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RESOLUTIONS OF HILBERT MODULES

VERN I. PAULSEN

1. Introduction. In the Sz.-Nagy-Foias model theory for a C_{00}-contraction T, all of the data for the model is contained in the essentially unique characteristic operator function Θ and T is unitarily equivalent to the compression of multiplication by Z to the space $H_F^2(\mathcal{D}) \ominus \Theta H_E^2(\mathcal{D})$. Here E and F are Hilbert spaces and $H_E^2(\mathcal{D})$ denotes the Hardy space of E-valued analytic functions on the unit disk $\mathcal{D}$. If T acts on a Hilbert space H, then we may regard H as a contractive module over the disk algebra $A(\mathcal{D})$ with the module action given by $f \cdot h = f(T)h$. If we regard $H_E^2(\mathcal{D})$ and $H_F^2(\mathcal{D})$ as modules over the disk algebra via pointwise multiplication, then the fact that T is unitarily equivalent to the compression of multiplication by Z to $H_F^2(\mathcal{D}) \ominus \Theta H_E^2(\mathcal{D})$ means that we have a short exact sequence of Hilbert $A(\mathcal{D})$-modules,

$$0 \longrightarrow H_F^2(\mathcal{D}) \xrightarrow{\theta} H_E^2(\mathcal{D}) \xrightarrow{p} H \longrightarrow 0,$$

where θ is the isometric inclusion induced by multiplication by Θ and p is unitarily equivalent to the quotient map. Conversely if we are given a short exact sequence of $A(\mathcal{D})$-module maps as above with θ isometric and P unitarily equivalent to the quotient map, then necessarily θ is given by multiplication by a function Θ and we are back in the situation given by the Sz.-Nagy-Foias model theory.

Thus, the Sz.-Nagy-Foias model theory can be regarded as a statement about the existence and uniqueness of certain special resolutions of the Hilbert $A(\mathcal{D})$-module H. In [**9**] we call these Silov resolutions for reasons which will be explained later.

Abrahamse and Douglas [**1**] succeeded in generalizing some of this theory to the case of contractive Hilbert modules over other function algebras. Let Ω be an *n-holed analytic Cauchy domain*, that is, Ω is an open connected subset of the complex plane which is bounded by $n+1$ nonintersecting analytic Jordan curves. Let $\mathbf{C}(\partial\Omega)$ be the space

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of continuous functions on the boundary of Ω, and let $R(\Omega) \subseteq \mathbf{C}(\partial\Omega)$ be the uniform closure of the set of rational functions with poles off Ω^-. If H is a Hilbert space, then Abrahamse and Douglas define a unital contractive homomorphism $\sigma : R(\Omega) \to B(H)$ to be C_{00} if it is continuous from the topology of bounded pointwise convergence to the double strong operator topology. They prove [**1**, Theorem 2] that every such Hilbert $R(\Omega)$-module has a resolution, $0 \to H^2_E(\Omega) \xrightarrow{\theta} H^2_F(\Omega) \xrightarrow{p} H \to 0$.

Here E and F are a special type of complex analytic vector bundle over Ω, called *Hermitian holomorphic bundles*, $H^2_E(\Omega)$ denotes a "Hardy" space of square-integrable sections of E, θ is an isometric inclusion induced by multiplication by Θ which is a bounded holomorphic bundle map from E to F with isometric boundary values, and p is unitarily equivalent to the quotient map.

However, even under reasonable minimality conditions, these resolutions are far from unique. We will see later that even the one-dimensional modules which arise from a fixed homomorphism $\sigma(f) = f(w)$, for some $w \in \Omega$, have a family of resolutions naturally parametrized by an n-torus. Ball [**4**] shows that even the ranks of the vector bundles is not uniquely determined.

The purpose of this paper is to attempt to describe the set of all possible resolutions of a given module, which satisfy some additional minimality hypotheses, modulo a natural equivalence relation. We accomplish this by applying some of Arveson's ideas for studying dilations of subalgebras of C^*-algebras [**3**]. We call a subalgebra A of a C^*-algebra B *sub-Dirichlet* if the linear span of the set $\{a^*b : a, b \in A\}$ is dense in B. In particular, every function algebra $A \subseteq C(X)$ is sub-Dirichlet on $C(X)$. When A is sub-Dirichlet on B, we can describe the set of equivalence classes of minimal resolutions as a fibered set over a base space which consists of a set of completely positive mappings. This base space is a convex compact set in a topological vector space.

Reconciling our description of the set of all resolutions with the Abrahamse-Douglas description leads to interesting analytical questions. For example, if Ω is an n-holed analytic Cauchy domain and we consider $\sigma(f) = f(w)$, w fixed, then our description yields the set of all resolutions as a space fibered over an n-dimensional compact convex set of positive measures, i.e., topologically a ball in $\mathbf{R}^n$; while

Abrahamse-Douglas realize it as an n-torus. Thus, the projection map from the total space to the base projects an n-torus onto a closed ball in n-space. Thus, we would expect the variation in the behavior of the fibers above measures to be quite complex. When Ω is an annulus, we are able to describe this fibering quite concretely. Our space of measures is represented by a closed interval, and the map from the Abrahamse-Douglas parameter space, which is a circle to our interval, is two-to-one over the interior of the interval and one-to-one at the endpoints. Clancey [**8**] using the theory of Riemann surfaces and theta functions is able to better explain this fibering.

Our paper is organized as follows. Section 2 contains some preliminary results on semi-invariant subspaces together with some key analytic examples. Section 3 contains a careful discussion of resolutions, equivalence of resolutions and sub-Dirichlet algebras. Section 4 concerns C_{00}-representations.

2. Semi-invariant subspaces. Let $\mathcal{K}$ be a Hilbert space, $\mathcal{H}$ a closed subspace and $P_{\mathcal{H}}$ the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. If A is an operator on $\mathcal{K}$, then the operator on $\mathcal{H}$ defined by $A_{\mathcal{H}} = P_{\mathcal{H}} A\,|_{\mathcal{H}}$ is called the *compression of A to $\mathcal{H}$.*

Let $\mathcal{L}(\mathcal{K})$ denote the algebra of bounded linear operators on $\mathcal{K}$, and let $\mathcal{A} \subset \mathcal{L}(\mathcal{K})$ be a subalgebra. A closed subspace $\mathcal{H}$ of $\mathcal{K}$ is said to be *invariant* for A if $A\mathcal{H} \subseteq \mathcal{H}$ for all $A \in \mathcal{A}$. A closed subspace $\mathcal{H}$ of $\mathcal{K}$ is said to be *semi-invariant* for $\mathcal{A}$ if the compression map $A \to A_{\mathcal{H}}$ is a homomorphism from $\mathcal{A}$ into $\mathrm{L}(\mathcal{H})$. Sarason [**13**] showed that a subspace $\mathcal{H}$ is semi-invariant for $\mathcal{A}$ if and only if there exists a nested pair of invariant subspaces $\mathcal{M} \subseteq \mathcal{N}$ for $\mathcal{A}$ such that $\mathcal{M} \oplus \mathcal{H} = \mathcal{N}$. We shall call $(\mathcal{M}, \mathcal{N})$ a *representing pair* for $\mathcal{H}$.

In general, this representation of $\mathcal{H}$ is not unique, and the purpose of this section is to give a complete description of the distinct pairs $(\mathcal{M}, \mathcal{N})$ of invariant subspaces that give rise to such a representation of $\mathcal{H}$. We close this section with a number of examples motivated by complex analysis. We begin by noting that if $\mathcal{H}$ is a semi-invariant subspace for $\mathcal{A}$, then the set of all nested pairs $(\mathcal{M}, \mathcal{N})$ of invariant subspaces for $\mathcal{A}$, $\mathcal{M} \subseteq \mathcal{N}$ such that $\mathcal{M} \oplus H = \mathcal{N}$, forms a lattice under the partial ordering $(\mathcal{M}, \mathcal{N}) \leq (\mathcal{M}', \mathcal{N}')$ if and only if $\mathcal{N} \subseteq \mathcal{N}'$. If $\{(\mathcal{M}_\alpha, \mathcal{N}_\alpha)\}$ is any family of such pairs, then $(\cap_\alpha \mathcal{M}_\alpha, \cap_\alpha \mathcal{N}_\alpha)$ and

$(\vee_\alpha \mathcal{M}_\alpha, \vee_\alpha \mathcal{N}_\alpha)$ are also representing pairs for $\mathcal{H}$, where $\cap$ and $\vee$ denote the intersection and span, respectively. Setting

$$\mathcal{M}_m = \bigcap_\alpha \mathcal{M}_\alpha, \qquad \mathcal{N}_m = \bigcap_\alpha \mathcal{N}_\alpha$$
$$\mathcal{M}_M = \bigvee_\alpha \mathcal{M}_\alpha, \qquad \mathcal{N}_M = \bigvee_\alpha \mathcal{N}\alpha$$

where $\{(\mathcal{M}_\alpha, \mathcal{N}_\alpha)\}$ is the family of all representing pairs for $\mathcal{H}$, we obtain two distinguished representing pairs for $\mathcal{H}, (\mathcal{M}_m, \mathcal{N}_m)$ and $(\mathcal{M}_M, \mathcal{N}_M)$ called, respectively, the *minimal* and *maximal* representing pair of $\mathcal{H}$. Clearly, if $(\mathcal{M}, \mathcal{N})$ is any representing pair for $\mathcal{H}$, then $\mathcal{M}_m \subseteq \mathcal{M} \subseteq \mathcal{M}_M$ and $N_m \subseteq \mathcal{N} \subseteq \mathcal{N}_M$.

We begin by characterizing these minimal and maximal elements of the lattice of representing pairs for $\mathcal{H}$. For $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ we let $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ and let $\mathcal{A}\mathcal{H}$ denote the closed subspace spanned by the vectors Ah, $A \in \mathcal{A}$, $h \in \mathcal{H}$.

