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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THREE-TERM POINCARÉ DIFFERENCE EQUATIONS

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ABSTRACT. Let $\{X_n\}$ be a solution of the difference equation

$$X_n = (b+\delta_n)X_{n-1} + (a+\varepsilon_n)X_{n-2}.$$

Then it is well known that the limit $\lim X_n/(b+x)^n$ exists (and is finite) if $\sum|\varepsilon_n| < \infty$, $\sum|\delta_n| < \infty$ and $x(b+x) = a$ with $|x| < |b+x|$. We generalize this result to cases where $\sum \varepsilon_n$ and $\sum \delta_n$ only converge conditionally. Such results have applications to the study of asymptotic behavior of orthogonal polynomials and orthogonal functions, and to the study of separate convergence of continued fractions.

1. Introduction. We shall study solutions $\{Z_n\}_{n=-1}^{\infty}$ of the Poincaré difference equation

$$(1.1) \qquad Z_n = b_n Z_{n-1} + a_n Z_{n-2}; \qquad n = 1, 2, 3, \dots;\ a_n \neq 0,$$

where a_n and b_n are either complex numbers or complex valued functions such that

$$(1.2) \qquad \begin{gathered} a_n \longrightarrow a \in \mathbf{C}, \quad b_n \longrightarrow b \in \mathbf{C}\backslash\{0\}, \\ a/b^2 \in \mathbf{C}\backslash(-\infty, -1/4]. \end{gathered}$$

The characteristic equation of (1.1), $\lambda^2 = b\lambda + a$, has two solutions $x_1 = -x$ and $x_2 = b + x$, where

$$(1.3) \qquad x := b(\sqrt{1+4a/b^2} - 1)/2; \qquad \Re\sqrt{1+4a/b^2} > 0,$$

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and thus $|x_1| < |x_2|$. It is a consequence of the Poincaré-Pincherle-Perron theorem [**34, 36, 37, 41**, p. 45] that the solution space of (1.1) has a basis $\{\{X_n\}, \{Y_n\}\}$ such that $X_{n+1}/X_n \to -x$, $Y_{n+1}/Y_n \to b+x$, and thus $X_n/Y_n \to 0$. We say that a solution

$$\{Z_n\} = \alpha\{X_n\} + \beta\{Y_n\} \tag{1.4}$$

is *minimal* if $\alpha \neq 0$ and $\beta = 0$, and *dominant* if $\beta \neq 0$. (It is *trivial* if $\alpha = \beta = 0$.)

It is a classical result that if now

$$\sum |a_n - a| < \infty \quad \text{and} \quad \sum |b_n - b| < \infty, \tag{1.5}$$

then the (finite) limit

$$Z := \lim_{n\to\infty} \frac{Z_n}{(b+x)^n} \tag{1.6}$$

exists, and $Z \neq 0$ if and only if $\{Z_n\}$ is a dominant solution. Moreover, if $\{X_n\}$ is a minimal solution of (1.1) and $a \neq 0$, then the finite limit $X := \lim_{n\to\infty} X_n/(-x)^n$ also exists and is nonzero. Finally, if a_n and b_n are analytic functions converging locally uniformly in some domain D, (1.2) holds in D and the convergence in (1.5) is locally uniform in D, then the convergence in (1.6) is also locally uniform in D. (See, for example, [**20, 32, 40, 42**] for references and historical remarks.)

There has also been derived asymptotics for $\{Z_n\}$ in a series of particular cases where $\{a_n\}$ and $\{b_n\}$ behave regularly, although (1.5) fails. Birkhoff's method [**4–6**] to solve (1.1) when $\{a_n\}$ and $\{b_n\}$ have asymptotic expansions of the form

$$a_n \sim n^{k_1/\omega} \sum_{j=0}^{\infty} \alpha_j n^{-j/\omega}, \qquad (n \to \infty)$$
$$b_n \sim n^{k_2/\omega} \sum_{j=0}^{\infty} \beta_j n^{-j/\omega}, \qquad (n \to \infty)$$

for some natural number ω and real constants k_1, k_2 is of particular interest. This method is very well described in [**45**] or [**49**].

Recurrence relations where a_n and b_n are rational functions of n are sometimes related to hypergeometric functions and can thus be solved explicitly in terms of such functions. See, for instance, [**2, 8, 9–19, 25–30, 44–48**] and references therein.

The purpose of this paper is to show that the finite limit (1.6) also exists in cases where the series $\sum(a_n - a)$ and $\sum(b_n - b)$ only converge conditionally, and to derive upper bounds for the truncation error $|Z - Z_n/(b+x)^n|$. Trench [**42**] made a recent breakthrough on this problem. He considered $(N+1)$-term difference equations

$$\sum_{k=0}^{N} a_k^{(k)} Z_{n-k}; \qquad n = 1, 2, 3, \dots ; \qquad a_n^{(0)} = 1, \qquad a_n^{(N)} \neq 0, \tag{1.7}$$

where

$$\lim_{n\to\infty} a_n^{(k)} = a^{(k)} \in \mathbf{C}; \qquad k = 1, 2, \dots , N \tag{1.8}$$

and the corresponding characteristic equation $\sum_{k=0}^{N} a^{(k)} x^{N-k} = 0$ has distinct solutions $x_1, x_2, \dots , x_N$. He proved:

Theorem (Trench [**42**]). *In addition to the conditions above, suppose the following holds*:

(i) $0 < |x_1| \le |x_2| \le \cdots \le |x_N|$,

(ii) $\sum_{n=1}^{\infty}(a_n^{(k)} - a^{(k)})$ *converges* (*to a finite limit*) *for* $k = 1, 2, \dots , N$,

(iii) *there exist positive, nonincreasing null sequences* $\{\lambda_n\}$ *and* $\{\psi_n\}$ *such that* $\psi_n = o(\lambda_n)$ *as* $n \to \infty$ *and*

$$\begin{aligned} \sum_{j=m}^{\infty} (a_j^{(k)} - a^{(k)}) &= \mathcal{O}(\lambda_m), \qquad (m \to \infty) \\ \sum_{j=m}^{\infty} |a_j^{(k)} - a^{(k)}| \lambda_{N+j-k} &= \mathcal{O}(\psi_m), \qquad (m \to \infty) \end{aligned} \tag{1.9}$$

for $k = 1, 2 \dots , N$.

Let r be an integer in $\{1, 2, \dots, N\}$ such that $|x_r| \neq |x_k|$ if $r \neq k$. If $r > 1$, suppose also that there are an integer n_0 and a number R such that $1 < R < |x_r/x_{r-1}|$ and $\{R^n \lambda_n\}$ and $\{R^n \psi_n\}$ are nondecreasing for $n \geq n_0$. Then (1.7) *has a solution $\{Z_n\}$ such that*

$$Z_n = x_r^n(1 + \mathcal{O}(\lambda_n)) \quad \text{as } n \to \infty. \tag{1.10}$$

We restrict our investigations to the case $N = 2$ with $|x_1| < |x_2|$, but we allow $x = x_1 = 0$. We are mainly concerned with the limit (1.6) which corresponds to the case $r = 2$ in Trench's theorem. This is a particularly interesting case in applications, such as orthogonal polynomials and continued fractions.

Let us write

$$\begin{aligned} a_n &= a(1 - \varepsilon_n), \qquad b_n = b(1 + \delta_n), \\ \beta_n &:= \delta_{n-1} + \delta_n + \delta_{n-1}\delta_n + \varepsilon_n \end{aligned} \tag{1.11}$$

if $a \neq 0$ and

$$b_n = b(1 + \delta_n) \quad \text{if } a = 0. \tag{1.12}$$

Although the two cases are similar, it is convenient to treat them separately. The case where $a \neq 0$, and thus $x \neq 0$, corresponds to Trench's situation. For this we shall prove:

Theorem 1.1. *Let* (1.2) *hold with $a \neq 0$, and let there exist a positive nonincreasing null sequence $\{\lambda_n\}$ such that from some $m \geq m_0 \geq 0$ on, $\delta_m \neq -1$ and*

$$\begin{aligned} \left| \prod_{j=m+1}^{\infty} (1 + \delta_j) - 1 \right| &\leq (1 + d + C_1)\lambda_m, \\ \left| \sum_{j=m+2}^{\infty} \beta_j \prod_{\nu=m+1}^{j-2} (1 + \delta_\nu) \right| &\leq C_1 \lambda_m, \end{aligned} \tag{1.13}$$

$$|\delta_{m+1}| \leq d\lambda_m \quad \text{and} \quad \sum_{j=m+2}^{\infty} |\beta_j| \lambda_j \prod_{\nu=m+1}^{j-2} |1 + \delta_\nu| \leq C_2 \lambda_m$$

for some positive constants d, C_1 *and* C_2 *such that* $|q| := |-x/(b+x)| < 1/(1+C_2)$. *If* $\prod_{j=m+1}^{m+n}(1+\delta_j)$ *is uniformly bounded with respect to* m *and* n, *then* $\{Z_n/(b+x)^n\}$ *converges to a finite limit* Z, *and*

$$|Z - Z_n/(b+x)^n| \leq C \sum_{r=0}^{n} \lambda_{n-r}(r+1)^2(|q|Q)^r \tag{1.14}$$

for every $Q > 1 + C_2$ *for some constant* C *independent of* n, q *and* Q.

Remarks 1.1. 1. Theorem 1.1 is a special case of a more general result presented in Section 5 of this paper, Theorem 5.1. Its proof is also given in Section 5.

2. It is a consequence of Lemma 2.5, to come, that the righthand side of (1.14) converges to 0 as $n \to \infty$. If, in addition, there exists an $R < 1/|q|Q$ such that $\{R^n\lambda_n\}$ is nondecreasing, then

$$\begin{aligned}\sum_{r=0}^{n} \lambda_{n-r}(r+1)^2(|q|Q)^r &= \frac{1}{R^n}\sum_{r=0}^{n} \lambda_{n-r}R^{n-r}(r+1)^2(R|q|Q)^r \\ &\leq \frac{1}{R^n}\sum_{r=0}^{n} \lambda_n R^n (r+1)^2 (R|q|Q)^r \\ &\leq \lambda_n \frac{1+R|q|Q}{(1-R|q|Q)^3},\end{aligned}$$

and thus $|Z - Z_n/(b+x)^n| = \mathcal{O}(\lambda_n)$, as in Trench's theorem.

3. If a_n and b_n are complex valued functions, then also the constants d, C_1, C_2 and λ_m may be chosen as functions of the same variables. The bound (1.14) may then be used to prove uniform convergence of $\{Z_n/(b+x)^n\}$. An expression for the constant C in (1.14) can be derived from the proof of Theorem 5.1 and (2.4).

4. In frequent cases, the constant C_2 in Theorem 1.1 can be chosen smaller; the larger one chooses m_0. In the extreme case, these values of C_2 approach 0 as $m_0 \to \infty$. Then C_2 can be replaced by $\varphi_m = o(1)$ in (1.13), and we can conclude that $\{Z_n/(b+x)^n\}$ converges locally uniformly with respect to q, for $0 < |q| < 1$. Compared to Trench's

conditions,

$$(1.9)' \qquad \begin{aligned} \sum_{j=m}^{\infty} \delta_j &= \mathcal{O}(\lambda_m), \\ \sum_{j=m}^{\infty} |\delta_j|\lambda_{j+1} &= \mathcal{O}(\varphi_m \lambda_m), \\ \sum_{j=m}^{\infty} \varepsilon_j &= \mathcal{O}(\lambda_m), \\ \sum_{j=m}^{\infty} |\varepsilon_j|\lambda_{j+1} &= \mathcal{O}(\varphi_m \lambda_m), \end{aligned}$$

conditions (1.13) are weaker.

