) \geq E(2r, r; a_{i-1}, a_i),$$

which implies

$$\begin{aligned} a_i^{r+1} - a_{i-1}^{r+1} &\geq \frac{r+1}{2}(a_i - a_{i-1})(a_i^r + a_{i-1}^r) \\ &\geq \frac{r+1}{2} p_i (a_i^r + a_{i-1}^r). \end{aligned}$$

Hence, we get, for $j \in \{1, \dots, n-1\}$ that

$$\begin{aligned} a_j^{r+1} &= \sum_{i=1}^{j} (a_i^{r+1} - a_{i-1}^{r+1}) \\ &\geq \frac{r+1}{2} \sum_{i=1}^{j} p_i (a_i^r + a_{i-1}^r) \\ &= (r+1)\left[A_j - \frac{1}{2} p_{j+1} a_j^r\right] \\ &= (r+1)\left[A_{j-1} + \frac{1}{2} p_j a_j^r\right], \end{aligned}$$

where

$$A_j = \sum_{i=1}^{j} a_i^r (p_i + p_{i+1})/2$$

and we set $A_0 = 0$ (the usual convention).

This implies

$$\frac{1}{2}(A_{j-1} + A_j) \leq \frac{1}{r+1} a_j^{r+1} \left[1 + \frac{r+1}{4a_j}(p_{j+1} - p_j)\right]. \tag{2.4}$$

Let $t = (s+1)/(r+1)$. Since $1 \leq t \leq 2$, we obtain

$$E(t, 1; A_{j-1}, A_j) \leq E(2, 1; A_{j-1}, A_j),$$

which leads to

$$(A_j^t - A_{j-1}^t)/[t(A_j - A_{j-1})] \leq ((A_j + A_{j-1})/2)^{t-1}. \tag{2.5}$$

From (2.4) we have

$$1 + \frac{r+1}{4a_j}(p_{j+1} - p_j) > 0,$$

so we conclude from (2.4) and (2.5) that

$$(2.6)\qquad (r+1)^t(A_j^t - A_{j-1}^t) \le t(r+1)(A_j - A_{j-1})a_j^{(r+1)(t-1)} \cdot \left[1 + \frac{r+1}{4a_j}(p_{j+1} - p_j)\right]^{t-1}.$$

On applying Bernoulli's inequality

$$(1+x)^\alpha \le 1 + \alpha x, \quad x > -1, \quad 0 \le \alpha \le 1$$

with

$$x = \frac{r+1}{4a_j}(p_{j+1} - p_j) \quad \text{and} \quad \alpha = t - 1 = \frac{s-r}{r+1},$$

we get from (2.6) that

$$(r+1)^t(A_j^t - A_{j-1}^t) \le t(r+1)(A_j - A_{j-1})a_j^{(r+1)(t-1)} \cdot \left[1 + (t-1)\frac{r+1}{4a_j}(p_{j+1} - p_j)\right],$$

whence

$$(r+1)^t(A_j^t - A_{j-1}^t) \le \frac{s+1}{2}a_j^s(p_j + p_{j+1}) + \frac{(s+1)(s-r)}{8}a_j^{s-1}(p_{j+1}^2 - p_j^2).$$

Summation over $j = 1, 2, \dots, n-1$ finally yields the desired reverse inequality to (1.6). □

Corollary 1. *Let* $a_0, a_1, \dots, a_{n-1}$ *and* $p_1, p_2, \dots, p_n$ *be nonnegative real numbers satisfying* (1.1), (2.3) *and*

$$(2.7)\qquad p_1 \ge p_2 \ge \dots \ge p_n.$$

If r *and* s *are real numbers with* $0 < r \le 1$ *and* $r \le s \le 2r+1$, *then* (1.4) *holds with the inequality reversed.*

Proof. It is clear that (2.7) gives a reverse inequality in (2.1). □

Corollary 2. *Let* $a_0, a_1, \dots, a_{n-1}$ *and* $p_1, p_2, \dots, p_n$ *be nonnegative real numbers satisfying* (1.1) *and* (2.3). *Then reverse inequalities apply to* (1.4) *and* (1.7) *if either*

(i) $0 < r \le 1 \le s \le 2r+1$ *and* $p_i \ge p_n$, $i = 1, \dots, n-1$, *or*

(ii) $0 < r \le s \le 1$ *and* $p_i \le p_1$, $i = 2, \dots, n$ *holds.*

Proof. We have the reverse inequality in (2.1) in case (i) as a simple consequence of (2.2), while if (ii) holds, we have a reverse inequality in (2.1) as a consequence of the formula

$$\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) = a_{n-1}^{s-1}(p_n^2 - p_1^2) + \sum_{k=1}^{n-2}(p_{k+1}^2 - p_1^2)(a_k^{s-1} - a_{k+1}^{s-1}). \qquad \square$$

Remark. For a result related to Corollary 1, see [**1**].

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