Proposition 2.1. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ be an algebra, and let $\mathcal{H} \subseteq \mathcal{K}$ be a semi-invariant subspace for $\mathcal{A}$, with $(\mathcal{M}_m, \mathcal{N}_m)$ and $(\mathcal{M}_M, \mathcal{N}_M)$ the minimal and maximal representing pairs for $\mathcal{H}$; then*

$$\mathcal{N}_m = \mathcal{A}\mathcal{H} \quad \textit{and} \quad \mathcal{M}_M = [A^*\mathcal{H}]^\perp.$$

Proof. Sarason [**13**] shows that if $\mathcal{N} = \mathcal{A}\mathcal{H}$ and $\mathcal{M} = \mathcal{N} \cap (\mathcal{H}^\perp)$, then $\mathcal{M}$ and $\mathcal{N}$ are both invariant subspaces for $\mathcal{A}$. Thus, $(\mathcal{M}, \mathcal{N})$ is a representing pair for $\mathcal{H}$. Clearly, if $\mathcal{N}'$ is any invariant subspace for $\mathcal{A}$ containing $\mathcal{H}$, then $\mathcal{A}\mathcal{H} \subseteq \mathcal{N}'$, so that $\mathcal{N}_m = \mathcal{A}\mathcal{H}$.

On the other hand, note that if $\mathcal{N}$ is any invariant subspace for $\mathcal{A}$, then $\mathcal{N}^\perp$ is an invariant subspace for $\mathcal{A}^*$. Thus, if $(\mathcal{M}, \mathcal{N})$ is any representing pair for $\mathcal{H}$, then $\mathcal{N}^\perp \subseteq \mathcal{M}^\perp$ is a pair of nested invariant subspaces for $\mathcal{A}^*$ with $\mathcal{N}^\perp \oplus \mathcal{H} = \mathcal{M}^\perp$. Hence, $\mathcal{H}$ is also semi-invariant for $\mathcal{A}^*$ and $(\mathcal{N}^\perp, \mathcal{M}^\perp)$ is a representing pair for $\mathcal{H}$ with respect to $\mathcal{A}^*$.

Clearly, if $(\mathcal{M}_M, \mathcal{N}_M)$ is the maximal representing pair for $\mathcal{H}$ with respect to $\mathcal{A}$, then $(\mathcal{N}_M^\perp, \mathcal{M}_M^\perp)$ is the minimal representing pair for $\mathcal{H}$, with respect to $\mathcal{A}^*$. Thus, by the above argument, $\mathcal{M}_M^\perp = \mathcal{A}_\mathcal{H}^*$. $\square$

Now that we have identified the two extreme representing pairs for a semi-invariant subspace, it is possible to describe all representing pairs. Note that if $\mathcal{H}$ is a semi-invariant subspace and $\mathcal{H} \subset \mathcal{N}$ an invariant subspace, then $\mathcal{M} = \mathcal{N} \cap \mathcal{H}^\perp$ is not necessarily invariant. So, in general, choosing an invariant subspace containing $\mathcal{H}$ is not sufficient to determine a representing pair. However, in some cases it is.

Theorem 2.2. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ be an algebra, and let $\mathcal{H} \subseteq \mathcal{K}$ be a semi-invariant subspace for $\mathcal{A}$, with $(\mathcal{M}_m, \mathcal{N}_m)$ and $(\mathcal{M}_M, \mathcal{N}_M)$ the corresponding minimal and maximal representing pairs for $\mathcal{H}$. Then the following are equivalent*:

(i) *$(\mathcal{M}, \mathcal{N})$ is a representing pair for $\mathcal{H}$.*

(ii) *$\mathcal{N}$ is an invariant subspace for $\mathcal{A}$, with $\mathcal{N}_m \subseteq \mathcal{N} \subseteq \mathcal{N}_M$ and $\mathcal{M} \oplus \mathcal{H} = \mathcal{N}$.*

(iii) *$\mathcal{M}$ is an invariant subspace for $\mathcal{A}$, with $\mathcal{M}_m \subseteq \mathcal{M} \subseteq \mathcal{M}_M$ and $\mathcal{M} \oplus \mathcal{H} = \mathcal{N}$.*

Proof. That (i) implies (ii) is clear from the definition. To see that (ii) implies (iii), note that

$$\mathcal{M}_m = \mathcal{N}_m \cap \mathcal{H}^\perp \subseteq \mathcal{N} \cap \mathcal{H}^\perp = \mathcal{M} \subseteq \mathcal{N}_M \cap \mathcal{H}^\perp = \mathcal{M}_M.$$

Now if $h \in \mathcal{M}$ and $A \in \mathcal{A}$, then $Ah \in \mathcal{M}_M \cap \mathcal{N}$ since both spaces are invariant and contain $\mathcal{M}$. But $\mathcal{M} \subseteq \mathcal{M}_M \cap \mathcal{N} \subseteq \mathcal{H}^\perp \cap \mathcal{N} = \mathcal{M}$, so $Ah \in \mathcal{M}$ and $\mathcal{M}$ is invariant.

Finally, to see that (iii) implies (i), it will be enough to show that $\mathcal{N}$ is invariant for $\mathcal{A}$. So assume that $k \in \mathcal{N}$ and $A \in \mathcal{A}$. Then $k = k_1 + h, k_1 \in \mathcal{M}, h \in H \subseteq \mathcal{N}_m$. Thus, $Ak = Ak_1 + Ah \in \mathcal{M} + \mathcal{N}_m = \mathcal{M} + \mathcal{M}_m + \mathcal{H} = \mathcal{N}$, from which it follows that $\mathcal{N}$ is invariant. □

In order to identify all of the invariant subspaces for $\mathcal{A}$ between $\mathcal{N}_m$ and $\mathcal{N}_M$, the following observations are useful. Note that the space $\mathcal{D} = \mathcal{N}_M \cap \mathcal{N}_m^\perp = \mathcal{M}_M \cap \mathcal{M}_m^\perp$ is also semi-invariant for $\mathcal{A}$, so that compression yields a homomorphism $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{D})$. The following result is easily checked.

Proposition 2.3. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ be an algebra and $\mathcal{H}$ be a semi-*

invariant subspace for $\mathcal{A}$ with $(\mathcal{M}_m, \mathcal{N}_m)$ and $(\mathcal{M}_M, \mathcal{N}_M)$ the minimal and maximal representations of $\mathcal{H}$, respectively. Then a subspace $\mathcal{N}, \mathcal{N}_m \subseteq \mathcal{N} \subseteq \mathcal{N}_M$ is invariant for $\mathcal{A}$ if and only if $\mathcal{N} = \mathcal{N}_m \oplus \mathcal{L}$ where $\mathcal{L} \subseteq \mathcal{D}$ is an invariant subspace for $\rho(\mathcal{A})$.

We shall call $\mathcal{D}$ the *defect space* for $\mathcal{A}$ with respect to $\mathcal{H}$. The following result gives an estimate of its size.

Proposition 2.4. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{K})$ be an algebra, and let $\mathcal{H}$ be a semi-invariant subspace for $\mathcal{A}$ with $(\mathcal{M}_m, \mathcal{N}_m)$ and $(\mathcal{M}_M, \mathcal{N}_M)$ the minimal and maximal representing pairs for $\mathcal{H}$, respectively. Then* $\dim(\mathcal{D}) = \dim(\mathcal{M}_M/\mathcal{M}_m) = \dim(\mathcal{N}_M/\mathcal{N}_m) = \dim([\mathcal{A}\mathcal{H} + \mathcal{A}^*\mathcal{H}]^\perp)$.

Furthermore, if $\mathcal{B}$ is the C^*-algebra generated by $\mathcal{A}$ and $n =$ $\dim(\mathcal{B}/[\mathcal{A}+\mathcal{A}^*]^-)$ *and $\mathcal{B}\mathcal{H} = \mathcal{K}$, then* $\dim(\mathcal{D}) \leq n \cdot \dim \mathcal{H}$.

Proof. $[\mathcal{A}\mathcal{H} + \mathcal{A}^*\mathcal{H}]^\perp = (\mathcal{A}^*\mathcal{H})^\perp \cap (\mathcal{A}\mathcal{H})^\perp = \mathcal{M}_M \cap (\mathcal{N}_m^\perp) = \mathcal{N}_M \cap (\mathcal{N}_m^\perp) = \mathcal{D} = \mathcal{M}_M \cap (\mathcal{M}_m^\perp)$, from which the first equalities follow. The inequality is obvious. □

Note in particular that if $\mathcal{A} + \mathcal{A}^*$ is dense in $\mathcal{B}$, then the above result shows that $\dim(\mathcal{D}) = 0$ and, consequently, there is a unique representing pair for $\mathcal{H}$.

We now wish to illustrate these concepts with some examples motivated by complex analysis and the theory of subnormal models. In what follows, Ω denotes a bounded open set in the complex plane whose boundary Γ consists of $m+1$ disjoint, simple, closed, analytic curves. We shall call such a set an *m-holed analytic Cauchy domain.*

Example 2.5. Fix $z_0 \in \Omega$, and let ω denote harmonic measure for the point z_0. It is well-known that ω and arc-length measure ds are mutually bounded absolutely continuous, and in fact that

$$\frac{dw}{ds} = \frac{-1}{2\pi}\frac{\partial g(\cdot,\lambda)}{\partial n},$$

where g is the Green's function for Ω and n is the outward normal.

Consider the Hilbert space of square-integrable functions on Γ with respect to harmonic measure at z_0, $\mathbf{L}^2(\Gamma, \omega)$, and for $f \in \mathbf{L}^\infty(\Gamma, \omega)$ let

$M_f \in \mathcal{L}(\mathbf{L}^2(\Gamma,\omega))$ be the operator of multiplication by f. Let $\mathcal{R}(\Omega)$ denote the closure of the algebra of quotients of polynomials p/q where the zeros of q lie off Ω^-, and finally let $\mathcal{A} = \{M_f : f \in \mathcal{R}(\Omega)\}$. Let $\mathcal{H}$ be the one-dimensional subspace of $\mathbf{L}^2(\Gamma,\omega)$ spanned by the constant function 1. Note that $\mathcal{H}$ is semi-invariant for $\mathcal{A}$ since compression to $\mathcal{H}$ is given by

$$P_{\mathcal{H}} M_f \mid_{\mathcal{H}} = \langle M_f \cdot 1, 1\rangle = \int f \, dw = f(z_0)$$

which is clearly a homomorphism.

If $(\mathcal{M}_m, \mathcal{N}_m)$ is the minimal representing pair for $\mathcal{H}$, then $\mathcal{N} = \mathcal{A} \cdot 1^- = \{f : f \in \mathcal{R}(\Omega)\}^- = H^2(\Gamma,\omega)$, which is commonly called the *Hardy space* on Γ. We also have $\mathcal{M}_m = \{f \in H^2(\Gamma,\omega) : f(z_0) = 0\}$, which we denote by $H_0^2(\Gamma,\omega)$.