For the case where $a = 0$ we prove:

Theorem 1.2. *Let* (1.2) *hold with* $a = 0$, *let* δ_n *be given by* (1.12), *and let there exist a nonincreasing positive null sequence* $\{\lambda_n\}$ *such that*

$$(1.15) \qquad \begin{aligned} \left| \sum_{j=m+1}^{\infty} \delta_j \right| &\le \lambda_m, \\ \left| \sum_{j=m+1}^{\infty} \left(\frac{a_{j+1}}{b^2} + \delta_j \sum_{\nu=j+1}^{\infty} \delta_\nu \right) \right| &\le \rho \lambda_m, \\ \sum_{j=m+1}^{\infty} \left(\rho |\delta_j| \lambda_j + \left| \frac{a_{j+1}}{b^2} \right| \lambda_{j+1} \right) &\le \rho^2 \lambda_m \end{aligned}$$

for all $m \ge m_0$ *for some constants* $0 < \rho < 1$ *and* $m_0 \in \mathbf{N}_0$. *Then* $Z_n/(b+x)^n$ *converges to a finite value* Z, *and*

$$(1.16) \qquad |Z - Z_n b^{-n}| \le \frac{(|Z_{m_0}| + |a_{m_0+1} Z_{m_0-1}|) \max\{M\lambda_{m_0}, \rho\}}{\rho(1-\rho)} \sum_{r=1}^{n+1} \lambda_{n+1-r} \rho^r,$$

where $M := 2 + 2\lambda_{m_0}/\rho$.

Remarks 1.2. 1. Theorem 1.2 is a special case of the main result in Section 4, Theorem 4.1. If we add the condition that there exists an $M > 0$ such that

$$\left|\sum_{j=m+1}^{m+n}\delta_j\right| \le M\lambda_m, \qquad (1.17)$$
$$\left|\sum_{j=m+1}^{m+n-1}\left(\frac{a_{j+1}}{b^2}+\delta_j\sum_{\nu=j+1}^{m+n}\delta_\nu\right)\right| \le M\rho\lambda_m$$

for all $n \ge 1$ and $m \ge m_0$, we no longer need that $\{\lambda_n\}$ is nonincreasing. In this case we use this value for M in (1.16).

2. Also in this case one can often find smaller constants ρ if m_0 is increased. The result for the extreme case where $\rho\lambda_m$ can be replaced by $\psi_m = o(\lambda_m)$ is described in Corollary 4.2.

3. Theorem 1.2 generalizes the classical result that $P_n b^{-n}$ converges if $\sum|\delta_n| < \infty$ and $\sum|a_n| < \infty$. Just let $0 < R < 1/2$ be arbitrarily chosen, and let $\lambda_m := \max\{\sum_{j=m+1}^{\infty}|\delta_j|, R^{-1}\sum_{j=m+2}^{\infty}|a_j b^{-2}|\}$ for all m. Then there exists an $m_0 \in \mathbf{N}_0$ such that (1.15) holds with $\rho = 2R$. Since R was arbitrarily chosen, it follows that $Z_n b^{-n}$ converges.

4. As in 1.1 Remark 2, the bound in (1.16) vanishes as $n \to \infty$. If, moreover, there exists an $R < 1/\rho$ such that $\{R^n\lambda_n\}$ is nondecreasing, then $|Z - Z_n b^{-n}| \le C\rho\lambda_n/(1-\rho R)$ by (1.16), where C represents the constant factor in front of the summation in (1.16). Tighter bounds can be obtained from (4.4) in Section 4 .

The conditions in Theorems 1.1 and 1.2 are quite involved, as one would expect. An alternative approach to the problem is based on the classical result that if $\{Z_n\}$ is a solution of (1.1) and all $b_{2n} \ne 0$, then $\{Z_{2n}\}_{n=0}^{\infty}$ is a solution of

$$Z_{2n} = \tilde{b}_n Z_{2n-2} + \tilde{a}_n Z_{2n-4} \quad \text{for } n = 2, 3, 4, \dots, \qquad (1.18)$$

where

$$\tilde{b}_n := b_{2n-1}b_{2n} + a_{2n} + \frac{b_{2n}}{b_{2n-2}}a_{2n-1}, \qquad (1.19)$$
$$\tilde{a}_n := -\frac{b_{2n}}{b_{2n-2}}a_{2n-2}a_{2n-1}.$$

Similarly, if all $b_{2n-1} \neq 0$, then $\{Z_{2n+1}\}_{n=-1}^{\infty}$ is a solution of

$$Z_{2n+1} = b_n^* Z_{2n-1} + a_n^* Z_{2n-3} \quad \text{for } n = 1, 2, 3, \dots, \tag{1.20}$$

where

$$\begin{aligned} b_n^* &:= b_{2n} b_{2n+1} + a_{2n+1} + \frac{b_{2n+1}}{b_{2n-1}} a_{2n}, \\ a_n^* &:= -\frac{b_{2n+1}}{b_{2n-1}} a_{2n-1} a_{2n}. \end{aligned} \tag{1.21}$$

(See, for instance, [**35**, pp. 12–13, **22**, pp. 83–85].) If one of these recurrence relations, say (1.18), satisfies for instance the classical condition (1.5), then we also find that the limit Z in (1.6) exists:

Theorem 1.3. *Let* $\{Z_n\}$ *be the solution of* (1.1), *where* (1.2)–(1.3) *hold. Let* $b_{2n} \neq 0$ *for all* n, *let* $\tilde{a}_n$ *and* $\tilde{b}_n$ *be given by* (1.19), *and let* $\tilde{a} := \lim \tilde{a}_n$, $\tilde{b} := \lim \tilde{b}_n$ *and* $\tilde{x} := \tilde{b}\left(\sqrt{1 + 4\tilde{a}/\tilde{b}^2} - 1\right)/2$ *where* $\Re\sqrt{1 + 4\tilde{a}/\tilde{b}^2} > 0$. *If* $\{Z_{2n}/(\tilde{b} + \tilde{x})^n\}$ *converges to a finite limit* Z, *then* $\{Z_n/(b + x)^n\}$ *also converges to* Z.

Remarks 1.3. 1. If all $b_{2n+1} \neq 0$, then we can use the recurrence relation (1.20) instead of (1.18) and get an analogous result.

2. If, in particular, a_n and b_n are functions satisfying (1.2)–(1.3) in some domain D, and the convergence of $\{Z_{2n}/(\tilde{b} + \tilde{x})^n\}$, $\{a_{2n}\}$ and $\{b_{2n}\}$ are locally uniform in D, and $|b + x|$ is locally bounded away from zero in D, then also $\{Z_n/(b + x)^n\}$ converges locally uniformly in D to Z.

Theorem 1.3 applies to several interesting examples, some of which are presented in Section 3, along with the proof of Theorem 1.3.

Section 2 contains some auxiliary results for later reference. Section 4 is devoted to the case where $a = 0$ and $b \neq 0, \infty$, whereas Section 5 contains the case where $a \neq 0, \infty$ and $b \neq 0, \infty$. Section 6 contains some examples of applications.

2. Auxiliary results. It follows from (1.4) that if the limit Z in (1.6) exists and is nonzero for one solution of (1.1), then all solutions of (1.1) have the property (1.6). Actually, if $|Z - Z_n/(b+x)^n| = \mathcal{O}(\lambda_n)$ for one solution with $Z \neq 0$, then this holds for all solutions. Moreover, if a_n and b_n are complex valued functions and $X_n/(b+x)^n \to 0$, $Y_n/(b+x)^n \to Y$ locally uniformly in some domain D, then $Z_n/(b+x)^n \to Z = \beta Y$ locally uniformly in D as long as α and β are locally bounded in D. Hence, if the particular solution $\{P_n\}$ of (1.1) with initial values

$$P_{-1} := 0, \qquad P_0 := 1 \tag{2.1}$$

is dominant, we may restrict the investigations to $\{P_n\}$.

If $\{P_n\}$ happens to be minimal, then the solution $\{P_n^{(m)}\}_{n=-1}^{\infty}$ of the shifted recurrence relation

$$\begin{gathered} P_n^{(m)} = b_{m+n} P_{n-1}^{(m)} + a_{m+n} P_{n-2}^{(m)} \\ \text{where} \quad a_{m+n} \neq 0;\ n = 1, 2, 3, \ldots \end{gathered} \tag{2.2}$$

($m \in \mathbf{N}_0$ fixed), with initial values

$$P_{-1}^{(m)} = 0, \qquad P_0^{(m)} = 1 \tag{2.3}$$

is dominant for $m := 1$, since $\{P_n\}$ and $\{P_n^{(1)}\}$ are linearly independent. We may therefore assume that $\{P_n\}$ is dominant.

Throughout this paper, $\{P_n^{(m)}\}$ shall always denote the solution of (2.2) with initial values (2.3). Similarly, $\{P_n\}$ shall always be the solution of (1.1) with initial values (2.1), so that $P_n = P_n^{(0)}$. A general solution of (1.1) can then be written

$$\begin{gathered} Z_n = \alpha P_{n-m}^{(m)} + \beta P_{n-m-1}^{(m+1)} \quad \text{for } n \geq m, \\ \text{where} \quad \alpha := Z_m,\ \beta := a_{m+1} Z_{m-1}. \end{gathered} \tag{2.4}$$

Hence, results for $\{P_n^{(m)}/(b+x)^n\}$ imply results for $\{Z_n/(b+x)^n\}$.

By induction it follows that

$$P_n^{(m)} = b_{m+1} P_{n-1}^{(m+1)} + a_{m+2} P_{n-2}^{(m+2)} \quad \text{for all } m, n \in \mathbf{N}.$$

See, for instance, [**35**, p. 1] or [**22**, p. 58]. This means that

$$\frac{P_n^{(m)}}{(b+x)^n} = \frac{b_{m+1}}{b+x} \frac{P_{n-1}^{(m+1)}}{(b+x)^{n-1}} + \frac{a_{m+2}}{(b+x)^2} \frac{P_{n-2}^{(m+2)}}{(b+x)^{n-2}}. \tag{2.5}$$

The simple idea behind Theorems 1.1 and 1.2 is to write $P_n^{(m)}/(b+x)^n$ as a polynomial

$$\frac{P_n^{(m)}}{(b+x)^n} = \sum_{k=0}^{n} p(m; m+n; k) w^k \quad \text{for all } m, n \in \mathbf{N}_0. \tag{2.6}$$

If $a \neq 0$ (Theorem 1.1), we choose $w := q$ given by

$$q := \frac{-x}{b+x} = \frac{1-\sqrt{1+4a/b^2}}{1+\sqrt{1+4a/b^2}} \quad \text{where} \quad \Re\sqrt{1+4a/b^2} > 0, \tag{2.7}$$

and if $a = 0$ (Theorem 1.2), we choose $w := b^{-1}$. One can then prove (with the convention that an empty sum has value 0 and an empty product has value 1):

Lemma 2.1. *If $a \neq 0$, then $P_n^{(m)}/(b+x)^n$ can be written as a polynomial* (2.6) *with $w := q$. These polynomials are symmetric in the sense that*

$$p(m; m+n; k) = p(m; m+n; n-k) \quad \textit{for all } m, n, k \in \mathbf{N}_0, \tag{2.8}$$

and $p(m; m+n; k)$ is given recursively by

(2.9)

$$\begin{aligned} p(m; m+n; 0) &:= \prod_{j=m+1}^{m+n} (1+\delta_j) \quad \textit{for all } m, n \in \mathbf{N}_0, \\ p(m; m+n; k) &:= (1+\delta_{m+1}) \cdot p(m+1; m+n; k-1) \\ &\quad + \sum_{j=m+2}^{m+n-k+1} \beta_j \cdot p(j; m+n; k-1) \prod_{\nu=m+1}^{j-2} (1+\delta_\nu) \end{aligned}$$

for all $m, n \in \mathbf{N}_0$ for $k = 1, 2, 3, \ldots$, where δ_n and β_n are given by (1.11).

Proof. We shall first see that $P_n^{(m)}/(b+x)^n$ is a symmetric polynomial in q of degree $\le n$. This is clearly true for $n=0$ and $n=1$ since

$$\frac{P_0^{(m)}}{(b+x)^0} = 1$$

and

$$\frac{P_1^{(m)}}{(b+x)^1} = \frac{b(1+\delta_{m+1})}{b+x} = (1+q)(1+\delta_{m+1}).$$

Since $x(b+x)=a$, and thus, by (2.2),

$$\begin{aligned}\frac{P_n^{(m)}}{(b+x)^n} &= \frac{b(1+\delta_{m+n})}{b+x}\frac{P_{n-1}^{(m)}}{(b+x)^{n-1}} + \frac{a(1-\varepsilon_{m+n})}{(b+x)^2}\frac{P_{n-2}^{(m)}}{(b+x)^{n-2}} \\ &= (1+q)(1+\delta_{m+n})\frac{P_{n-1}^{(m)}}{(b+x)^{n-1}} - q(1-\varepsilon_{m+n})\frac{P_{n-2}^{(m)}}{(b+x)^{n-2}},\end{aligned}$$

the result follows by induction on n.