To identify the maximal representing pair for $\mathcal{H}$ we need to first recall [**10**, p. 94] that there is a subspace N such that $\mathbf{L}^2(\Gamma,\omega) = H^2(\Gamma,\omega) \oplus \overline{H_0^2(\Gamma,\omega)} \oplus N$, where $\overline{H_0^2(\Gamma,\omega)}$ denotes the space of complex conjugates of functions in $H_0^2(\Gamma,\omega)$. The space N is the complex span of n real functions [**10**, p. 93], and consequently, $N = \overline{N}$ and also $\mathbf{L}^2(\Gamma,\omega) = [\overline{H^2(\Gamma,\omega)} \oplus N \oplus H_0^2(\Gamma,\omega)$.Thus, if $(\mathcal{M}_M, \mathcal{N}_M)$ is the maximal representing pair for $\mathcal{H}$ we have, by Proposition 2.1, that $\mathcal{M}_M = [\mathcal{A}^* \cdot 1]^\perp = [\overline{H^2(\Gamma,\omega)}]^\perp = N \oplus H_0^2(\Gamma,\omega)$and so $\mathcal{N}_M = H^2(\Gamma,\omega) \oplus \mathbf{N}$.

To complete the description of all representing pairs for $\mathcal{H}$, we apply Proposition 2.3. First we note that the defect space $\mathcal{D} = N$, and so we need to find all invariant subspaces of the representation $f \to P_N M_f \mid_N$ for $f \in \mathbf{R}(\Omega)$.

By [**10**, Theorem 4.8], $[\overline{H_0^2(\Gamma,\omega)}]^\perp = \{f \in \mathbf{L}^2(\Gamma,\omega) : f = F/P, F \in H^2(\Gamma,\omega)\}$, where $P(z) = (z - z_1)\cdots(z - z_m)$ and $z_1, \ldots, z_m$ are the critical points of the Green's function, $g(z, z_0)$ counted with multiplicity. If we assume that these points are distinct, then $H^2(\Gamma,\omega) \oplus N = [\overline{H_0^2(\Gamma,\omega)}]^\perp = H^2(\Gamma,\omega) + \operatorname{spn}\{1/(z - z_j) : j = 1, \ldots, m\}$. Let $f_j = P_N(1/(z - z_j))$, so $f_j = 1/(z - z_j) + h_j$, $h_j \in H^2(\Gamma,\omega)$, and calculate, for $f \in \mathcal{R}(\Omega)$,

$$\begin{aligned} P_N M_f f_j &= P_N(f/z - z_j) \\ &= P_N\left(\frac{f(z) - f(z_j)}{z - z_j} + f(z_j) \cdot \frac{1}{z - z_j}\right) \\ &= f(z_j) \cdot f_j. \end{aligned}$$

Thus, the representation of compression to N is diagonal with respect to the (nonorthogonal) basis $\{f_1, \dots, f_m\}$. The invariant subspaces of this representation are clearly determined by choosing some subset of the basis.

In conclusion, we see that there are 2^m possible representing pairs for this one-dimensional semi-invariant subspace and that the lattice of representing pairs is isomorphic to the lattice of subsets of $\{1, \dots, m\}$ with containment for the order.

When the critical points of the Green's functions are not distinct, the lattice is more complicated.

Example 2.6. Fix $z_0 \in \Omega$, and let ds denote arc length measure on Γ. Again we consider the Hilbert space of square-integrable functions on Γ with respect to arc length measure, $L^2(\Gamma, ds)$, and we let $H^2(\Gamma, ds)$ denote the closure of the analytic functions on Ω^-. It is well known that the map $f \to f(z_0)$ defines a bounded linear functional on $H^2(\Gamma, ds)$ and hence there exists a function $k_{z_0} \in H^2(\Gamma, ds)$ such that $f(z_0) = \langle f, k_{z_0}\rangle = \int f\bar{k}_{z_0}\, ds$. We shall call $k_{z_0}(z)$ the *Szego kernel*.

The space $\mathcal{H}$ spanned by k_{z_0} is semi-invariant for $\mathcal{A} = \{M_f \mid f \in \mathcal{R}(\Omega)\}$. To see this, note that if we set $H_0^2(\Gamma, ds) = \{g \in H^2(\Gamma, ds) : g(z_0) = 0\}$, then $H^2(\Gamma, ds) = H_0^2(\Gamma, ds) \oplus \mathcal{H}$ with $H_0^2(\Gamma, ds)$ and $H^2(\Gamma, ds)$ both invariant for $\mathcal{A}$. The homomorphism obtained by compressing to $\mathcal{H}$ is easily seen to be $M_f \to f(z_0)$ since if we let $h = k_{z_0}/k_{z_0}(z_0)^{1/2}$ be a unit vector in $\mathcal{H}$, then

$$P_{\mathcal{H}} M_f \mid_{\mathcal{H}} = \langle fh, h\rangle = f(z_0).$$

We shall show that the representing pair for $\mathcal{H}$, given by $(H_0^2(\Gamma, ds), H^2(\Gamma, ds))$ is in fact the maximal representing pair.

First, we use the fact that k_{z_0} has m distinct zeros $z_1, \dots, z_m$. In fact, these can be seen to be the zeros of the Ahlfors function for Ω and

z_0, with z_0 deleted. See [**6**, p. 87] and [**10**, Theorem 5.16]. From this, we see that the minimal representing pair $(\mathcal{M}_m, \mathcal{N}_m)$ satisfies

$$\mathcal{N}_m = \mathcal{A} \cdot k_{z_0} = \{g \in H^2(\Gamma, ds) : g(z_i) = 0, i = 1, \dots, m\},$$
$$\mathcal{M}_m = \{g \in H^2(\Gamma, ds) : g(z_i) = 0, i = 0, \dots, m\}.$$

The C^*-algebra generated by $\mathcal{A}$ is clearly $\mathcal{B} = \{M_f : f \in C(\Gamma)\}$. Thus, the codimension of the closure of $\mathcal{A} + \mathcal{A}^*$ in $\mathcal{B}$ is the same as the codimension of the closure of $\mathcal{R}(\Omega) + \overline{\mathcal{R}(\Omega)}$ in $C(\Gamma)$, which is m [**10**, Theorem 4.21].

Applying Proposition 2.4, we see that $\dim(\mathcal{N}_M/\mathcal{N}_m) \le m$. Since $\dim(\mathcal{H}^2(\Gamma, ds)/\mathcal{N}_m) = m$ we must have that $\mathcal{N}_M = H^2(\Gamma, ds)$. Thus, $(H_0^2(\Gamma, ds), H^2(\Gamma, ds))$ is indeed the maximal representing pair.

In order to identify all representing pairs, we again need to identify the invariant subspaces of the representation of $\mathcal{A}$ induced by compression to the defect space. Clearly, $H^2(\Gamma, ds) = \mathcal{N}_m + \operatorname{spn}\{k_{z_i} : i = 1, \dots, m\}$. Arguing as in Example 2.4, we see that the projection of k_{z_i} onto the defect space is an eigenvector for the compression of M_f with eigenvalue $f(z_i)$, $i = 1, \dots, m$.

Thus, as before, the invariant subspaces of the compression of $\mathcal{A}$ are 2^m in number and correspond to the lattice of all subsets of $\{1, \dots, m\}$.

3. Resolutions. Throughout this section $\mathcal{B}$ will denote a C^*-algebra with unit 1, and $\mathcal{A} \subseteq \mathcal{B}$ will denote an arbitrary subalgebra containing 1, but not necessarily norm closed or self-adjoint. By a *representation* of $\mathcal{A}$, we mean a unital, contractive homomorphism $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is some Hilbert space. By a $\mathcal{B}$*-dilation* $(\pi, \mathcal{K})$ of ρ we mean a *-representation $(\pi, \mathcal{K})$ of $\mathcal{B}$ such that $\mathcal{K}$ contains $\mathcal{H}$ as a subspace and $\rho(a) = P_{\mathcal{H}}\pi(a)\,|_{\mathcal{H}}$ for all $a \in \mathcal{A}$, that is, $\mathcal{H}$ is semi-invariant for $\pi(\mathcal{A})$ and ρ is the compression of $\pi(\mathcal{A})$ to $\mathcal{H}$.

Given a $\mathcal{B}$-dilation $(\pi, \mathcal{K})$ of ρ, if one sets $\mathcal{K}' = \pi(\mathcal{B})\mathcal{H}$, then $\mathcal{K}'$ is a reducing subspace for π and so the compression π' of π to $\mathcal{K}'$ gives another $\mathcal{B}$-dilation $(\pi', \mathcal{K}')$ of ρ satisfying $\mathcal{K}' = \pi'(\mathcal{B})\mathcal{H}$. We shall therefore always assume that our $\mathcal{B}$-dilations satisfy this minimality condition.

For the purposes of this paper we shall only be concerned with representations of $\mathcal{A}$ which have a $\mathcal{B}$-dilation. A characterization of these representations has been given by Arveson [**3**]:

Theorem 3.1 (Arveson). *Let $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a representation. The following are equivalent*:

(i) *ρ has a $\mathcal{B}$-dilation*

(ii) *ρ is completely contractive,*

(iii) *the map $\tilde{\rho} : \mathcal{A} + \mathcal{A}^* \to \mathcal{L}(\mathcal{H})$ given by $\tilde{\rho}(a + b^*) = \rho(a) + \rho(b)^*$ is completely positive.*

(iv) *the map ρ extends to a completely positive map from $\mathcal{B}$ to $\mathcal{L}(\mathcal{H})$.*

Let $\mathcal{A} \subseteq \mathcal{B}$, and let $(\rho, \mathcal{H})$ be a representation of $\mathcal{A}$ which has $\mathcal{B}$-dilations $(\pi, \mathcal{K})$ and $(\pi', \mathcal{K}')$. We say that these $\mathcal{B}$-dilations are *unitarily equivalent* provided that there exists $U : \mathcal{K} \to \mathcal{K}'$ unitary, such that $Uh = h$ for all $h \in \mathcal{H}$ and $U\pi(b) = \pi'(b)U$ for all $b \in \mathcal{B}$.

The following is elementary but plays a central role.