Formula (2.6) clearly holds for all m if $n=-1$ or $n=0$. Hence the result follows, since the proposed expression for $P_n^{(m)}/(b+x)^n$ satisfies the recurrence relation (2.5). □

Lemma 2.2. *If $a=0$, then $P_n^{(m)}/(b+x)^n$ can be written as a polynomial* (2.6) *in $w := b^{-1}$, whose coefficients $p(m;m+n;k)$ are given recursively by*

$$\begin{aligned} p(m;m+n;-1) &:= 0,\\ p(m;m+n;0) &:= 1\\ p(m;m+n;k) &:= \sum_{j=m+1}^{m+n-k+1} (\eta_j \cdot p(j;m+n;k-1) \\ &\qquad + a_{j+1}\cdot p(j+1;m+n;k-2)) \end{aligned} \tag{2.10}$$

for all $m,n \in \mathbf{N}_0$, for $k=1,2,3,\dots,$ where $\eta_n := b\delta_n$ with δ_n given by (1.12).

To see this, we just observe that it holds for all m if $n=-1$ or $n=0$ and that the expression (2.6) for $P_n^{(m)}/(b+x)^n$ satisfies the recurrence relation (2.5).

In both cases, if the limits

$$p(m;\infty;k) := \lim_{n\to\infty} p(m;n;k) \in \mathbf{C} \quad \text{for } k = 0,1,2,\dots \tag{2.11}$$

exist, and the formal power series

$$\sum_{k=0}^{\infty} p(m;\infty;k)w^k \tag{2.12}$$

converges for $|w| < R$ for some $R > 0$, and

$$\left(\sum_{k=0}^{\infty} p(m;\infty;k)w^k - \sum_{k=0}^{n} p(m;m+n;k)w^k\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{2.13}$$

for $|w| < R$, then (2.3) holds for $|w| < R$ with limit (2.12). This approach may seem very restrictive. Equation (2.11) is a strong condition. However, it is a necessary condition in important cases of uniform convergence of $\{P_n^{(m)}/(b+x)^n\}$ with respect to $w = q$ or $w = b^{-1}$ in some disk $|w| < R$:

Lemma 2.3. *Let $R > 0$ and $m \in \mathbf{N}_0$ be fixed. Let a_n, b_n, b and x be analytic functions of a complex variable w for $|w| < R$, and let $b(w)+x(w) \neq 0$ for $|w| < R$. Assume that $P_n^{(m)}/(b+x)^n$ can be written as a polynomial* (2.6) *in w and that $\{P_n^{(m)}/(b+x)^n\}$ converges locally uniformly in $|w| < R$. Then the limits* (2.11) *exist, and the power series* (2.12) *converges for $|w| < R$.*

Proof. Since the limit $P^{(m)}$ is an analytic function of w, it has a power series expansion of the form (2.12) which converges to $P^{(m)} =: P^{(m)}(w)$ for $|w| < R$. Since the convergence in (2.13) is locally uniform, it follows by Weierstrass's double series theorem that the coefficients of this power series satisfy (2.11). □

Of course, the expression (2.6) for $P_n^{(m)}/(b+x)^n$ gives an expression for the limit $P^{(m)}$, provided the quantities involved converge. Actually we shall see that

$$P^{(m)}(w) := P^{(m)} = \lim_{n\to\infty} \frac{P_n^{(m)}}{(b+x)^n} = \sum_{j=0}^{\infty} t_j^{(m)} w^j \tag{2.14}$$

provided the series converges, where $w = q$ and

$$\begin{aligned} t_0^{(m)} &:= \prod_{j=m+1}^{\infty} (1+\delta_j) \quad \text{for all } m \\ t_k^{(m)} &:= (1+\delta_{m+1}) t_{k-1}^{(m+1)} \\ &\quad + \sum_{j=m+2}^{\infty} \beta_j t_{k-1}^{(j)} \prod_{\nu=m+1}^{j-2} (1+\delta_\nu) \quad \text{for all } m, \end{aligned} \tag{2.15}$$

for $k = 1, 2, 3, \ldots$, if $a \neq 0$, and where $w := b^{-1}$ and

$$\begin{aligned} t_{-1}^{(m)} &:= 0, \qquad t_0^{(m)} := 1 \quad \text{for all } m \in \mathbf{N}_0 \\ t_k^{(m)} &:= \sum_{j=m+1}^{\infty} (\eta_j t_{k-1}^{(j)} + a_{j+1} t_{k-2}^{(j+1)}) \\ &\qquad \text{for all } m; \qquad \eta_j = b\delta_j \end{aligned} \tag{2.16}$$

for $k = 1, 2, 3, \ldots$, if $a = 0$, assuming that these quantities are well-defined complex numbers.

Lemma 2.4. *Let $k^* \in \hat{\mathbf{N}}$. The finite limits*

$$\begin{aligned} p(m;\infty;k) &:= \lim_{n\to\infty} p(m; m+n; k) \\ &\textit{for all } m \in \mathbf{N}_0 \textit{ and } k < k^* \end{aligned} \tag{2.17}$$

exist if and only if all $t_k^{(m)}$ are well defined (and finite) for $k < k^*$; by (2.15) if $a \neq 0$, by (2.16) if $a = 0$. The limits are then $p(m;\infty;k) = t_k^{(m)}$.

The proof is found in Section 4 for the case $a = 0$ and in Section 5 for the case $a \neq 0$.

Lemmas 2.1 and 2.2 are strongly related to other expressions for solutions of recurrence relations (1.1) and (1.7). See, for instance, Masson [**24**], Trench [**42**] and Runckel [**38, 39**]. Let is also be mentioned that, in a series of beautiful papers and results, Ramanujan (his results are further clarified and proved by Berndt [**3**]), Masson, Ismail, Gupta, Wimp [**9–19, 25–31, 44–48**], and others have derived expressions of a similar nature for solutions of particular recurrence relations related to hypergeometric functions.

We shall also apply the following standard result which, for instance, can be found in [**21**, p. 78]:

Lemma 2.5. *Let $\{x_n\}$ and $\{y_n\}$ be two real null sequences. If $\sum |y_n| < \infty$, then*

$$z_n := \sum_{r=0}^{n} x_r y_{n-r}; \quad n = 0, 1, 2, \dots$$

is also a null sequence.

3. Theorem 1.3 revisited. The simple idea of Theorem 1.3, to study $\{Z_{2n}\}$ and its recurrence relation (1.18) instead of $\{Z_n\}$, has many applications. The examples we show in this section can all be treated by Birkhoff's method, which then gives stronger asymptotics. (The strength of the present method is that it also applies to cases where Birkhoff's method fails.)

Example 3.1. Let $a_n := (-1)^{n+1} z/n$ and $b_n := 1$ for all n. Then $a_n \to a = 0$, $b_n = b = 1$ and $x = 0$, and we are therefore concerned with the possible convergence of $Z_n(z)$ or $P_n(z)$. The coefficients of (1.18) are

$$\tilde{b}_n(z) = 1 + \frac{z}{4n^2 - 2n},$$
$$\tilde{a}_n(z) = \frac{z^2}{4n^2 - 6n + 2}.$$

Since $\sum |\tilde{b}_n(z) - 1| < \infty$ and $\sum |\tilde{a}_n(z)| < \infty$, it follows that $\{P_{2n}(z)\}$ converges locally uniformly to an entire function of z in $\mathbf{C}$. (Continuity arguments show that $z = 0$ does not have to be excluded in this case.)

Hence $\{P_n(z)\}$ converges locally uniformly in $\mathbf{C}$ to an entire function by Theorem 1.3. This function is not identically zero, since $P_n(0) = 1$ for all n. It follows similarly that $\{P_n^{(m)}(z)\}$ converges locally uniformly to an entire function not identically zero for every $m \in \mathbf{N}_0$.

Example 3.2. Let $a_n := (-1)^{n+1} z/n^\alpha$ and $b_n := 1$ for all n, where $1/2 < \alpha < 1$. Then

$$\begin{aligned} \tilde{b}_n(z) &= 1 + \frac{\alpha z}{2n(2n-1)^\alpha} + \mathcal{O}(n^{-2-\alpha}), \\ \tilde{a}_n(z) &= \frac{z^2}{(2n-2)^\alpha (2n-1)^\alpha} \end{aligned}$$

for $n \geq 2$. Since, thus, $\sum |\tilde{b}_n(z) - 1| < \infty$ and $\sum |\tilde{a}_n(z)| < \infty$, it follows that $P_n(z)$ and $P_n^{(m)}(z)$ converge locally uniformly to entire functions not identically zero, by the same arguments as in the previous example.

Example 3.3. Let $a_n := (a + (-1)^{n+1}/n^\alpha)z$ and $b_n := b \neq 0, \infty$ for all n, where $\alpha > 1/2$ and $az/b^2 \notin (-\infty, -1/4]$. We consider a and b as fixed constants and z as a complex variable. Let $x(z) := b(\sqrt{1 + 4az/b^2} - 1)/2$ where $|b + x| > |x|$. Then

$$\begin{aligned} \tilde{b}_n(z) &= b^2 + 2az + \frac{\alpha z}{2n(2n-1)^\alpha} + \mathcal{O}(n^{-2-\alpha}), \\ \tilde{a}_n(z) &= -a^2 z^2 + \frac{z^2}{(2n-2)^\alpha} \\ &\quad \times \left(\frac{a\alpha}{2n-1} + \frac{1}{(2n-1)^\alpha} + \mathcal{O}(n^{-2}) \right). \end{aligned}$$

Since $\sum |\tilde{b}_n(z) - b^2 - 2az| < \infty$ and $\sum |\tilde{a}_n(z) + a^2 z^2| < \infty$, it follows that $P_n^{(m)}(z)/(b + x(z))^n$ converges locally uniformly to an analytic function in the cut plane where $az/b^2 \notin (-\infty, -1/4]$ for every $m \in \mathbf{N}_0$. This function is not identically zero, since $P_n^{(m)}(0)/(b + x(0))^n = 1$ for all n and m. (Again, continuity arguments show that the point $z = 0$ does not have to be excluded.)

Example 3.4. The Kummer function is given by

$$M(\beta;\gamma;z) := {}_1F_1(\beta;\gamma;z) := \sum_{k=0}^{\infty} \frac{(\beta)_k}{(\gamma)_k} \cdot \frac{z^k}{k!},$$

where $(d)_k$ denotes the usual Pochhammer symbol. We assume that $\beta+1, \gamma, \gamma-\beta \in \mathbf{C}\backslash\{0,-1,-2,\dots\}$. According to Gauss's contiguous relations for hypergeometric functions, the sequence $\{f_n(z)\}$ where

$$\begin{aligned} f_{2n}(z) &:= M(\beta+n;\gamma+2n;z), \\ f_{2n+1}(z) &:= M(\beta+n+1;\gamma+2n+1;z), \end{aligned}$$

is a solution of the recurrence relation

$$f_n(z) = f_{n+1}(z) + a_{n+1}zf_{n+2}(z); \quad n = 0,1,2,\dots,$$

where

$$\begin{aligned} a_{2n}z &= \frac{(\beta+n)z}{(\gamma+2n-1)(\gamma+2n)}, \\ a_{2n+1}z &= -\frac{(\gamma-\beta+n)z}{(\gamma+2n)(\gamma+2n+1)} \end{aligned}$$

for all n. See, for instance, [**1**, p. 507, **20**, p. 206] or [**22**, p. 313]. It follows (see, for instance, [**20**, p. 206] or [**22**, p. 313]) that

$$\frac{M(\beta;\gamma;z)}{M(\beta+1;\gamma+1;z)} = 1 - \frac{\gamma-\beta}{\gamma(\gamma+1)} z \lim_{n\to\infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)},$$

where $\{P_n^{(1)}\}$ and $\{P_n\}$ are the special solutions of the recurrence relation

$$Z_n(z) = Z_{n-1}(z) + a_n z_n Z_{n-2}(z); \quad n = 1,2,3,\dots,$$

which we are considering here. The coefficients of (1.18) are now

$$\begin{aligned} \tilde{b}_n(z) &= 1 + \frac{(2\beta-\gamma)z}{(2n+\gamma)(2n+\gamma-2)} = 1 + \mathcal{O}(n^{-2}), \\ \tilde{a}_n(z) &= \mathcal{O}(n^{-2}). \end{aligned}$$

This means that $\sum |\tilde{a}_n(z)| < \infty$ and $\sum |\tilde{b}_n(z) - 1| < \infty$, and thus $\{P_n^{(1)}(z)\}$ and $\{P_n(z)\}$ converge locally uniformly in $\mathbf{C}$ to entire functions $P^{(1)}(z)$ and $P(z)$. As before, $z = 0$ represents no problem. Since $P_n^{(m)}(0) = 1$ for all m and n, it follows that neither $P^{(1)}(z)$ nor $P(z)$ are identically equal to zero.

This result is not new, of course. Actually, it follows from the recurrence relation for $\{f_n(z)\}$ above that $\{f_{2n}(z)\}$ is a solution of

$$f_{2n-2}(z) = \tilde{b}_n(z) f_{2n}(z) + \tilde{a}_{n+1}(z) f_{2n+2}(z)$$

and thus, by standard arguments, that

$$\begin{aligned} F_{2n}(z) &= M(-\beta - n; \gamma - 2n; -z) \\ G_{2n}(z) &= (-1)^n z^{2n} \frac{\Gamma(n+\beta+1)\Gamma(n+\gamma-\beta+1)}{\Gamma(2n+\gamma+1)\Gamma(2n+\gamma+2)} \\ &\qquad M(\gamma - \beta + n + 1; \gamma + 2n + 2; -z) \end{aligned}$$

are solutions of the recurrence relation (1.18). (See, for instance, [**25, 44, 46, 22**, p. 198].) The general solution of (1.18) is therefore $Z_{2n}(z) = C_1 F_{2n}(z) + C_2 G_{2n}(z)$ in the present case. From [**23**, p. 133] we find that both $M(-\beta - n; -\gamma - 2n; -z)$ and $M(\gamma - \beta + n + 1; \gamma + 2n + 2; -z)$ converge locally uniformly to $e^{-z/2}$. Since Stirling's formula shows that

$$\begin{aligned} &\frac{\Gamma(n+\beta+1)\Gamma(n+\gamma-\beta+1)}{\Gamma(2n+\gamma+1)\Gamma(2n+\gamma+2)} \\ &\qquad = e^{-\gamma-1}\left(\frac{\varepsilon}{4n}\right)^{2n+\gamma+1} (1 + \mathcal{O}(n^{-1})) \quad \text{as } n \to \infty, \end{aligned}$$

this means that $G_n(z)$ converges locally uniformly to 0. More accurate results can also be obtained from these sources.

Example 3.5. Let $a_n := a + (-1)^{n+1} c/n^\alpha$ and $b_n := z + (-1)^n d/n^\beta$, where $\alpha > 1/2$, $\beta > 1/2$ and $a/z^2 \notin [-\infty, -1/4]$. We assume that $c \in \mathbf{C}$ is chosen such that all $a_n \neq 0$. Then

$$\begin{aligned} \tilde{b}_n &= z^2 + 2a - \frac{d\beta z}{2n(2n-1)^\beta} - \frac{d^2}{(2n-1)^\beta (2n)^\beta} \\ &\quad + \frac{c\alpha}{2n(2n-1)^\alpha} - \frac{da\beta/z}{n(2n-2)^\beta} + \mathcal{O}(n^{-1-\varepsilon}) \end{aligned}$$

for an $\varepsilon > 0$, and

$$\begin{aligned}\tilde{a}_n = -a^2 + \frac{ca\alpha}{2n(2n-1)^\alpha} + \frac{da^2\beta/z}{n(2n-2)^\beta} \\ + \frac{c^2}{(2n-1)^\alpha(2n-2)^\alpha} + \mathcal{O}(n^{-1-\mu})\end{aligned}$$

for a $\mu > 0$. Hence $\{P_n^{(m)}/(z+x)^n\}$, where $x := z(\sqrt{1+4a/z^2} - 1)/2; \Re\sqrt{\cdots} > 0$, and thus $|z+x| > |x|$, converges locally uniformly in the cut plane where $a/z^2 \notin [-\infty, -1/4]$, to an analytic function for every $m \in \mathbf{N}_0$. Since $\lim_{z\to\infty} P_n^{(m)}/(z+x)^n = 1$ at $z = 0$ for all $m, n \in \mathbf{N}_0$, this function is not identically zero.

Example 3.6. Similarly to the Kummer function in Example 3.4, the hypergeometric function

$$\begin{gathered}{}_2F_1(\alpha, \beta; \gamma; z) := \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}; \\ \alpha, \beta+1, \gamma, \gamma-\beta, \gamma-\alpha+1 \in \mathbf{C}\backslash\{0, -1, -2, \dots\}\end{gathered}$$

satisfies the recurrence relation

$$f_n(z) = f_{n+1}(z) + a_{n+1}zf_{n+2}(z); \quad n = 0, 1, 2, \dots,$$

where

$$\begin{aligned}a_{2n+1} &= -\frac{(\alpha+n)(\gamma-\beta+n)}{(\gamma+2n)(\gamma+2n+1)}, \\ a_{2n} &= -\frac{(\beta+n)(\gamma-\alpha+n)}{(\gamma+2n-1)(\gamma+2n)},\end{aligned}$$

in the sense that

$$\begin{aligned}f_{2n}(z) &= {}_2F_1(\alpha+n, \beta+n; \gamma+2n; z), \\ f_{2n+1}(z) &= {}_2F_1(\alpha+n; \beta+n+1; \gamma+2n+1; z).\end{aligned}$$

See, for instance, [**20**, p. 200, **22**, p. 294] where it is also proved that

$$\frac{f_0(z)}{f_1(z)} = 1 - \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}z \lim_{n\to\infty} \frac{P_n^{(1)}(z)}{P_n(z)},$$

where $\{P_n^{(1)}\}$ and $\{P_n\}$ are our specific solutions of the recurrence relation

$$Z_n(z) = Z_{n-1}(z) + a_n z Z_{n-2}(z); \quad n = 1, 2, 3, \ldots .$$

The coefficients of (1.18) are now

$$\begin{aligned}\tilde{b}_n(z) &= 1 - \frac{z}{2} + \frac{2\alpha\beta - \gamma(\alpha+\beta-\gamma/2)}{(\gamma+2n-2)(\gamma+2n)} z, \\ \tilde{a}_n(z) &= -\frac{z^2}{16} \\ &\quad - \frac{(\beta-\alpha+1/2+\gamma(\alpha+\beta-\gamma/2)-2\alpha\beta)n + \mathcal{O}(1)}{2(\gamma+2n-3)(\gamma+2n-2)(\gamma+2n-1)} z^2 \\ &\quad + \frac{(\beta-\alpha+1/2)^2(n^2+(\gamma-2)n)+\mathcal{O}(1)}{(\gamma+2n-3)(\gamma+2n-2)^2(\gamma+2n-1)} z^2.\end{aligned}$$

Hence, $\sum|\tilde{b}_n(z) - 1 + z/2| < \infty$ and $\sum|\tilde{a}_n(z) + z^2/16| < \infty$, which means that $\{P_n(z)/(1+x(z))^n\}$, where $x(z) := (\sqrt{1-z}-1)/2$, converges locally uniformly in the cut plane $z \in D := \mathbf{C}\backslash[1,\infty)$ to an analytic function $P(z)$. (The point $z = 0$ is included by a continuity argument.) Since $P(0) = 1$, this function is not identically zero. Similarly, $P_n^{(1)}(z) \to P^{(1)}(z) \not\equiv 0$ locally uniformly in D.

Also, this is already well known. In fact, also in this case, we know the general solution (1.18). It is $Z_{2n}(z) = C_1 F_{2n}(z) + C_2 G_{2n}(z)$ where

$$\begin{aligned}F_{2n}(z) &= {}_2F_1(-\alpha-n, -\beta-n; -\gamma-2n; z) \\ G_{2n}(z) &= z^{2n} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\gamma+2n+1)\Gamma(\gamma+2n+2)} \\ &\quad \times \Gamma(\gamma-\alpha+n+1)\Gamma(\gamma-\beta+n+1) \\ &\quad \times {}_2F_1(\alpha+n+1, \beta+n+1; \gamma+2n+2; z).\end{aligned}$$

According to Watson's asymptotic formula [**43**] (see, for instance, [**7**, p. 77])

$$\begin{aligned}&\left(\frac{z-1}{2}\right)^{-a-\lambda} {}_2F_1\left(a+\lambda, a-c+1+\lambda; a-b+1+2\lambda : \frac{2}{1-z}\right) \\ &\qquad = 2^{a+b}\sqrt{\frac{\pi}{\lambda}} \frac{\Gamma(a-b+1+2\lambda)}{\Gamma(a-c+1+\lambda)\Gamma(c-b+\lambda)} e^{-(a+\lambda)\xi} \\ &\qquad\quad \times (1-e^{-\xi})^{-c+1/2}(1+e^{-\xi})^{c-a-b-1/2} \\ &\qquad\quad \times (1+\mathcal{O}(\lambda^{-1})) \quad \text{as } |\lambda| \to 0\end{aligned}$$

for $|\arg\lambda| \le \pi-\delta$ for a $\delta>0$, where ξ is defined by $z\pm\sqrt{z^2-1}=e^{\pm\xi}$. By setting $a:=\alpha$, $b:=\alpha-\gamma+1$, $c:=\alpha-\beta+1$, $\lambda:=n$ and $z:=1-2/w$ in Watson's formula, we get $e^{-\xi}=-(1+\sqrt{1-w})/(1-\sqrt{1-w})$, and thus

$$\begin{aligned}
{}_2F_1(\alpha+n,\beta+n;\gamma+2n;w)\\
&=\left(-\frac{1}{w}\right)^{\alpha+n}2^{2\alpha-\gamma+1}\sqrt{\frac{\pi}{n}}\frac{\Gamma(\gamma+2n)}{\Gamma(\beta+n)\Gamma(\gamma-\beta+n)}\\
&\quad\times\left(-\frac{1+\sqrt{1-w}}{1-\sqrt{1-w}}\right)^{\alpha+n}\left(\frac{2}{1-\sqrt{1-w}}\right)^{\beta-\alpha-1/2}\\
&\quad\times\left(\frac{-2\sqrt{1-w}}{1-\sqrt{1-w}}\right)^{\gamma-\alpha-b-1/2}(1+\mathcal{O}(n^{-1}))
\end{aligned}$$

where Stirling's formula shows that

$$\frac{\Gamma(\gamma+2n)}{\Gamma(\beta+n)\Gamma(\gamma-\beta+n)}=2^{2n+\gamma-1}\sqrt{\frac{n}{\pi}}(1+\mathcal{O}(n^{-1})).$$

Hence

$$\begin{aligned}
{}_2F_1(\alpha+n,\beta+n;\gamma+2n;z)\\
&=\left(\frac{1}{x(z)}\right)^{2n+\gamma-1}(\sqrt{1-z})^{\gamma-\alpha-\beta-1/2}\\
&\quad\times((-1)^{\beta-\alpha-1/2}+\mathcal{O}(n^{-1})\\
&=\left(-\frac{4}{z}\right)^{2n+\gamma-1}(1+x(z))^{2n+\gamma-1}\\
&\quad\times(\sqrt{1-z})^{\gamma-\alpha-b-1/2}((-1)^{\beta-\alpha-1/2}+\mathcal{O}(n^{-1})).
\end{aligned}$$

Proof of Theorem 1.3. We have $\tilde{a}=-a^2$, $\tilde{b}=b^2+2a$ and

$$\tilde{x}=(b^2\sqrt{1+4a/b^2}-b^2-2a)/2\quad\text{where } |\tilde{b}+\tilde{x}|>|\tilde{x}|.$$

Since $a=x(b+x)$ by (1.3), it thus follows that $\tilde{x}=-x^2$ and $(\tilde{b}+\tilde{x})=(b+x)^2$, and thus $|\tilde{b}+\tilde{x}|>|\tilde{x}|$ if and only if $|b+x|>|x|$.