Proposition 3.2. *Let $\mathcal{A} \subseteq \mathcal{B}$, and let $(\rho, \mathcal{H})$ be a representation of $\mathcal{A}$ which has a $\mathcal{B}$-dilation. Then there is a one-to-one correspondence between completely positive extensions of $\rho, \tau : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ and unitary equivalence classes of $\mathcal{B}$-dilations $(\pi, \mathcal{K})$ of ρ, given by $\tau(b) = P_{\mathcal{H}}\pi(b) \mid_{\mathcal{H}}$ for $b \in \mathcal{B}$.*

Proof. Given a $\mathcal{B}$-dilation $(\pi, \mathcal{K})$ of ρ, setting $\tau(b) = P_{\mathcal{H}}\pi(b) \mid_{\mathcal{H}}$ defines a completely positive extension of ρ. It is easily checked that if two $\mathcal{B}$-dilations are unitarily equivalent then these extensions are the same.

Conversely, given any completely positive extension τ of $\bar{\rho}$, by taking its Stinespring representation one obtains a $\mathcal{B}$-dilation of ρ. □

We note that when $\mathcal{B}$ is commutative, the word "completely" can be dropped in the above theorem since by [**14**] every positive map is automatically completely positive.

A representation $(\tau, \mathcal{N})$ of $\mathcal{A}$ is called $\mathcal{B}$-subnormal if there is a $\mathcal{B}$-dilation $(\pi, \mathcal{K})$ of τ such that $\mathcal{N}$ is an invariant subspace for $\pi(\mathcal{A})$. We shall call such a $\mathcal{B}$-dilation a *$\mathcal{B}$-extension* of $(\tau, \mathcal{N})$. Let ρ be a representation of $\mathcal{A}$ on $\mathcal{H}$. Again we assume that all $\mathcal{B}$-extensions are minimal. By a *$\mathcal{B}$-resolution* $(\tau, \mathcal{M}, \mathcal{N})$ of ρ, we mean that $(\tau, \mathcal{N})$ is a

$\mathcal{B}$-subnormal representation of $\mathcal{A}$, that $\mathcal{M} \subseteq \mathcal{N}$ is an invariant subspace for $\tau(\mathcal{A})$ (so that τ compressed to $\mathcal{M}$ is also $\mathcal{B}$-subnormal), and that $\mathcal{H} \subseteq \mathcal{N}$ with $\mathcal{M} \oplus \mathcal{H} = \mathcal{N}$ and $\rho(a) = P_{\mathcal{H}}\tau(a) \mid_{\mathcal{H}}$. We call a $\mathcal{B}$-subnormal representation *cyclic* if $\tau(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{N}$.

Note that having a $\mathcal{B}$-resolution $(\tau, \mathcal{M}, \mathcal{N})$ of $(\rho, \mathcal{H})$ is equivalent to having a short exact sequence of contractive Hilbert $\mathcal{A}$-modules, $0 \to \mathcal{M} \stackrel{\theta}{\to} \mathcal{N} \stackrel{\rho}{\to} \mathcal{H} \to 0$, i.e., a resolution, where θ is an isometry, ρ is the quotient map, and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{B}$-subnormal. The terminology $\mathcal{B}$-subnormal comes from the case of a function algebra $\mathcal{A} \subseteq C(X)$ for then a $C(X)$-subnormal representation $(\tau, \mathcal{N})$ is subnormal in the usual sense, i.e., $\tau(\mathcal{A})$ is a commuting family of subnormal operators with normal extension $\pi(A)$.

When $\mathcal{A}$ is a function algebra and X is its Silov boundary, then $\mathcal{N}$ is a *Silov module* in the sense introduced by [**9**] exactly when $(\tau, \mathcal{N})$ is $C(X)$-subnormal, and consequently $C(X)$-resolutions are exactly what we termed *Silov resolutions* in [**9**].

We say that two $\mathcal{B}$-resolutions $(\tau_i, \mathcal{M}_i, \mathcal{N}_i)$, $i = 1, 2$, of a representation $(\rho, \mathcal{H})$ of $\mathcal{A}$ are *unitarily equivalent* if there exists a unitary $U : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\tau_2(a)U = U\tau_1(a)$ for all $a \in \mathcal{A}$ and $Uh = h$ for all $h \in \mathcal{H}$. Note that necessarily $U_1 = U \mid_{\mathcal{M}_1}: \mathcal{M}_1 \to \mathcal{M}_2$ is also unitary and defines a unitary equivalence between the compression of τ_1 to $\mathcal{M}_1$ and the compression of τ_2 to $\mathcal{M}_2$.

From the module viewpoint this means that we have a commuting diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{N}_1 & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{N}_2 & \longrightarrow & \mathcal{H} & \longrightarrow & 0, \end{array}$$

of isometric module isomorphisms where the last vertical arrow is the identity map.

The goal of this section is to classify up to unitary equivalence all "reasonable" $\mathcal{B}$-resolutions of $(\rho, \mathcal{H})$ of $\mathcal{A}$ in as great a generality as possible. We begin with a couple of observations on subnormal representations.

Note that there are certain standard $\mathcal{B}$-resolutions of $(\rho, \mathcal{H})$. First choose a completely positive map $\psi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$, which extends ρ and consider its (minimal) Stinespring representation $(\pi, \mathcal{K})$. Since $\rho(a) = P_{\mathcal{H}}\pi(a) \mid_{\mathcal{H}}$, for all $a \in \mathcal{A}$, we have that $\mathcal{H}$ is a semi-invariant subspace for $\pi(\mathcal{A})$. If $(\mathcal{M}, \mathcal{N})$ is any representing pair for $\mathcal{H}$, then clearly $(\tau, \mathcal{M}, \mathcal{N})$ is a subnormal resolution of $(\rho, \mathcal{H})$. Where $\tau(a)$ is the compression of $\pi(a)$ to $\mathcal{N}$. We call these the *canonical* $\mathcal{B}$-resolutions of $(\rho, \mathcal{H})$. To produce a noncanonical $\mathcal{B}$-resolution, we can simply start with a canonical $\mathcal{B}$-resolution $(\tau, \mathcal{M}, \mathcal{N})$ and form $(\tau \oplus \tau, \mathcal{M} \oplus \mathcal{N}, \mathcal{N} \oplus \mathcal{N})$.

Let $\mathcal{A}^*\mathcal{A} \subseteq \mathcal{B}$ denote the complex linear span of the elements of the form $a^*b, a, b \in \mathcal{A}$. Note that $\mathcal{A} + \mathcal{A}^* \subseteq \mathcal{A}^*\mathcal{A}$ since $1 \in \mathcal{A}$.

Proposition 3.3. *A representation $(\rho, \mathcal{N})$ of $\mathcal{A}$ is $\mathcal{B}$-subnormal if and only if there is a completely positive map $\phi : \mathcal{A}^*\mathcal{A} \to \mathcal{L}(\mathcal{N})$ satisfying $\phi(a^*b) = \rho(a)^*\rho(b)$, for all $a, b \in \mathcal{A}$. Moreover, there is a one-to-one correspondence between the completely positive extensions of ϕ from $\mathcal{A}^*\mathcal{A}$ to $\mathcal{B}$ and the unitary equivalence classes of $\mathcal{B}$-extensions of $(\rho, \mathcal{N})$.*

Proof. If $(\rho, \mathcal{N})$ is $\mathcal{B}$-subnormal there exists a $\mathcal{B}$-extension $(\pi, \mathcal{K})$ of ρ. Clearly, the map $\phi : \mathcal{B} \to \mathcal{L}(\mathcal{N})$ defined by $\phi(b) = P_{\mathcal{N}}\pi(b) \mid_{\mathcal{N}}$ is completely positive. Since $\mathcal{N}$ is invariant for $\pi(\mathcal{A}), \phi(a^*b) = \rho(a)^*\rho(b)$ for $a, b \in \mathcal{N}$.

Conversely, given such a ϕ we may extend it to a completely positive map on all of $\mathcal{B}$, by Arveson's extension theorem. Let $(\pi, \mathcal{K})$ be the minimal Stinespring representation of this completely positive map, then $\rho(a)^*\rho(a) = \phi(a^*a) = P_{\mathcal{N}}\pi(a^*a) \mid_{\mathcal{N}}$ and $\rho(a) = P_{\mathcal{N}}\pi(a) \mid_{\mathcal{N}}$ implies that $\mathcal{N}$ is invariant for $\pi(a)$.

Thus, given a $\mathcal{B}$-subnormal representation $(\rho, \mathcal{N})$, we see that every completely positive extension of $\phi, \psi : \mathcal{B} \to \mathcal{L}(\mathcal{N})$ gives rise to a $\mathcal{B}$-extension $(\pi, \mathcal{K})$ of ρ, by taking its (minimal) Stinespring dilation. Given distinct extensions ψ and ψ' with Stinespring dilations $(\pi, \mathcal{K})$ and $(\pi', \mathcal{K}')$, respectively, we have that these $\mathcal{B}$-extensions are not unitarily equivalent. For if they were, then there would exist a unitary $U : \mathcal{K} \to \mathcal{K}'$ such that $U\pi(b)U^* = \pi^*(b)$, $Uh = h$, for $h \in \mathcal{N}$. Hence, $\psi(b) = P_{\mathcal{N}}\pi(b) \mid_{\mathcal{N}} = P_{\mathcal{N}}U\pi(b)U^* \mid_{\mathcal{N}} = P_{\mathcal{N}}\pi'(b)|_{\mathcal{N}} = \psi'(b)$, for all $b \in \mathcal{B}$, a clear contradiction.

Conversely, every $\mathcal{B}$-extension $(\pi, \mathcal{K})$ determines a completely positive map $\psi : \mathcal{B} \to \mathcal{L}(\mathcal{N})$ via $\psi(b) = P_{\mathcal{N}}\pi(b)\mid_{\mathcal{N}}$. Since $\mathcal{N}$ is invariant for $\pi(\mathcal{A})$, $\psi(a^*b) = \rho(a)^*\rho(b)$, and so ψ is an extension of ϕ. □

We define an algebra $\mathcal{A} \subset \mathcal{B}$ to be *sub-Dirichlet* in $\mathcal{B}$ provided that the linear space $\mathcal{A}^*\mathcal{A}$ is dense in $\mathcal{B}$. By the above result, if $\mathcal{A}$ is sub-Dirichlet in $\mathcal{B}$, then every subnormal representation of $\mathcal{A}$ possesses a unique (up to unitary equivalence), (minimal) $\mathcal{B}$-extension.

In particular, note that if $\mathcal{A}$ is a uniform algebra on Y, for a compact Hausdorff space Y, then $\mathcal{A}$ is sub-Dirichlet (in $C(Y)$) by the Stone-Weierstrass theorem. Other examples are the upper (or lower) triangular matrices which are sub-Dirichlet in M_n and the upper triangular matrices whose entries come from a fixed uniform subalgebra of $C(Y)$ which is sub-Dirichlet in $M_n(C(Y))$. On the other hand, there are many subalgebras of M_n which are not sub-Dirichlet.