If $Z_{2n}/(\tilde{b}+\tilde{x})^n = Z_{2n}/(b+x)^{2n}$ converges to Z, then it follows from (1.1) that

$$\frac{b_{2n}}{b+x}\cdot\frac{Z_{2n-1}}{(b+x)^{2n-1}} = \frac{Z_{2n}}{(b+x)^{2n}} - \frac{a_{2n}}{(b+x)^2}\cdot\frac{Z_{2n-2}}{(b+x)^{2n-2}}$$
$$\longrightarrow Z - \frac{a}{(b+x)^2}Z = \frac{b}{b+x}Z.$$

Since $b_{2n}\to b\neq 0$, this proves the assertions. □

Theorem 1.3 was based on even (or odd) contractions. One can also use other contractions than the even part $\{P_{2n}\}$ or the odd part $\{P_{2n+1}\}$ of $\{P_n\}$. If $\{n_k\}$ is a subsequence of $\mathbf{N}$, $n_{-1} := -1$, and $P^{n_{k-1}+1}_{n_k-n_{k-1}-1}\neq 0$ for all k, then

(3.1) $$P_{n_k} = \tilde{b}_k P_{n_{k-1}} + \tilde{a}_k P_{n_{k-2}} \quad \text{for } k = 1,2,3,\dots,$$

where

(3.2) $$\begin{aligned}\tilde{b}_1 &:= P_{n_1}/P_{n_0}\\ \tilde{a}_k &:= -\frac{P^{(n_{k-1}+1)}_{n_k-n_{k-1}-1}}{P^{(n_{k-2}+1)}_{n_{k-1}-n_{k-2}-1}}\prod_{j=n_{k-2}+2}^{n_{k-1}+1}(-a_j),\\ \tilde{b}_k &:= \frac{P^{(n_{k-2}+1)}_{n_k-n_{k-2}-1}}{P^{(n_{k-2}+1)}_{n_{k-1}-n_{k-2}-1}}\end{aligned}$$

for $k = 2,3,4,\dots$. (See [**35**, p. 12].)

Another tool of this type is extensions, [**35**, p. 13]. Also other transformations can be used, such as the Bauer-Muir transformation: If $\{Z_n\}$ is a solution of (1.1), and $\{w_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that

(3.3) $$\varphi_n := a_n - w_{n-1}(b_n+w_n)\neq 0 \quad \text{for all } n\in\mathbf{N},$$

then $\{Z_n + Z_{n-1}w_n\}$ is a solution of the recurrence relation

(3.4) $$X_n = \left(b_n + w_n - \frac{\varphi_n}{\varphi_{n-1}}w_{n-2}\right)X_{n-1} + a_{n-1}\frac{\varphi_n}{\varphi_{n-1}}X_{n-2}; \quad n = 2,3,4,\dots.$$

(See, for instance, [**35**, p. 25, **22**, p. 76].)

4. Theorem 1.2 revisited. In this section we consider the case where $a_n \to 0$ and $b_n \to b \neq 0, \infty$, and we use the notation

$$b_n = b(1+\delta_n) = b + \eta_n. \tag{4.1}$$

Since x given by (1.3) is equal to zero, we want to find sufficient conditions for the existence of the limit $Z := \lim_{n\to\infty} Z_n b^{-n}$. Our main result is:

Theorem 4.1. *Let* (1.2) *hold with* $a = 0$ *and* $b \neq 0, \infty$. *Let there exist positive constants* $\rho < 1$ *and* M, *integers* $k_0 \geq 1$ *and* $m_0 \geq 0$, *and a positive null sequence* $\{\lambda_n\}$ *such that the following three conditions all hold with the notation from* (4.1), *Lemma* 2.2 *and* (2.16):

(i) $\sum_{m=1}^{\infty}(\eta_m t_k^{(m)} + a_{m+1} t_{k-1}^{(m+1)})$ *converges to a finite value for* $k = 0, 1, \dots, k_0$.

(ii) $|t_{k_0}^{(m)}| \leq \lambda_m$, $|t_{k_0+1}^{(m)}| \leq |b|\rho\lambda_m$, $\sum_{j=m+1}^{\infty}(\rho|\delta_j|\lambda_j + |a_{j+1}b^{-2}|\lambda_{j+1}) \leq \rho^2\lambda_m$ *for all* $m \geq m_0$.

(iii) $|p(m; m+n; k_0)| \leq M\lambda_m$, $|p(m; m+n; k_0+1)| \leq M|b|\rho\lambda_m$ *for all* $m \geq m_0$ *and* $n \geq k_0$.

Then $\{P_n^{(m)} b^{-n}\}_{n=0}^{\infty}$ *converges to the finite limit* $P^{(m)}$ *in* (2.14) *for every* $m \in \mathbf{N}_0$, *and*

$$|P^{(m)} - P_n^{(m)} b^{-n}| \leq \frac{H_m}{C^{2k_0}(1-\rho)} \times \left\{ \sum_{r=1}^{k_0-1} \mu_{m+n+1-r}\rho^r + \sum_{r=k_0}^{n+1} \lambda_{m+n+1-r}\rho^r \right\} \tag{4.2}$$

for $m \geq m_0$ *and* $n \geq k_0 - 1$, *where* $C = \rho|b|$, $\mu_m := \max_{1 \leq r < k_0} |t_r^{(m)}|$ *and*

$$K_m := \max\{|p(m; m+n; r)| : 0 \leq r < k_0 \text{ and } n \in \mathbf{N}_0\},$$
$$H_m := \max\{M\lambda_m, M\lambda_m C, M\lambda_m C^{k_0-1}, K_m C, K_m C^{k_0+1}, K_m C^{2k_0-1}, C^{k_0}\}.$$

Remarks 4.1. Condition (i) is just that $t_k^{(m)}$ exists for $k = 0, 1, \dots, k_0 + 1$. Condition (ii) ensures the existence of all $t_k^{(m)}$ for larger k.

2. Theorem 4.1 is not so easy to apply if $k_0 > 1$. Its conditions involve both $p(m; m+n; k)$ and $t_k^{(m)}$ for $k = 0, 1, \dots, k_0$. It should be regarded as an auxiliary result. The useful results are its corollaries, such as Theorem 1.2.

3. The righthand side of (4.2) converges to 0 as $n \to \infty$. This is a consequence of Lemma 2.5. If there exists an $R < 1/\rho$ such that $\{R^n \lambda_n\}$ is nondecreasing, then it can be replaced by $\mathcal{O}(\lambda_{m+n})$ as in Remark 1.2.4. From the proof of Theorem 4.1 one also gets the following tighter bound in Theorem 4.1:

$$
\begin{aligned}
|P^{(m)} - P_n^{(m)} b^{-n}| \le \frac{1}{C^{2k_0}} \sum_{k=1}^{n} \Bigg\{ & M\lambda_m \\
& \times \sum_{r=1}^{\min\{k-k_0, k_0-1\}} |t_r^{(m+n+1-r)}| C^{k_0-r} \\
& + M\lambda_m \sum_{r=k_0}^{k-k_0} \lambda_{m+n+1-r} \\
& + \sum_{r=\max\{1, k-k_0+1\}}^{\min\{k, k_0-1\}} |t_r^{(m+n+1-r)} \\
& \cdot p(m; m+n-r; k-r)| C^{2k_0-k} \\
& + \sum_{r=\max\{k_0, k-k_0+1\}}^{k} \lambda_{m+n+1-r} \\
& \times |p(m; m+n-r; k-r)| C^{r+k_0-k} \Bigg\} \rho^k \\
& + \frac{\lambda_m}{C^{k_0}} \frac{\rho^{n+1}}{1-\rho}
\end{aligned}
\tag{4.3}
$$

for $m \ge m_0$ and $n \ge k_0 - 1$.

The proof of Theorem 4.1 is deferred to the end of this section. We shall rather look at some implications. The conditions in Theorem 4.1 simplify considerably if we require that $k_0 := 1$, which is the case described in Theorem 1.2.

Proof of Theorem 1.2. Let $k_0 := 1$ and replace $\{\lambda_n\}$ by $\{|b|\lambda_n\}$ in Theorem 4.1. Assume that the conditions in Theorem 1.2 hold. Since $t_0^{(m)} = 1$ and $t_{-1}^{(m)} = 0$ and, since $\sum \eta_m$ and $\sum(\eta_m t_1^{(m)} + a_{m+1})$ converge by (1.15), Condition (i) in Theorem 4.1 holds. Also Condition (ii) holds, since by the first two inequalities in (1.15) we have

$$|t_1^{(m)}| = \left| \sum_{j=m+1}^{\infty} \eta_j \right| \leq |b|\lambda_m,$$

$$|t_2^{(m)}| = \left| \sum_{j=m+1}^{\infty} (\eta_j t_1^{(j)} + a_{j+1}) \right| \leq |b|^2 \rho \lambda_m,$$

and the third inequality carries over by multiplication of $|b|$. To check Condition (iii) we observe that, by (2.10), $p(m; m+n; 1) = \sum_{j=m+1}^{m+n} \eta_j = t_1^{(m)} - t_1^{(m+n)}$, which means that $|p(m; m+n; 1)| \leq 2|b|\lambda_m$, when $\{\lambda_n\}$ is nonincreasing.

In the same way, we find that $p(m; m+n; 2) = t_2^{(m)} - t_1^{(m+n)} \cdot p(m; m+n-1; 1) - t_2^{(m+n-1)}$, and thus $|p(m; m+n; 2)| \leq (2\rho + 2\lambda_{m+n})|b|^2 \lambda_m$. Hence, Condition (iii) holds with $M := 2 + 2\lambda_{m_0}/\rho$. We therefore get from (4.3) that

$$\begin{aligned} |P^{(m)} - P_n^{(m)} b^{-n}| &\leq \frac{1}{\rho^2 |b|^2} \sum_{k=1}^{n} \Big\{ M|b|\lambda_m \\ &\qquad \times \sum_{r=1}^{k-1} |b|\lambda_{m+n+1-r} + |b|\lambda_{m+n+1-k}|b|\rho \Big\} \rho^k \\ &\qquad + \frac{|b|\lambda_m}{|b|\rho} \frac{\rho^{n+1}}{1-\rho} \\ &= \sum_{k=1}^{n} \Big\{ \frac{M}{\rho} \lambda_m \\ &\qquad \times \sum_{r=1}^{k-1} \lambda_{m+n+1-r} + \lambda_{m+n+1-k} \Big\} \rho^{k-1} \\ &\qquad + \frac{\lambda_m \rho^n}{1-\rho} \\ &\leq \frac{\max\{M\lambda_m, \rho\}}{\rho(1-\rho)} \sum_{r=0}^{n} \lambda_{m+n-r} \rho^r \end{aligned} \tag{4.4}$$

for all $n \in \mathbf{N}_0$ and $m \geq m_0$. It follows therefore by (2.4) that (1.16) holds. □

If the constant ρ in Theorem 4.1 can be chosen smaller the larger one chooses m_0, such that this value of ρ approaches 0 as $m_0 \to \infty$, then we are in a situation similar to the one described by Trench [**42**]:

Corollary 4.2. *Let there exist a positive constant M, integers $k_0 \geq 1$ and $m_0 \geq 0$, and two positive null sequences $\{\lambda_n\}$ and $\{\kappa_n\}$ such that the following three conditions all hold*:

(i) $\sum_{m=1}^{\infty}(\eta_m t_k^{(m)} + a_{m+1}t_{k-1}^{(m+1)})$ *converge to finite values for* $k = 0, 1, \ldots, k_0$.

(ii) $|t_{k_0}^{(m)}| \leq \lambda_m$, $|t_{k_0+1}^{(m)}| \leq k_m\lambda_m$, $\sum_{j=m+1}^{\infty}(\kappa_j|\eta_j|\lambda_j + |a_{j+1}|\lambda_{j+1}) \leq \kappa_m^2\lambda_m$ *for all* $m \geq m_0$.

(iii) $|p(m; m+n; k_0)| \leq M\lambda_m$, $|p(m; m+n; k_0+1)| \leq M\kappa_m\lambda_m$ *for all* $m \geq m_0$ *and* $n \geq k_0$.

Then $\{P_n^{(m)}b^{-n}\}_{n=0}^{\infty}$ *converges locally uniformly for all* $b^{-1} \in \mathbf{C}$ *to the analytic function* $P^{(m)}(b^{-1}) \not\equiv 0$ *in* (2.14) *for every* $m \in \mathbf{N}_0$, *and*

$$|P^{(m)}(b^{-1}) - P_n^{(m)}b^{-n}| = \mathcal{O}\left((k_0-1)\hat{\mu}_{m+n+2-k_0} + \sum_{r=k_0}^{n+1}\lambda_{m+n+1-r}(\hat{\kappa}_m/|b|)^r\right), \tag{4.5}$$

where $\hat{\mu}_j := \max\{\mu_r : r \geq j\}$ *and* $\hat{\kappa}_j := \max\{\kappa_r : r \geq j\}$. *If, moreover, there exists an* $R > 0$ *such that* $\{\lambda_n R^n\}$ *is nondecreasing, then*

$$|P^{(m)}(b^{-1}) - P_n^{(m)}b^{-n}| = \mathcal{O}((k_0-1)\tilde{\mu}_{m+n+2-k_0} + \lambda_{m+n}). \tag{4.6}$$

Proof. Let $R^* > 0$ be arbitrarily chosen, and let $m_0^* \geq m_0$ be chosen such that $\kappa_m < R^*$ for all $m \geq m_0^*$. Then the conditions of Theorem 4.1 hold for $m \geq m_0^*$ with $\tilde{\kappa}_{m_0^*}/|b|$ as the new constant ρ. It follows, therefore, that $\{P_n^{(m)}b^{-n}\}$ converges if $|b| \geq R^*$. Equation (4.5) follows by (4.2), and (4.6) follows by an argument similar to Remark 1.1.2. □

For the special case where all $b_n = b$, which is an important case in continued fraction theory, Theorem 1.2 simplifies further. Then the recurrence relation (1.1) can be written

$$Q_n^{(m)} = Q_{n-1}^{(m)} + a_{m+n} w Q_{n-2}^{(m)}, \qquad a_{m+n} \neq 0; \quad n = 1, 2, 3, \dots, \tag{4.7}$$

where $w := b^{-2}$ and $Q_n^{(m)} := P_n^{(m)} b^{-n}$. Since all $\delta_n = 0$, we find that $p(m; m+n; k) = 0$ for odd values of k. Hence, the solution of (4.7) with initial values $Q_{-1}^{(m)} = 0$, $Q_0^{(m)} = 1$ can be written

$$Q_n^{(m)}(w) = \sum_{k=0}^{[n/2]} q(m; m+n; k) w^k; \quad q(m; m+n; k) = p(m; m+n; 2k), \tag{4.8}$$

where $[r] := \min\{m \in \mathbf{Z} : r \leq m\}$ for an $r \in \mathbf{R}$. We get:

Corollary 4.3. *Let $Q_n^{(m)}(w)$ be the solution of (4.7) with initial values $Q_{-1}^{(m)}(w) \equiv 0$, $Q_0^{(m)}(w) \equiv 1$. If there exist a positive constant C, an integer $m_0 \in \mathbf{N}_0$ and a positive nonincreasing null sequence $\{\lambda_n\}$ such that*

$$\left| \sum_{j=m+2}^{\infty} a_j \right| \leq \lambda_m, \qquad \sum_{j=m+2}^{\infty} |a_j| \lambda_j \leq C \lambda_m \tag{4.9}$$

for all $m \geq m_0$, then the sequence $\{Q_n^{(m)}(w)\}_{n=0}^{\infty}$ converges locally uniformly for $|w| < 1/C$ to the analytic function

$$Q^{(m)}(w) := \lim_{n \to \infty} Q_n^{(m)}(w) = \sum_{k=0}^{\infty} t_{2k}^{(m)} w^k \quad \textit{for } |w| < 1/C; \qquad Q^{(m)}(w) \not\equiv 0, \tag{4.10}$$

for every $m \in \mathbf{N}_0$, *and*

$$
(4.11) \quad |Q^{(m)}(w) - Q_n^{(m)}(w)| \\
\le \sum_{k=1}^{[n/2]} \left\{ \frac{2\lambda_m}{C} \sum_{r=1}^{k-1} \frac{\lambda_{m+n+1-r}}{C} + \frac{\lambda_{m+n+1-k}}{C} \right\} (C|w|)^k \\
+ \frac{\lambda_m}{C} \frac{(C|w|)^{[n/2]+1}}{1 - C|w|} \\
\le \frac{\max\{2\lambda_m, C\}}{C(1 - C|w|)} \sum_{r=1}^{[n/2]+1} \frac{\lambda_{m+n+1-r}}{C} (C|w|)^r
$$

for all $n \in \mathbf{N}_0$, $m \ge m_0$ *and* $|w| < 1/C$.

Proof. Set $\delta_n := 0$, $b := 1$ and replace a_n by $a_n w$ and λ_n by $|w|\lambda_n/\rho$ in Theorem 1.2. Then (1.15) reduces to (4.9) with $C := \rho^2/|w|$. Clearly $\rho < 1$ if and only if $|w| < 1/C$, and thus (4.10) follows. Since $Q_n^{(m)}(0) = 1$ for all n, it follows further that $Q^{(m)}(w) \not\equiv 0$. Moreover, it follows from the proof of Theorem 1.2 that we can use $M := 2$ in Theorem 4.1. The bound (4.11) follows then by the same type of arguments as in (4.3). □

To prove Theorem 4.1, we shall use a number of lemmas. The first one is easily proved by induction on k:

Lemma 4.4.

$$
p(m; m+n+N; k) - p(m; m+n; k) \\
= \sum_{r=1}^{k} p(m+n+1-r; m+n+N; r) p(m; m+n-r; k-r)
$$

for all $m, n, k, N \in \mathbf{N}_0$.

Remark 4.2. If $t_k^{(m)}$ is well defined for all $m \in \mathbf{N}_0$ and $k < k^*$ for some $k^* \in \bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$, then one also has, by the same method of proof, that

$$t_k^{(m)} - p(m; m+n; k) = \sum_{r=1}^{k} t_r^{(m+n+1-r)} \cdot p(m; m+n-r; k-r), \tag{4.12}$$

and thus also that

$$\sum_{j=m+1}^{m+n-k} (\eta_j t_k^{(j)} + a_{j+1} t_{k-1}^{(j+1)}) - p(m; m+n; k+1) = \sum_{r=1}^{k} t_r^{(m+n+1-r)} p(m; m+n-r; k+1-r) \tag{4.13}$$

for $k < k^*$.

Proof of Lemma 2.4 *for the case* $a = 0$. First let all $t_k^{(m)}; k < k^*$ be well defined. Equation (2.17) holds trivially for all m, with $p(m; \infty; k) = t_k^{(m)}$, if $k = 0$. Assume that it holds for $k = 0, 1, \ldots, k_0$ for some integer k_0, $0 \leq k_0 < k^* - 1$. If we let $k := k_0 + 1$ and $n \to \infty$ in (4.12), then $t_r^{(m+n+1-r)} \to 0$ and $p(m; m+n-r; k_0+1-r) \to t_{k_0+1-r}^{(m)} \neq \infty$ for $r = 1, 2, \ldots, k_0 + 1$. This proves (2.17) with $p(m; \infty; k) = t_k^{(m)}$ for $k = k_0 + 1$, and thus this follows for all $k < k^*$ by induction.

Now let the finite limits $p(m; \infty; k)$ exist. Then $p(m; \infty; 0) = 1$ which is equal to $t_0^{(m)}$, and $p(m; \infty; 1) = \sum_{j=m+1}^{\infty} \eta_j$ which is equal to $t_1^{(m)}$. Assume that $t_k^{(m)}$ is well defined for all $k \leq k_0$ for some k_0, $1 \leq k_0 < k^* - 1$. Since $t_r^{(m+n+1-r)} \to 0$ as $n \to \infty$ for $r \leq k_0$ and $p(m; m+n-r; k_0+1-r) \to t_{k_0+1-r}^{(m)} \neq \infty$ for $r \geq 1$, it follows from (4.13) that $\sum_{j=m+1}^{\infty} (\eta_j t_{k_0}^{(j)} + a_{j+1} t_{k_0-1}^{(j+1)}) = p(m; \infty; k_0 + 1)$. □

Proof of Theorem 4.1. We shall first prove that

$$|t_k^{(m)}| \leq \lambda_m C^{k-k_0} \quad \text{for all } m \geq m_0 \text{ and } k \geq k_0; \quad C := |b|\rho, \tag{4.14}$$

and thus that the series in (2.14) converges locally uniformly for $|w| = |b^{-1}| < 1/C$. Equation (4.14) holds for $k = k_0$ and $k = k_0 + 1$ by

Condition (ii), and it follows by induction on k for larger values of k, since the induction step goes (using (2.16) and Condition (ii))

$$
\begin{aligned}
|t_{k+1}^{(m)}| &\le \sum_{j=m+1}^{\infty} (|\eta_j t_k^{(j)}| + |a_{j+1} t_{k-1}^{(j+1)}|) \\
&\le C^{k-k_0-1} \sum_{j=m+1}^{\infty} (C|\eta_j|\lambda_j + |a_{j+1}|\lambda_{j+1}) \\
&\le \lambda_m C^{k+1-k_0}.
\end{aligned}
$$

In a similar manner we also find, using (2.10) and Condition (iii), that

$$
\begin{gathered}
|p(m; m+n; k)| \le M\lambda_m C^{k-k_0} \\
\text{for all } n \in \mathbf{N}_0, \quad m \ge m_0, \quad k \ge k_0.
\end{gathered}
\tag{4.15}
$$

Next we shall prove (4.3). According to Lemma 2.2 and formula (4.12), using (4.14), we have for fixed $m \ge m_0$ and $n \ge k_0 - 1$,

$$
\begin{aligned}
\left| \sum_{k=0}^{\infty} t_k^{(m)} b^{-k} - P_n^{(m)} b^{-n} \right| &\le \sum_{k=0}^{n} |t_k^{(m)} - p(m; m+n; k)| \\
&\quad \cdot |b|^{-k} + \sum_{k=n+1}^{\infty} |t_k^{(m)} b^{-k}| \\
&\le \sum_{k=1}^{n} \sum_{r=1}^{k} |t_r^{(m+n+1-r)} \\
&\qquad \cdot p(m; m+n-r; k-r)| \\
&\quad \cdot |b|^{-k} + \lambda_m \sum_{k=n+1}^{\infty} C^{k-k_0} |b|^{-k}.
\end{aligned}
$$

Using (4.14) and (4.15), this proves (4.3). If we apply the inequalities

$$
|t_r^{(m)}| \le \mu_m \quad \text{for } 1 \le r < k_0
$$

and

$$
|p(m; j; r)| \le K_m \quad \text{for } 0 \le r < k_0, \quad m \ge m_0
$$

for all $j \in \mathbf{N}_0$, we find that (4.3) is bounded by

$$\frac{1}{C^{2k_0}} \sum_{k=1}^{n} \Bigg\{ M\lambda_m \sum_{r=1}^{\min\{k-k_0,k_0-1\}} \mu_{m+n+1-r} C^{k_0-r}$$
$$+ M\lambda_m \sum_{r=k_0}^{k-k_0} \lambda_{m+n+1-r}$$
$$+ K_m C^{2k_0-k} \sum_{r=\max\{1,k-k_0+1\}}^{\min\{k,k_0-1\}} \mu_{m+n+1-r}$$
$$+ K_m \sum_{r=\max\{k_0,k-k_0+1\}}^{k} \lambda_{m+n+1-r} C^{k_0-k+r} \Bigg\}$$
$$\times \rho^k + \frac{\lambda_m \rho^{n+1}}{C^{k_0}(1-\rho)}$$
$$\leq \frac{H_m}{C^{2k_0}} \sum_{k=1}^{n} \Bigg\{ \sum_{r=1}^{\min\{k,k_0-1\}} \mu_{m+n+1-r} + \sum_{r=k_0}^{k} \lambda_{m+n+1-r} \Bigg\}$$
$$\times \rho^k + \frac{\lambda_m \rho^{n+1}}{C^{k_0}(1-\rho)}.$$

Changing the order of summation shows that this bound is less than the bound in (4.2). □

5. Theorem 1.1 revisited. In this section we consider the case where $a_n \to a \neq 0, \infty$ and $b_n \to b \neq 0, \infty$ and $a/b^2 \in \mathbf{C}\setminus\{0\}$ is not a real, negative number $\leq -1/4$. Let x be given by (1.3), and let δ_n and β_n be given by (1.11) and q be given by (2.7). Then $|q| < 1$. This time, $P_n^{(m)}/(b+x)^n$ can be written as a polynomial (2.6) in $w = q$, whose coefficients $p(m; m+n; k)$ are given recursively by (2.9). Our main result is now, with the notation from (2.15),

Theorem 5.1. *Let* (1.2) *hold with* $a \neq 0, \infty$ *and* $b \neq 0, \infty$. *Let there exist positive constants* M, C_1, C_2 *and* d, *integers* $k_0 \geq 0$ *and* $m_0 \geq 0$, *and a positive nonincreasing null sequence* $\{\lambda_n\}$ *such that the following four conditions all hold*:

(i) $\delta_m \neq -1$ *for* $m > m_0$, *and* $\prod_{j=m_0}^{\infty}(1+\delta_j)$ *and* $\sum_{j=m_0+2}^{\infty}\beta_j \times \prod_{\nu=m_0+1}^{j-2}(1+\delta_\nu)$ *converge.*

(ii) $\sum_{j=m_0+2}^{\infty}\beta_j t_k^{(j)}\prod_{\nu=m_0+1}^{j-2}(1+\delta_\nu)$ *converge to finite values for* $k = 0, \dots, k_0 - 1$.