Let $\mathcal{A} \subset \mathcal{B}$ be sub-Dirichlet, $(\rho, \mathcal{H})$ a representation, $(\tau, \mathcal{M}, \mathcal{N})$ a resolution for $(\rho, \mathcal{H})$ and $(\pi, \mathcal{K})$ the (unique) $\mathcal{B}$-extension of $(\tau, \mathcal{N})$. We say that $(\tau, \mathcal{M}, \mathcal{N})$ is a *minimal $\mathcal{B}$-resolution* if $\pi(\mathcal{B})\mathcal{H}$ is dense in $\mathcal{K}$. That is, $(\tau, \mathcal{M}, \mathcal{N})$ is minimal exactly when $(\pi, \mathcal{K})$ is a minimal $\mathcal{B}$-dilation of $(\rho, \mathcal{H})$.

I do not know whether or not the hypothesis that $(\tau, \mathcal{M}, \mathcal{N})$ is a minimal $\mathcal{B}$-resolution of $(\rho, \mathcal{H})$ is equivalent to assuming that $\mathcal{N}$ is the smallest reducing subspace for $\tau(\mathcal{A})$ that contains $\mathcal{H}$.

Proposition 3.4. *Let $\mathcal{A}$ be a sub-Dirichlet algebra of $\mathcal{B}$, and let $(\rho, \mathcal{H})$ be a representation of $\mathcal{A}$. If $(\tau, \mathcal{M}, \mathcal{N})$ is a cyclic $\mathcal{B}$-resolution of $(\rho, \mathcal{H})$, then $(\tau, \mathcal{M}, \mathcal{N})$ is minimal.*

Proof. Let $[\cdot]$ denote closure; then $[\pi(\mathcal{A}^*\mathcal{A})\mathcal{H}] = [\pi(\mathcal{A})^*\tau(\mathcal{A})\mathcal{H}] = [\pi(\mathcal{A})^*\mathcal{N}] = [\pi(\mathcal{A})^*\pi(\mathcal{A})\mathcal{N}] = \mathcal{K}$ since $\tau(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{N}$ and $\pi(\mathcal{B})(\mathcal{N})$ is dense in $\mathcal{K}$. □

While it is possible to restrict attention to the cyclic resolutions, many natural examples are not cyclic as was seen in Example 2.6.

Theorem 3.5. *Let $\mathcal{A} \subset \mathcal{B}$ be sub-Dirichlet, and let $(\rho, \mathcal{H})$ be a $\mathcal{B}$-dilatable representation of $\mathcal{A}$. Then every minimal $\mathcal{B}$-resolution of $(\rho, \mathcal{H})$ is unitarily equivalent to a canonical $\mathcal{B}$-resolution. Moreover, no two distinct canonical $\mathcal{B}$-resolutions are unitarily equivalent.*

Proof. Let $(\tau, \mathcal{M}, \mathcal{N})$ be a minimal $\mathcal{B}$-resolution for $(\rho, \mathcal{H})$, and let $(\pi, \mathcal{K})$ be the $\mathcal{B}$-extension of $(\tau, \mathcal{N})$. Define $\psi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ by $\psi(b) = P_{\mathcal{H}}\pi(b) \mid_{\mathcal{H}}$, so that ψ is a completely positive extension of ρ. Since $(\tau, \mathcal{M}, \mathcal{N})$ is minimal $(\pi, \mathcal{K})$ is the minimal Stinespring dilation of ψ, up to unitary equivalence. Since $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ are invariant subspaces for $\pi(\mathcal{A})$, under this unitary equivalence $(\tau, \mathcal{M}, \mathcal{N})$ becomes a canonical subnormal resolution.

Now let ψ_i be a completely positive extension of ρ, with $(\pi_i, \mathcal{K}_i)$ the minimal Stinespring representation, and let $\mathcal{M}_i \subseteq \mathcal{N}_i \subseteq \mathcal{K}_i$ be invariant subspaces for $\pi_i(\mathcal{A})$ with $\mathcal{M}_i \oplus \mathcal{H} = \mathcal{N}_i$, $i = 1, 2$. Assume that $(\psi_1, \mathcal{M}_1, \mathcal{N}_1)$ and $(\tau_2, \mathcal{M}_2, \mathcal{N}_2)$ are unitarily equivalent where $\tau_i(a) = P_{\mathcal{N}_i}\pi_i(a) \mid_{\mathcal{N}_i}$, $i = 1, 2$. We must first show that $\psi_1 = \psi_2$. Let $U : \mathcal{N}_1 \to \mathcal{N}_2$ implement the unitary equivalence, so that $uh = h$ for $h \in H$. Then for $a, b \in \mathcal{A}$,

$$\begin{aligned}\psi_1(a^*b) &= P_{\mathcal{H}}\pi_1(a)^*\pi_1(b) \mid_{\mathcal{H}} \\ &= P_{\mathcal{H}}\tau_1(a)^*\tau_1(b) \mid_{\mathcal{H}} \\ &= P_{\mathcal{H}}U\tau_1(a)^*\tau_1(b)U^* \mid_{\mathcal{H}} \\ &= P_{\mathcal{H}}\tau_2(a)^*\tau_2(b) \mid_{\mathcal{H}} \\ &= \psi_2(a^*b).\end{aligned}$$

Since $\mathcal{A}^*\mathcal{A}$ is dense in $\mathcal{B}$, this shows that $\psi_1 = \psi_2$.

Define completely positive maps $\gamma_i : \mathcal{B} \to \mathcal{L}(\mathcal{N}_i)$ via $\gamma_i(a^*b) = \tau_i(a)^*\tau_i(b)$, then $(\pi_i, \mathcal{K}_i)$ is the minimal Stinespring representation of γ_i, $i = 1, 2$, and $U\gamma_1U^* = \gamma_2$. By the uniqueness of the Stinespring representation, there exists $\tilde{U} : \mathcal{K}_1 \to \mathcal{K}_2$ a unitary, such that $\tilde{U}\pi_1(b) = \pi_2(b)\tilde{U}$ for all $b \in \mathcal{B}$ and $\tilde{U}h = Uh$ for $h \in \mathcal{N}_1$.

Thus, we may assume that $\pi_1 = \pi_2 = \pi$ and $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ and that $\tilde{U} : \mathcal{K} \to \mathcal{K}$ is unitary with $\tilde{U}\pi = \pi\tilde{U}$, and $\tilde{U}h = h$ for $h \in \mathcal{H}$. This implies that $\tilde{U}\pi(b)h = \pi(b)\tilde{U}h = \pi(b)h$ for $h \in \mathcal{H}$. Since the linear span of these vectors is dense in $\mathcal{K}$, $\tilde{U}$ is the identity on $\mathcal{K}$. Hence, $\mathcal{N}_2 = \tilde{U}\mathcal{N}_1 = N_1$ from which the result follows. □

Thus, we see that we can, for $\mathcal{A} \subseteq \mathcal{B}$ sub-Dirichlet, identify the unitary equivalence classes of minimal $\mathcal{B}$-resolutions of $(\rho, \mathcal{H})$ as a fibered set. The base is the collection of all completely positive maps $\phi : \mathcal{B} \to B(\mathcal{H})$ which extend ρ. The fiber above a particular ϕ is the lattice of representing pairs $(\mathcal{M}, \mathcal{N})$ for the semi-invariant subspace $\mathcal{H}$ of $\pi(\mathcal{A})$ inside the Stinespring dilation $(\pi, \mathcal{K})$ of ϕ.

We have enough at the moment to classify the cyclic subnormal resolutions in the sub-Dirichlet case.

Corollary 3.6. *Let $\mathcal{A} \subseteq \mathcal{B}$ be sub-Dirichlet, and let $(\rho, \mathcal{H})$ be a representation of $\mathcal{A}$ which has a $\mathcal{B}$-dilation. Then there is a one-to-one correspondence between unitary equivalence classes of cyclic $\mathcal{B}$-resolutions $(\tau, \mathcal{M}, \mathcal{N})$ for ρ and completely positive extensions of ρ to B.*

Proof. Consider the $\mathcal{B}$-extension, $(\pi, \mathcal{K})$ of $(\tau, \mathcal{N})$. We have that $\mathcal{H}$ is semi-invariant for $\pi(\mathcal{A})$ and that $\mathcal{N} = \mathcal{N}_m$, $\mathcal{M} = \mathcal{N}_m$ is the minimal representing pair for $\mathcal{H}$. Now apply Theorem 3.5. □

To illustrate the above concepts, consider an m-holed analytic Cauchy domain Ω, $z_0 \in \Omega$ and $\mathcal{R}(\Omega) \subseteq C(\Gamma)$ where $\Gamma = \partial\Omega$. If we consider the one-dimensional representation $\rho : \mathcal{R}(\Omega) \to \boldsymbol{C}$ given by $\rho(f) = f(z_0)$ and $\tau : \mathcal{R}(\Omega) \to \mathrm{L}(H^2(\Gamma, \omega))$ given by $\tau(f) = M_f$–the operator of multiplication by f, where ω denotes harmonic measure for the point z_0, then $(\tau, H_0^2(\Gamma, \omega), H^2(\Gamma, \omega))$ is a $\mathbf{C}(\partial\Omega)$-resolution of ρ, provided that we identify $\boldsymbol{C}$ with the space of constants. Similarly, $(\tau, H_0^2(\Gamma, ds), H^2(\Gamma, ds))$, where dS denotes arc-length measure, is a subnormal resolution of ρ, provided that we identify $\boldsymbol{C}$ with the span of the kernel function k_{z_0} and let τ again denote the representation of multiplication. Note that the first example is cyclic while the second is not.

To see which canonical resolutions these resolutions correspond to, we must first identify the completely positive maps that they lie in the fiber over. To do this, note that the $\mathbf{C}(\Gamma)$-extensions are given by multiplication operators on $\mathbf{L}^2(\Gamma, \omega)$ and $\mathbf{L}^2(\Gamma, ds)$, respectively. Thus, the positive maps are given by compression to the one-dimensional space spanned by 1 and k_{z_0}, respectively. Thus, the maps are

$$\phi(f) = \int f\,dw \quad \text{and} \quad \psi(f) = \frac{1}{||k_{z_0}||^2}\int f|k_{z_0}|^2\,ds,$$

respectively.

Since the Stinespring representation of ϕ is given as multiplication on $\mathbf{L}^2(\Gamma, \omega)$ compressed to the constants, we see that the first resolution is already canonical.

The Stinespring representation of ψ is given as multiplication on $\mathbf{L}^2(\Gamma, \mu)$, where $d\mu/ds = |k_{z_0}|^2/||k_{z_0}||^2$, compressed to the constants. Thus, the canonical resolutions above ψ correspond to all representing pairs for the semi-invariant subspace spanned by the constants for the algebra $R(\Gamma)$. Thus, we see that as $R(\Gamma)$-modules, $H^2(\Gamma, ds)$ is unitarily equivalent to one of the $\mathcal{R}(\Omega)$-invariant subspaces between $H^2(\Gamma, \mu)$ and $H^2(\Gamma, \mu) \oplus N = (\overline{H_0^2(\Gamma, d\mu)})^{\perp}$. Moreover, this unitary must carry k_{z_0} to a constant. From these considerations it follows easily that the unitary can be taken to be multiplication by $||k_{z_0}||/k_{z_0}$.

From Examples 2.4 and 2.5, we see that, inside the $*$-representations

$$\pi : C(\Gamma) \to \mathcal{L}(L^2(\Gamma, \omega)), \pi : C(\Gamma) \to \mathcal{L}(L^2(\Gamma, ds))$$

where $\pi(f) = M_f$, we generically expect to find 2^m $C(\Gamma)$-resolutions of the above representation ρ of $\mathcal{R}(\Omega)$. These lie above ϕ and ψ, respectively. Since $H^2(\Gamma, ds)$ is the maximal of these, we have that as an $\mathcal{R}(\Omega)$-module it is isomorphic to $H^2(\Gamma, \mu) \oplus N$ and hence $k_{z_0}(H^2(\Gamma, \mu) \oplus N) = H^2(\Gamma, ds)$.

Example 3.7. We wish to illustrate the above theory, again with an example from complex analysis. Let $\Omega = \{z : r < |z| < R\}$ with boundary $\Gamma = \Gamma_r \cup \Gamma_R$. It is well known [**15**] that the closure of $\mathcal{R}(\Omega) + \overline{\mathcal{R}(\Omega)}$ has codimension 1 in $C(\Gamma)$. In fact, if we set

$$h(s) = \begin{cases} +1/R & s \in \Gamma_R, \\ -1/r & s \in \Gamma_r, \end{cases}$$

then $h(s)\,ds$ is an annihilating measure for $\mathcal{R}(\Omega) + \overline{\mathcal{R}(\Omega)}$ where ds denotes arc length measure. To see this, note that for $f \in \mathcal{R}(\Omega)$,

$\int_\Gamma f(s)h(s)\,ds = 1/2\pi \int_0^{2\pi} f(Re^{i\theta})\,d\theta - 1/2\pi \int_0^{2\pi} f(re^{i\theta})\,d\theta = a_0 - a_0 = 0$, where a_0 is the constant term in the Laurent series expansion of f.

Fix a point $\lambda \in \Omega$ and consider $\mathcal{R}(\Omega) \subset C(\Gamma)$. We wish to describe all unitary equivalence classes of minimal $C(\Gamma)$-resolutions for the representation $\rho : \mathcal{R}(\Omega) \to \boldsymbol{C}$ given by $\rho(f) = f(\lambda)$. First, we need to describe all (completely) positive extensions $\psi : C(\Gamma) \to \boldsymbol{C}$ of ρ. Note that if ω_λ is an harmonic measure at λ then $u \to \hat{u}(\lambda) = \int u\,dw_\lambda$ gives a positive extension. To find all positive extensions, recall that $dw_\lambda/ds = P(s,\lambda)$, which is just the normal derivative of the Green's function at λ, is positive and bounded away from 0 on Γ. Thus, if we set $\psi_t(\mu) = \int_\Gamma \mu(s)(P(s,\lambda) + th(s))\,ds$, we have that $\psi_t(f) = f(\lambda)$ for $f \in \mathcal{R}(\Omega)$ and ψ_t will be positive, provided that we pick t sufficiently small so that $P(s,\lambda) + th(s) \geq 0$ for all s on Γ.

Thus, we see that there is some closed interval about 0, $\alpha \leq t \leq \beta$, such that ψ_t is positive in this interval. The endpoints α, β are the first points where $P(s,\lambda) + th(s)$ is 0 for some s.

Let $dm_t = (P(s,\lambda) + th(s))\,ds$, $\alpha \leq t \leq \beta$, be the positive measure which yields ψ_t. The minimal Stinespring representation of ψ_t is quite concrete. Consider $L^2(\Gamma, m_t)$ and represent $C(\Gamma)$ by $u \to M_u$ where M_u is the operator of multiplication by u, and $\boldsymbol{C}$ is identified with the subspace of constant functions. Indeed, since 1 is a unit vector,

$$P_{\mathcal{H}} M_u \,|_{\mathcal{H}} = \langle u \cdot 1, 1\rangle = \int u\,dm_t = \psi_t(u).$$

Since m_t is a regular measure, $C(\Gamma)$ is dense in $L^2(\Gamma, m_t)$ and so the closure $\mathcal{R}(\Omega) + \overline{\mathcal{R}(\Omega)}$ can be at most codimension 1. Let $H^2(\Gamma, m_t)$ and $H_0^2(\Gamma, m_t)$ denote the closures of $\mathcal{R}(\Omega)$ and $\{f \in \mathcal{R}(\Omega) : f(\lambda) = 0\}$, respectively. Note that $H^2(\Gamma, m_t)$ and $\overline{H_0^2(\Gamma, m_t)}$ are orthogonal. For $\alpha < t < \beta$, set $g_t(s) = h(s)(P(s,\lambda) + th(s))^{-1}$ so that $g_t(s)$ is continuous and hence in $L^2(\Gamma, m_t)$. Also, for $f \in \mathcal{R}(\Omega)$,

$$\int f g_t\,dm_t = \int f h\,ds = 0.$$

Thus, we can see that $L^2(\Gamma, m_t) = H^2(\Gamma, m_t) \oplus \overline{H_0^2(\Gamma, m_t)} \oplus N_t$, where N_t is the span of g_t, $\alpha < t < \beta$.

Arguing as in Example 2.5, we have that in $L^2(\Gamma, m_t)$ there are exactly two representing pairs for the semi-invariant subspace $\mathcal{H}$, $\alpha <$

$t < \beta$. The minimal pair is $\mathcal{N}_m = H^2(\Gamma, m_t)$, $\mathcal{M}_m = H_0^2(\Gamma, m_t)$ and the maximal pair is $\mathcal{N}_m = H^2(\Gamma, m_t) \oplus N_t$, $\mathcal{M}_M = H_0^2(\Gamma, m_t) \oplus N_t$. Consequently, there are two unitary equivalence classes of minimal resolutions above each point, $\alpha < t < \beta$.

Recall that $N_t = \mathcal{N}_M \ominus \mathcal{N}_m$ will also be semi-invariant for M_f, $f \in \mathcal{R}(\Omega)$. Thus, compression to this one-dimensional subspace yields a homomorphism of $\mathcal{R}(\Omega)$, $\rho_t : \mathcal{R}(\Omega) \to \boldsymbol{C}$. We have that

$$\begin{aligned}\rho_t(f) &= \frac{1}{||g_t^2||} \cdot \int f g_t^2 \, dm_t \\ &= \frac{1}{||g_t^2||} \int f(s)(P(s,\lambda) + th(s))^{-1} h(s)^2 \, ds.\end{aligned}$$

Since this homomorphism is clearly weak*-continuous, this implies that there exists a $\mu_t \in \Omega$ such that $\rho_t(f) = f(\mu_t)$. Hence,

$$||g_t||^{-2}(P(s,\lambda) + th(s))^{-1} h(s)^2 = P(s,\mu_t) + \gamma h(s),$$

where $P(s,\mu_t) = d\omega_u/ds, \omega_\mu$ is harmonic measure at μ_t, and γ is some suitably chosen real number. Thus, we see that, for each (λ, t), there exists a (μ, γ) so that $(P(s,\lambda) + th(s))/(P(s,\mu) + \gamma h(s))$ is a constant function of s on each circle.

It is difficult to derive this fact analytically.

We now wish to show that, for the two end points α and β, that there is only one unitary equivalence class of models above each point. We argue for α.

We still have that $H^2(\Gamma, m_\alpha)$ and $\overline{H_0^2(\Gamma, m_\alpha)}$ are orthogonal. We wish to show that their direct sum is all of L^2. Suppose not; then there exists $g_\alpha \in L^2(\Gamma, m_\alpha)$ with

$$\int (f + \bar{g}) g_\alpha \, dm_\alpha = \int (f + \bar{g}) g_\alpha (P(s,\lambda) + \alpha h(s)) \, ds = 0$$

for all $f \in H^2(\Gamma, m_\alpha)$, $g \in H_0^2(\Gamma, m_\alpha)$. Since $h(s)\, ds$ is the unique (up to multiples) annihilating measure for $\mathcal{R}(\Omega) + \overline{\mathcal{R}(\Omega)}$ we have that $g_\alpha(s)(P(s,\lambda) + \alpha h(s)) = c \cdot h(s)$, a.e. (ds), for some constant c. This implies that the function $h(s)(P(s,\lambda) + \alpha h(s))^{-1}$ belongs to $L^2(\Gamma, dm_\alpha)$.

However, this implies that

$$\int \frac{h(s)^2}{(P(s,\lambda)+\alpha h(s))^{-2}}\,dm_\alpha = \int (P(x,\lambda)+\alpha h(s))^{-1} h(s)^2\,ds \tag{3.1}$$

must be finite, which we claim is a contradiction. To see this, recall that there must exist $s_0 \in \Gamma$ with $P(s_0,\lambda)+\alpha h(s_0)=0$. But since the partial derivative in the tangent direction exists for $P(s,\lambda)+\alpha h(s)$, we have that there exists a constant M with

$$|P(s,\lambda)+\alpha h(s)| \le M|s-s_0|,$$

for s in some neighborhood of s_0 on Γ. This clearly contradicts the finiteness of (3.1).

Thus, the set of resolutions in this case can be naturally regarded as two copies of the interval $[\alpha,\beta]$, one copy for the minimal and one for the maximal representing pair, with their endpoints identified, i.e., topologically a circle. In Section 4 we'll see how the Abrahamse-Douglas theory gives another realization of the set of all resolutions as a circle.