(iii) $|t_{k_0}^{(m)}-1| \leq (1+d+C_1)\lambda_m$, $|\delta_{m+1}| \leq d\lambda_m$, $|\sum_{j=m+2}^{\infty}\beta_j\prod_{\nu=m+1}^{j-2}(1+\delta_\nu)| \leq C_1\lambda_m$ *and* $\sum_{j=m+2}^{\infty}|\beta_j|\lambda_j\prod_{\nu=m+1}^{j-2}|1+\delta_\nu| \leq C_2\lambda_m$ *for all* $m \geq m_0$.

(iv) $|p(m;m+n;k_0)-1| \leq M(1+d+C_1)\lambda_m$ *and* $|\sum_{j=m+2}^{m+n}\beta_j\prod_{\nu=m+1}^{j-2}(1+\delta_\nu)| \leq MC_1\lambda_m$ *for all* $n \in \mathbf{N}_0$ *and* $m \geq m_0$, *and* $p(m;m+n;k)$ *is uniformly bounded with respect to* m *and* n *for* $k = 0, \dots, k_0 - 1$.

Then, for every $m \in \mathbf{N}_0$, *the sequence* $\{P_n^{(m)}/(b+x)^n\}_{n=0}^{\infty}$ *converges to*

$$P^{(m)} := \sum_{k=0}^{\infty} t_k^{(m)} q^k \quad \text{for } 0 < |q| < 1/(1+C_2), \tag{5.1}$$

and

$$\left| P^{(m)} - \frac{P_n^{(m)}}{(b+x)^n} \right| \leq C\left(\sum_{r=0}^{n} \mu_{m+n-r}(r+1)^2(|q|Q)^r + n(|q|Q)^n \right), \tag{5.2}$$

for $|q| < 1/(1+C_2)$ *and* $Q > 1 + C_2$ *for some constants* $C > 0$ (*independent of* m, n, q *and* Q), *where* $\mu_n := \max\{\lambda_n, |t_k^{(m)} - 1|; m \geq n, 0 \leq k < k_0\}$.

Remarks 5.1. Conditions (i) and (ii) are required so that $t_0^{(m)}, \dots, t_{k_0}^{(m)}$ shall be well defined complex numbers. We shall see that Condition (iii) implies that $t_k^{(m)}$ exists for $k > k_0$. The condition $\delta_m \not\equiv -1$ for $m > m_0$ is to ensure that the series in Condition (i) converges also if m_0 is replaced by $m > m_0$. It always holds for m_0 sufficiently large, since $\delta_m \to 0$.

2. Theorem 5.1 should be regarded as an auxiliary result, the important results being its corollaries, such as Theorem 1.1.

3. From the proof of Theorem 5.1, one can also derive more precise upper bounds for $|P^{(m)} - P_n^{(m)}/(b+x)^n|$.

4. As in Remark 1.1.3, we may extend the domain of convergence to all q with $0 < |q| < 1$ if C_2 can be chosen arbitrarily small by increasing m_0. That is, if the conditions of Theorem 5.1 hold, and there exists a positive null sequence $\{\kappa_n\}$ such that Condition (iii) holds with C_2 replaced by κ_m, then $\{P_n^{(m)}/(b+x)^n\}$ converges locally uniformly for $0 < |q| < 1$ to the analytic function $P^{(m)} = P^{(m)}(q)$.

The proof of Theorem 5.1 is deferred to the end of this section. We shall first consider some special cases. If we set $k_0 := 0$ in Theorem 5.1, we have the situation described in Theorem 1.1.

Proof of Theorem 1.1. We set $k_0 := 0$ in Theorem 5.1. Conditions (i) and (iii) of Theorem 5.1 hold by (1.13). Condition (ii) is empty in this case. It remains to prove that Condition (iv) holds. Let $M_1 > 0$ be such that $\prod_{j=m+1}^{m+n} |1+\delta_j| \leq M_1$ for all m and n. Then

$$\begin{aligned}|p(m;m+n;0)-1| &= \left|\prod_{j=m+1}^{m+n}(1+\delta_j)-1\right| \\ &\leq |t_0^{(m)}-1| + |p(m;m+n;0)|\,|t_0^{(m+n)}-1| \\ &\leq \lambda_m(1+d+C_1) + M_1\lambda_{m+n}(1+d+C_1).\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\sum_{j=m+2}^{m+n}\beta_j\prod_{\nu=m+1}^{j-2}(1+\delta_\nu)\right| \leq{}& \left|\sum_{j=m+2}^{\infty}\beta_j\prod_{\nu=m+1}^{j-2}(1+\delta_\nu)\right| \\ &+\left|\prod_{\nu=m+1}^{m+n-1}(1+\delta_\nu)\right| \\ &\times\left|\sum_{j=m+n+1}^{\infty}\beta_j\prod_{\nu=m+n}^{j-2}(1+\delta_\nu)\right| \\ \leq{}& C_1\lambda_m + M_1C_1\lambda_{m+n-1}.\end{aligned}$$

Hence, Condition (iv) holds with any $M \geq 1+M_1$. The result follows now by means of (2.4). □

In the special case where all $b_n = b$, and thus all $\delta_n = 0$, the formulas for $p(m; m+n; k)$ and $t_k^{(m)}$ reduce to

$$\begin{aligned} p(m; m+n; 0) &:= 1 \\ p(m; m+n; k) &:= p(m+1; m+n; k-1) \\ &\qquad + \sum_{j=m+2}^{m+n-k+1} \varepsilon_j \cdot p(j; m+n; k-1), \end{aligned} \tag{2.9$'$}$$

and

$$\begin{aligned} t_0^{(m)} &:= 1, \\ t_k^{(m)} &:= t_{k-1}^{(m+1)} + \sum_{j=m+2}^{\infty} \varepsilon_j t_{k-1}^{(j)} \\ &\qquad \text{for } k = 1, 2, 3, \ldots . \end{aligned} \tag{2.15$'$}$$

For this special case Theorem 1.1 reduces to

Corollary 5.2. *Let all $\delta_n = 0$. If there exist a positive constant $C < -1 + 1/|q|$, an integer $m_0 \in \mathbf{N}_0$ and a positive nonincreasing null sequence $\{\lambda_n\}$ such that*

$$\left| \sum_{j=m+2}^{\infty} \varepsilon_j \right| \le \lambda_m \tag{5.3}$$

and

$$\sum_{j=m+2}^{\infty} |\varepsilon_j| \lambda_j \le C \lambda_m \quad \textit{for all } m \ge m_0,$$

then $\{P_n^{(m)}/(b+x)^n\}$ converges for every $m \in \mathbf{N}_0$, and

$$\left| P^{(m)} - \frac{P_n^{(m)}}{(b+x)^n} \right| \le \tilde{C} \left(\sum_{r=0}^{n} \lambda_{m+n-r} (r+1)^2 (|q|Q)^r + n(|q|Q)^n \right) \tag{5.4}$$

for every $Q > 1 + C$, for some constant $\tilde{C} > 0$ independent of n, m, q and Q.

To prove Theorem 5.1 we shall use the following analogue to Lemma 4.4. It is easily proved by induction on k, although it requires a fair amount of computation.

Lemma 5.3.

$$\begin{aligned} p(m;m+n+N;k) &- p(m;m+n;k) \\ &= p(m;m+n;k)\{p(m+n;m+n+N;0)-1\} \\ &\quad + \sum_{r=1}^{k} p(m;m+n-r;k-r) \\ &\quad \times \{p(m+n-r;m+n+N;r) - (1+\delta_{m+n-r+1}) \\ &\quad \cdot p(m+n-r+1;m+n+N;r-1)\} \end{aligned}$$

for all $m, n, k, N \in \mathbf{N}_0$.

Remark 5.2. Let $k^* \in \hat{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. If $t_k^{(m)}$ is well defined for all m and $k < k^*$, then one also has, by similar arguments,

$$\begin{aligned} t_k^{(m)} &- p(m;m+n;k) \\ &= p(m;m+n;k)\{t_0^{(m+n)}-1\} \\ &\quad + \sum_{r=1}^{k} p(m;m+n-r;k-r) \\ &\qquad \times \{t_r^{(m+n-r)} - (1+\delta_{m+n-r+1}) t_{r-1}^{(m+n-r+1)}\} \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \Bigg\{(1+\delta_{m+1})t_{k-1}^{(m+1)} &+ \sum_{j=m+2}^{m+n+1-k} \beta_j t_{k-1}^{(j)} \prod_{\nu=m+1}^{j-2} (1+\delta_\nu)\Bigg\} \\ &- p(m;m+n;k) \\ &= p(m;m+n;k)(t_0^{(m+n)}-1) \\ &\quad + \sum_{r=1}^{k-1} p(m;m+n-r;k-r)(t_r^{(m+n-r)} \\ &\qquad - (1+\delta_{m+n-r+1}) t_{r-1}^{(m+n-r+1)}) \end{aligned} \tag{5.6}$$

for all $m, n \in \mathbf{N}_0$ and $0 \le k < k^*$.

Proof of Lemma 2.4 *for the case* $a \neq 0$. Assume first that all $t_k^{(m)}$ are well defined by (2.15) for $k < k^*$. Then $t_k^{(m+n)} \to 1$ as $n \to \infty$ for all m and $k < k^*$, and $p(m; m+n; 0) \to t_0^{(m)}$ as $n \to \infty$, by definition. Assume that $p(m; m+n; k) \to t_k^{(m)}$ for all m for $k = 0, \dots, k_0 - 1$ for a k_0, $1 \le k_0 < k^*$. Let $m \in \mathbf{N}_0$ be fixed, and let $M > 0$ be arbitrarily chosen. Let $\mathbf{N}$ be separated into two monotone sequences, $\{n_\nu\}$ and $\{\tilde{n}_\nu\}$, such that $|p(m; m+n_\nu; k_0)| \le M$ for all ν and $|p(m; m+\tilde{n}_\nu; k_0)| > M$ for all ν. If $\{n_\nu\}$ contains infinitely many values, then $p(m; m+n_\nu; k_0) \to t_{k_0}^{(m)}$ as $\nu \to \infty$ by (5.5). If $\{\tilde{n}_\nu\}$ contains infinitely many values, then (5.5) implies that

$$\begin{aligned}\left|\frac{t_{k_0}^{(m)}}{p(m; m+\tilde{n}_\nu; k_0)} - 1\right| &\le |t_0^{(m+\tilde{n}_\nu)} - 1| \\ &\quad + \frac{1}{M}\sum_{r=1}^{k_0} |\, p(m; m+\tilde{n}_\nu - r; k_0 - r) \\ &\quad \cdot \{t_r^{(m+\tilde{n}_\nu - r)} - (1+\delta_{m+\tilde{n}_\nu - r+1}) t_{r-1}^{(m+\tilde{n}_\nu - r+1)}\}| \\ &\longrightarrow 0 \quad \text{as} \quad \nu \longrightarrow \infty.\end{aligned}$$

Hence also $p(m; m+\tilde{n}_\nu; k_0) \to t_{k_0}^{(m)}$ as $\nu \to \infty$. In this way it follows that $p(m; m+n; k) \to t_k^{(m)}$ for all m for $k = k_0$ as $n \to \infty$, and thus for all $k < k^*$ by induction.