Example 3.8. Keeping the same notations as above, fix $\lambda_1, \lambda_2 \in \Omega$, and consider

$$\rho : \mathcal{R}(\Omega) \to \mathbf{L}(\mathbf{C}^2) = M_2,$$

defined by

$$\rho(f) = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}.$$

We wish to describe the resolutions of ρ. Let ω_1, ω_2 be the corresponding harmonic measures, $P(s,\lambda_1) = dw_1/ds$, $P(s,\lambda_2) = dw_2/ds$, and let $H = H^*$ be a 2×2 matrix. Note that, for H sufficiently small, the function

$$G_H(s) = \begin{pmatrix} P(x,\lambda_1) & 0 \\ 0 & P(s,\lambda_2) \end{pmatrix} + h(s)\cdot H,$$

will assume values which are positive matrices for each point $s \in \Gamma$. Thus, if for such an H we define

$$\psi_H : \mathbf{C}(\Gamma) \longrightarrow M_2, \quad \text{via}$$
$$\psi_H(u) = \int u(s)\cdot G_H(s)\,ds$$

then ψ_H will be (completely) positive and, for $f \in \mathcal{R}(\Omega)$, $\psi_H(f) = \rho(f)$. Since any linear map from $\mathbf{C}(\Gamma)$ to M_2 can be represented by integration against such a matrix-valued function, we see that the set of completely positive extensions of ρ is parametrized by a certain convex set of 2×2 Hermitian matrices H, determined by the condition that $G_H(s) \geq 0$ for all $s \in \Gamma$.

Again the Stinespring representation of ψ_H is very concrete. Consider the space of $\boldsymbol{C}^2$-valued functions on Γ with inner-product,

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \int \left\langle G_H(s) \begin{pmatrix} f_1(s) \\ f_1(s) \end{pmatrix}, \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} \right\rangle_{\boldsymbol{C}^2} ds$$

We denote this space by $\mathbf{L}^2(\Gamma, G_H(s)\, ds)$. Embed $\boldsymbol{C}^2$ inside this space by identifying it with the constant functions, and represent $\mathbf{C}(\Gamma)$ on this space by $u \to M_u$ where

$$M_u \begin{pmatrix} f_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} uf_1 \\ uf_2 \end{pmatrix}.$$

If e_1, e_2 are the canonical basis vectors for $\boldsymbol{C}^2$, then

$$\langle M_u e_j, e_i \rangle = \langle \psi_H(u) e_j, e_i \rangle_{\boldsymbol{C}^2}$$

and so,

$$P_{\boldsymbol{C}^2} M_\mu \mid_{\boldsymbol{C}^2} = \psi_H(u).$$

Let

$$H^2(\Gamma, G_H(s)\, ds) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \in \mathcal{R}(\Omega) \right\}^-,$$

$$H_0^2(\Gamma, G_H(s)\, ds) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \in \mathcal{R}(\Omega), f_1(\lambda_1) = f_2(\lambda_2) = 0 \right\},$$

so that $\mathbf{L}^2(\Gamma, G_H(s)\, ds) = H^2(\Gamma, G_H(s)\, ds) \oplus \overline{H_0^2(\Gamma, G_H(s)\, ds)} \oplus N_H$, where possibly $N_H = (0)$.

Following the methods of Section 2, in order to understand all possible representing pairs for the semi-invariant subspace $\boldsymbol{C}^2$ of the algebra of M_f, $f \in \mathcal{R}(\Omega)$, we must find all invariant subspaces of the representation of $\mathcal{R}(\Omega)$ obtained by compressing M_f to N_H.

Assume that $G_H(s)$ is invertible for all s, i.e., the "generic" case, then N_H is spanned by

$$G_H(s)^{*^{-1}} \begin{pmatrix} h(s) \\ 0 \end{pmatrix}$$

and

$$G_H(s)^{*^{-1}} \begin{pmatrix} 0 \\ h(s) \end{pmatrix}$$

and so N_H is two-dimensional.

When H is assumed to be diagonal, say

$$H = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$$

it is possible to do a more detailed analysis. For H to satisfy $G_H(s) \geq 0$, we must have that $\alpha_1 \leq \varepsilon_1 \leq \beta_1$, $\alpha_2 \leq \varepsilon_2 \leq \beta_2$, where (α_1, β_2) and (α_2, β_2) are determined by λ_1 and λ_2, respectively, as in the above example. When $(\varepsilon_1, \varepsilon_2)$ is in the interior of this rectangle, N_H is two-dimensional and is spanned by $\begin{pmatrix} g_1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ g_2 \end{pmatrix}$ where $g_i(s) = h(s)(P(s, \lambda_i) + \varepsilon_i h(s))^{-1}$, $i = 1, 2$, which are orthogonal. Thus, from the above example, we see that compression of M_f, $f \in \mathcal{R}(\Omega)$ to N_H yields a homomorphism

$$\hat{\rho} : \mathcal{R}(\Omega) \longrightarrow M_2 = \mathcal{L}(N_H), \quad \hat{\rho}(f) = \begin{pmatrix} f(\mu_1) & 0 \\ 0 & f(\mu_2) \end{pmatrix},$$

for some μ_1, μ_2 in Ω.

When $\mu_1 \neq \mu_2$, then $\hat{\rho}$ will have exactly four invariant subspaces, which corresponds to four representing pairs for our semi-invariant subspace, and by Theorem 3.3, four distinct unitary equivalence classes of minimal subnormal models for ρ above this particular extension. However, when $\mu_1 = \mu_2$, which can occur, then $\hat{\rho}$ will have uncountably many invariant subspaces.

Thus, we see that for a single completely positive map it is possible to have uncountably many distinct unitary equivalence classes of minimal resolutions, all lying above the same completely positive map. This is perhaps surprising, since for Dirichlet algebras we showed that the fiber above a completely positive map is always a singleton, and this algebra falls only one dimension short of being Dirichlet.

4. $\mathbf{C}_{00}$-representations. Let Ω be an n-holed analytic Cauchy domain, and consider $\mathcal{R}(\Omega) \subseteq \mathbf{C}(\partial\Omega)$. Abrahamse and Douglas [**1**] define $\mathbf{C}_{00}$-representations of $\mathcal{R}(\Omega)$ and proved that every $\mathbf{C}(\partial\Omega)$-dilatable $\mathbf{C}_{00}$-representation $\rho : \mathcal{R}(\Omega) \to \mathcal{L}(\mathcal{H})$ has resolutions of the form

$$0 \longrightarrow H^2_E(\Omega) \longrightarrow H^2_F(\Omega) \longrightarrow H \longrightarrow 0 \tag{4.1}$$

where E and F are special types of complex analytic vector bundles over Ω. Thus, we see that *some* of the minimal $\mathbf{C}(\partial\Omega)$-resolutions of (ρ, H) considered in Section 3 have this type of realization. The principle result of this section is to prove that *every* minimal $\mathbf{C}(\partial\Omega)$-resolution of a $\mathbf{C}_{00}$-representation has the form given by Abrahamse-Douglas. We then apply this fact to obtain alternate descriptions of the set of all unitary equivalence classes of minimal $\mathbf{C}(\partial\Omega)$-resolution for some of the examples studied in Section 3.

Recall that a representation $\rho : R(\Omega) \to \mathcal{L}(\mathcal{H})$ is called a $\mathbf{C}_{00}$-representation if it is continuous from the topology of uniform convergence on compact subsets of Ω to the double strong operator topology. Let $\pi : \mathbf{C}(\partial\Omega) \to \mathcal{L}(\mathcal{K})$ be a $*$-homomorphism, and let $\mathcal{N} \subseteq \mathcal{K}$ be a subspace which is invariant for $\pi(\mathcal{R}(\Omega))$. If $\mathcal{N}$ reduces $\pi(\mathbf{C}(\partial\Omega))$, then $\mathcal{N}$ is called $\partial\Omega$-*normal.* If no subspace of $\mathcal{N}$ (other than (0)) reduces $\pi(\mathbf{C}(\partial\Omega))$, then $\mathcal{N}$ is called $\partial\Omega$-*pure* or a *pure* $\mathbf{C}(\partial\Omega)$-*subnormal module.*

It is well known that every $\mathbf{C}(\partial\Omega)$-representation $\psi : \mathcal{R}(\Omega) \to B(\mathcal{N})$ decomposes as $\psi = \psi_1 \oplus \psi_2$, $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ where $\mathcal{N}_1$ is $\partial\Omega$-normal and $\mathcal{N}_2$ is $\partial\Omega$-pure [**1**].

Theorem 4.1. *Let Ω be an n-holed analytic Cauchy domain, and let $\rho : \mathcal{R}(\Omega) \to \mathcal{L}(\mathcal{H})$ be a completely contractive $\mathbf{C}_{00}$-representation. If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{H} \to 0$ is a minimal $\mathbf{C}(\partial\Omega)$-resolution, then $\mathcal{M}$ and $\mathcal{N}$ are $\partial\Omega$-pure.*

Proof. Since $\mathcal{R}(\Omega)$ is sub-Dirichlet and the property of being $\partial\Omega$-pure is preserved under unitary equivalence, it will be enough by Theorem

3.5 to prove that every canonical resolution has this property. To this end, let $\phi : \mathbf{C}(\partial\Omega) \to B(\mathcal{H})$ be a completely positive map which extends ρ, let $\pi : \mathbf{C}(\partial\Omega) \to B(\mathcal{K})$, $\mathcal{H} \subseteq \mathcal{K}$ be its minimal Stinespring dilation, let $\mathcal{N} \subseteq \mathcal{K}$ be a subspace which is invariant for $\pi(\mathcal{R}(\Omega))$, and let $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ be its decomposition into its $\partial\Omega$-normal and $\partial\Omega$-pure parts. Since $\mathcal{N}_1$ reduces $\pi(\mathbf{C}(\partial\Omega))$, if we show that $\mathcal{H} \subseteq \mathcal{N}_2$, then $\pi(\mathbf{C}(\partial\Omega))\mathcal{H} \subseteq \mathcal{N}_1^{\perp}$, and since this set is dense in $\mathcal{K}$ we will have that $\mathcal{N}_1 = (0)$, i.e., that $\mathcal{N}$ is $\partial\Omega$-pure. Clearly, if $\mathcal{N}$ is $\partial\Omega$ pure then $\mathcal{M}$ is also.

Thus, it remains to show that $\mathcal{H} \subseteq \mathcal{N}_2$. The key idea comes from [**5**, Lemma 3.2]. Let $f : \Omega^- \to \mathcal{D}^-$ be the Ahlfors function (or any nonconstant inner function). Then $\psi(f) = \psi_1(f) \oplus \psi_2(f) = \psi(f) \mid_{\mathcal{N}}$ must be an isometry and $\psi_1(f)$ must be a unitary. Hence, for any $h \in \mathcal{H}$, writing $h = h_1 \oplus h_2$ relative to the decomposition of $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ we have that, for all n, $||h_1|| = ||\psi_1(f)^{*^n} h_1|| \leq ||\psi(f)^{*^n} h|| = ||\rho(f)^{*^n} h||$, but this last quantity tends to 0 as $n \to \infty$ because f^n tends to 0 uniformly on compact subsets of Ω. Hence, $h_1 = 0$ and $\mathcal{H} \subseteq \mathcal{N}_2$ as desired. □

Remark 4.2. Let $A \subseteq C(X)$ be a uniform algebra, and let $S \subseteq X$ denote the Silov boundary. We can define a representation $\rho : A \to \mathcal{L}(\mathcal{H})$ to be C_{00} if it is continuous from the topology of uniform convergence on compact subsets of $X \backslash S$ to the double strong operator topology. We also can define S-normal and S-pure as above. It would be interesting to know under what conditions the conclusions of Theorem 4.1 hold. If A contains a function f with $|f(x)| = 1$ for all $x \in S$ and $|f(x)| < 1$ if $x \notin S$, then the same proof given above applies; but it is not clear when this latter condition is met or if some weaker condition might be sufficient. For example, is it sufficient to assume that every point in S is a strong boundary point?

Remark 4.3. Abrahamse-Douglas [**2**] prove that every $\partial\Omega$-pure $\mathbf{C}(\partial\Omega)$-representation of $\mathcal{R}(\Omega)$ is unitarily equivalent to the representation given by pointwise multiplication of sections on a space of the form $H^2_E(\Omega)$ for some Hermitian holomorphic vector bundle E over Ω. Thus, Theorem 4.1 shows that every minimal $\mathbf{C}(\partial\Omega)$-subnormal resolution is of the form (1) when $\mathcal{H}$ is given by a $\mathbf{C}_{00}$-representation.

Before proceeding with our next results, it will be necessary to have a very concrete realization of the spaces $H^2_E(\Omega)$. For our purposes, the description given by Ball [**4**] will be most useful. To this end fix cut lines $C_1, \dots, C_n$ which are disjoint and such that $\Omega_0 = \Omega \backslash C_1 \cup \cdots C_n$ is simply connected. Fix k, $1 \leq k \leq \aleph_0$ and a Hilbert space H_k of dimension k. Let $U = (U_1, \dots, U_n)$ be an arbitrary n-tuple of unitaries on H_k. Given an analytic function $f : \Omega_0 \to H_k$, for each j and each $z \in C_j$, let $f(z^+) = \lim^+_{w \to z} f(w)$, $f(z^-) = \lim^-_{w \to z} f(w)$ denote the limits as $w \in \Omega_0$ where w is restricted to belong (locally) to the clockwise and counterclockwise side of C_j, respectively. We let $H_U(\Omega_0)$ denote the set of analytic functions $f : \Omega_0 \to H_k$ such that for each j and each $z \in C_j$ the limits $f(z^+)$ and $f(z^-)$ exist and satisfy $f(z^+) = U_j f(z^-)$. We let $H^2_U(\Omega)$ denote the set of functions in $H_U(\Omega_0)$ whose boundary values exist almost everywhere on $\partial\Omega$ and are norm square-integrable with respect to arc-length measure. We let $\mathbf{L}^2_k(\partial\Omega)$ denote the usual space of H_k-valued functions on $\partial\Omega$ which are norm square integrable with respect to arc-length measure. It follows from the work of Abrahamse-Douglas [**2**] that $H^2_U(\Omega)$ is a closed subspace of $\mathbf{L}^2_k(\partial\Omega)$ which is an $\mathcal{R}(\Omega)$-module under pointwise multiplication and is a $\partial\Omega$-pure $\mathbf{C}(\partial\Omega)$-subnormal module, i.e., a Silov module, whose $\mathbf{C}(\partial\Omega)$-extension is given by $\mathbf{L}^2_k(\partial\Omega)$. Moreover, every $\partial\Omega$-pure $\mathbf{C}(\partial\Omega)$-subnormal module is unitarily equivalent to one of these. The space $H^2_U(\Omega)$ can be identified with $H^2_E(\Omega)$ where E is the Hermitian holomorphic bundle over Ω with fiber H_k and "gluing" data given by U. Moreover, two such spaces $H^2_U(\Omega)$, $H^2_W(\Omega)$, $U = (U_1, \dots, U_n)$, $W = (W_1, \dots, W_n)$ are unitarily equivalent as $\mathcal{R}(\Omega)$-modules if and only if there exists a unitary V such that $U_j = V W_j V^*$ for all j [**2**]. (Actually, Abrahamse-Douglas [**2**] and Ball [**4**] do not use arc-length measure but instead use harmonic measure for a fixed point t in Ω. However, these measures are mutually boundedly absolutely continuous.)

Example 4.4. Consider the representation $\rho : \mathcal{R}(\Omega) \to \boldsymbol{C}$, $\rho(f) = f(w)$ for a fixed $w \in \Omega$. We wish to describe all $\mathbf{C}(\partial\Omega)$-resolutions. Let $\boldsymbol{C}_w$ denote $\boldsymbol{C}$ with this module action. Clearly, ρ is a $\mathbf{C}_{00}$-representation, so by Theorem 4.1 and the remarks above, every resolution has the form $0 \to H^2_V(\Omega) \to H^2_U(\Omega) \to \boldsymbol{C}_w \to 0$ where $U = (U_1, \dots, U_m)$ is an n-tuple of unitary operators on some space

H_k. Since $\mathbf{L}^2_k(\partial\Omega)$ is the $\mathbf{C}(\partial\Omega)$-extension of $H^2_U(\Omega)$ our minimality assumption, i.e., $\mathbf{C}(\partial\Omega)\mathcal{H}$ dense in $\mathbf{L}^2_k(\partial\Omega)$, and the fact that in this case $\mathcal{H} = \boldsymbol{C}_w$ is one-dimensional implies that $k = 1$. Hence, each U_j is a point on the unit circle. Conversely, if we start with a U_j on the circle, then $H^2_U(\Omega)$ is a reproducing kernel Hilbert space, and we have a resolution $0 \to \{k^U_w\}^\perp \to H^2_U(\Omega) \to \boldsymbol{C}_w \to 0$ where k^U_w denotes the reproducing kernel function for the point w.

It is easily seen that two n-tuples $U = (U_1, \ldots, U_n)$ and $V = (V_1, \ldots, V_n)$ of points from the unit circle are unitarily equivalent if and only if they are equal. Thus, we have a parameterization of the set of all $\mathbf{C}(\partial\Omega)$-resolutions of $\boldsymbol{C}_w$ by the n-torus.

Our other realization of the set of all $\mathbf{C}(\partial\Omega)$-resolutions of $\boldsymbol{C}_\omega$ is as a fibered set over the set of all (completely) positive extensions $\phi : \mathbf{C}(\partial\Omega) \to \boldsymbol{C}$ of ρ. Since $\boldsymbol{C}_\omega$ is being identified with the span of $k^u_w/|k^u_w| = h_u$ in the above resolutions, we see that the associated positive map is given by

$$\begin{aligned}\phi(f) &= P_{h_u} M_f \mid_{h_u} = \langle M_f h_u, h_u\rangle \\ &= \int_{\partial\Omega} f(s)|h_u(s)|^2\, ds\end{aligned}$$

where ds denotes arc-length measure.

In the case when Ω is an annulus, i.e., $n = 1$, we saw in Example 3.6 that the space of positive extensions was a closed interval with two resolutions above each interior point and one above each endpoint. Thus, we see that, with the exception of two distinguished points on the circle for every other point u, there is exactly one point v such that $|h_u| = |h_v|$. Moreover, for each representing measure, one of these resolutions must correspond to the cyclic (i.e., minimal representing pair) and the other to the maximal representing pair. Thus, either $\mathcal{R}(\Omega)k^u_w$ is dense in $H^2_U(\Omega)$ or $\mathcal{R}(\Omega)k^v_w$ is dense in $H^2_V(\Omega)$.

Let's say that $\mathcal{R}(\Omega)k^u_w$ is dense in $H^2_U(\Omega)$ and, returning to the notation of Example 3.6, let $dm_t = |h_u|^2\, ds = |h_v|^2\, ds$. The two resolutions above m_t are given by the representing pairs $(H^2_0(\partial\Omega, m_t), H^2(\partial\Omega, m_t))$, which is the cyclic one, and $(H^2_0(\partial\Omega, m_t) \oplus N_t, H^2(\partial\Omega, m_t) \oplus N_t)$, both inside $\mathbf{L}^2(\partial\Omega, m_t)$.

Consider the unitary $\Theta : \mathbf{L}^2(\partial\Omega, m_t) \to \mathbf{L}^2(\partial\Omega, ds)$ given by multiplication by h_u. Since this is a $\mathbf{C}(\partial\Omega)$-module map, $\boldsymbol{C}_w$ is identified with the span of 1 in the first space and the span of h_u in the second, we get that

$$\Theta \cdot H^2(\partial\Omega, m_t) = H^2_U(\Omega).$$

Similarly, looking at the maximal representing pair and letting Φ denote the unitary obtained as multiplication by h_v, we find $\Phi(H^2(\partial\Omega, m_t) \oplus N_t) = H^2_V(\Omega)$.

Finally we see that the unitary operator on $\mathbf{L}^2(\partial\Omega, ds)$ given by multiplication by h_u/h_v carries $H^2_V(\Omega)$ to $H^2_U(\Omega) \oplus \Theta N_t$ while multiplication by h_v/h_u carries $H^2_U(\Omega)$ to the cyclic subspace of $H^2_V(\Omega)$ generated by h_v.

When $n > 1$, the problem of understanding how the n-torus of all $\mathbf{C}(\partial\Omega)$-resolutions fibers over the space of all (completely) positive extensions, i.e., positive representing measures for $\rho(f) = f(\omega)$, requires considerably more sophistication, but it is worked out in [**8**].

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Department of Mathematics, University of Houston, Houston, Texas 77204-3476
E-mail address: vern@math.uh.edu