Conversely, assume that the finite limits $p(m; \infty; k)$ exist. Then $p(m; \infty; 0) = \prod_{j=m+1}^{\infty}(1+\delta_j) = t_0^{(m)}$, so $t_0^{(m)}$ is well defined. Assume that $t_k^{(m)}$ is well defined and equal to $p(m; \infty; k)$ for all $m \in \mathbf{N}_0$ for $k = 0, \dots, k_0 < k^* - 1$. Then it follows from (5.6) that $t_{k_0+1}^{(m)}$ is well defined and equal to $p(m; \infty; k_0+1)$. Hence the lemma follows by induction. □

Proof of Theorem 5.1. Let $m_0^* \ge m_0$ be chosen such that $1 + C_2 + d\lambda_{m+1} \le Q$ for all $m \ge m_0^*$. $t_k^{(m)}$ is well defined in $\mathbf{C}$ for all $k \le k_0$ by Conditions (i) and (ii). We shall see by induction on k that this also holds for $k > k_0$. Simultaneously we also prove that

$$|t_k^{(m)} - 1| \le \lambda_m (1 + d + C_1) \sum_{r=0}^{k-k_0} Q^r \quad \text{for } m \ge m_0^* \text{ and } k \ge k_0. \tag{5.7}$$

Both claims hold for $k = k_0$ by Conditions (ii) and (iii). Assume that they hold for all k up to, but not including some value of k, $k > k_0$. Then it follows that the expression (2.15) for $t_k^{(m)}$ converges, since

$$\sum_{j=m+2}^{\infty} \beta_j t_{k-1}^{(j)} \prod_{\nu=m+1}^{j-2} (1+\delta_\nu) = \sum_{j=m+2}^{\infty} \beta_j \prod_{\nu=m+1}^{j-2} (1+\delta_\nu) + \sum_{j=m+2}^{\infty} \beta_j (t_{k-1}^{(j)} - 1) \prod_{\nu=m+1}^{j-2} (1+\delta_\nu),$$

where the first series converges by Condition (i) and the second series converges absolutely by (5.7) and Condition (iii). Moreover,

$$\begin{aligned} |t_k^{(m)} - 1| \le {} & |t_{k-1}^{(m+1)} - 1| + |\delta_{m+1}(t_{k-1}^{(m+1)} - 1)| + |\delta_{m+1}| \\ & + \left| \sum_{j=m+2}^{\infty} \beta_j \prod_{\nu=m+1}^{j-2} (1+\delta_\nu) \right| \\ & + \sum_{j=m+2}^{\infty} |\beta_j|\, |t_{k-1}^{(j)} - 1| \prod_{\nu=m+1}^{j-2} |1+\delta_\nu|. \end{aligned}$$

Inserting (5.7) for $|t_{k-1}^{(j)} - 1|$ proves that (5.7) also holds for $t_k^{(m)}$.

Without loss of generality, we assume that $M \ge d/(1+d)$. Then we similarly find from Conditions (iii) and (iv) that

$$|p(m; m+n; k) - 1| \le M \lambda_m (1 + d + C_1) \sum_{r=0}^{k-k_0} Q^r. \tag{5.8}$$

We shall see that this means that

$$\begin{aligned} |t_k^{(m)} - 1| &\le A \mu_m \sum_{r=0}^{k} Q^r \le A(k+1) \mu_m Q^k, \\ |p(m; m+n; k) - 1| &\le B(k+1) \mu_m Q^k \end{aligned} \tag{5.9}$$

for all $n \in \mathbf{N}_0$, $k \in \mathbf{N}_0$ and $m \geq m_0^*$ for some appropriately chosen constants A and B independent of m, n, k, q and Q. The inequality for $t_k^{(m)}$ follows trivially from (5.7). The inequality for $p(m; m+n; k)$ needs a little more consideration for $k < k_0$ if $k_0 > 0$. Assume that the inequality for $t_k^{(m)}$ holds with $A\mu_m := \tilde{\mu}_m$. By Condition (iv) there exists a constant $M_1 > 0$ such that $|p(m; m+n; k)| \leq M_1$ for all $m \geq m_0^*$, $n \geq 0$ and $0 \leq k < k_0$. Hence, by (5.5) we find that

$$\begin{aligned}
|p(m; m+n; k) - 1| \leq |t_k^{(m)} - 1 + |p(m; m+n; k)| \\
\cdot |t_0^{(m+n)} - 1| + \sum_{r=1}^{k} |p(m; m+n-r; k-r)| \\
\times \{|t_r^{(m+n-r)} - 1| + (1 + d\lambda_{m+n-r}) \\
\times |t_{r-1}^{(m+n-r+1)} - 1| + d\lambda_{m+n-r}\} \\
\leq (k+1)\tilde{\mu}_m Q^k + M_1\tilde{\mu}_m + \sum_{r=1}^{k} M_1\{(r+1)\tilde{\mu}_m Q^r \\
+ (1 + d\lambda_m) r\tilde{\mu}_m Q^{r-1} + d\lambda_m\} \\
\leq \tilde{\mu}_m(k_0 + M_1 + k_0(k_0+1)M_1 \\
+ M_1(1 + d\lambda_0)k_0^2 + dk_0)Q^k
\end{aligned}$$

for $k < k_0$ and $n \geq k$. This proves (5.9). It follows therefore that $\sum t_k^{(m)} q^k = \sum q^k + \sum (t_k^{(m)} - 1) q^k$ converges, locally uniformly for $|q| < 1/Q$ to a limit $P^{(m)}$. We shall see that this also is the limit of $\{P_n^{(m)}/(b+x)^n\}$, i.e., that

$$\left| P^{(m)} - \frac{P_n^{(m)}}{(b+x)^n} \right| \leq \sum_{k=0}^{n} |t_k^{(m)} - p(m; m+n; k)|\, |q|^k + \left| \sum_{k=n+1}^{\infty} t_k^{(m)} q^k \right| \tag{5.10}$$

approaches 0 as $n \to \infty$. It follows from (5.5) and (5.9) that

$$\begin{aligned} |t_k^{(m)} - p(m; m+n; k)| &\le (1 + (k+1)B\mu_m Q^k)A\mu_{m+n} \\ &\quad + \sum_{r=1}^{k} \{1 + (k-r+1)B\mu_m Q^{k-r}\} \\ &\quad \times \{(r+1)B\mu_{m+n-r}Q^r + rB\mu_{m+n-r+1}Q^{r-1} \\ &\quad + d\lambda_{m+n-r}(1 + rB\mu_{m+n-r+1}Q^{r-1})\} \\ &\le C(k+1)^2 Q^k \sum_{r=0}^{k} \mu_{m+n-r} \end{aligned}$$

for some appropriately chosen constant $C > 0$, independent of m, n, k, q and Q. Hence, for $|q| < 1/Q$,

$$\begin{aligned} \sum_{k=0}^{n} |t_k^{(m)} - p(m; m+n; k)|\, |q|^k &\le C \sum_{k=0}^{n} (k+1)^2 \left(\sum_{r=0}^{k} \mu_{m+n-r} \right) (|q|Q)^k \\ &\le C \sum_{r=0}^{n} \mu_{m+n-r} \sum_{k=r}^{\infty} (k+1)^2 (|q|Q)^k \\ &\le \tilde{C} \sum_{r=0}^{n} \mu_{m+n-r} (r+1)^2 (|q|Q)^r \end{aligned}$$

for some new constant $\tilde{C} > 0$. This converges to 0 by Lemma 2.5. The last term in (5.10) satisfies

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} t_k^{(m)} q^k \right| &\le \sum_{k=n+1}^{\infty} (1 + |t_k^{(m)} - 1|)|q|^k \\ &\le \sum_{k=n+1}^{\infty} (1 + A(k+1)\mu_m Q^k)|q|^k \\ &\le Dn(Q|q|)^n \end{aligned}$$

for some absolute constant $D > 0$. Hence, $\{P_n^{(m)}/(b+x)^n\}$ converges to the limit (5.1) for every $m \ge m_0^*$, and thus also for all $m < m_0^*$ if $|q| < 1/Q$. Since $Q > 1 + C_2$ was arbitrarily chosen, this proves the theorem. □

6. Some applications. Let

$$K\left(\frac{a_n}{b_n}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}; \quad a_n \neq 0, \tag{6.1}$$

be a continued fraction whose elements a_n, b_n are either complex numbers or complex valued functions. The approximants

$$f_n = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \in \hat{\mathbf{C}}, \quad n = 1, 2, 3, \ldots \tag{6.2}$$

of $K(a_n/b_n)$ can be written $f_n = A_n/B_n$, where A_n and B_n are solutions of the recurrence relation

$$X_n = b_n X_{n-1} + a_n X_{n-2}; \quad n = 1, 2, 3, \ldots \tag{6.3}$$

with initial values

$$A_{-1} = 1, \qquad A_0 = 0, \qquad B_{-1} = 0, \qquad B_0 = 1. \tag{6.4}$$

We say that $\{A_n\}$ and $\{B_n\}$ are the *canonical numerators and denominators* of $K(a_n/b_n)$. If the (finite) limits $A := \lim A_n$ and $B := \lim B_n$ both exist and are not both zero, we say that $K(a_n/b_n)$ converges *strongly separately*. More generally, if the finite limits

$$A := \lim_{n\to\infty} A_n/\Gamma_n, \qquad B := \lim_{n\to\infty} B_n/\Gamma_n \tag{6.5}$$

both exist and are not both zero for some given sequence $\Gamma := \{\Gamma_n\}$, we say that $K(a_n/b_n)$ converges *separately modulo* Γ. Since

$$B_n = P_n = P_n^{(0)} \quad \text{and} \quad A_n = a_1 P_{n-1}^{(1)} \tag{6.6}$$

with our previous notation, we can conclude separate convergence modulo $\{(b+x)^n\}$ from results in this paper. From Lemma 2.3, we find immediately:

Lemma 6.1. *Let $m \in \mathbf{N}_0$ be arbitrarily chosen. Then the continued fraction $K(a_n/b_n)$ converges separately if and only if its mth tail converges separately.*

It is evident that the results in Section 4 can be used to conclude strongly separate convergence if $b = 1$ and that the results in Section 5 implies separate convergence modulo $\Gamma = \{(b+x)^n\}$. But also the technique in Section 3 gives interesting results.

Example 6.1. Let $a_n := (a+(-1)^{n+1})z/n^\alpha$ and $b_n := (b+(-1)^n)w/n^\beta$ where $\alpha > 1/2$ and $\beta > 1/2$, and where a, b, z, w are complex parameters and/or variables such that $bw \neq 0$ and all $a_n \neq 0$. Then it follows as in Example 3.5 that $K(a_n/b_n)$ converges separately modulo $\Gamma = \{(bw+x)^n\}$, where $x := bw(\sqrt{1+4az/b^2w^2}-1)/2$; $|bw+x| > |x|$. The convergence is locally uniform with respect to a, b, z and w as long as $|bw + x| > |x|$.

Example 6.2. The regular C-fraction expansion $1 + K(a_n z/1)$ of the following ratio $M(\beta;\gamma;z)/M(\beta+1;\gamma+1;z)$ of Kummer functions, has coefficients a_n as given in Example 3.4 [**20**, p. 206, **22**, p. 313]. Hence, it follows from this example that this C-fraction converges strongly separately in the entire complex plane. The convergence is locally uniform with respect to z in $\mathbf{C}$.

Example 6.3. The regular C-fraction expansion $1 + K(a_n z/1)$ of the following ratio ${}_2F_1(\alpha,\beta;\gamma;z)/{}_2F_1(\alpha,\beta+1;\gamma+1;z)$ of hypergeometric functions, has coefficients a_n as given in Example 3.6, [**20**, p. 199, **22**, p. 295]. Hence, this C-fraction converges separately modulo $\Gamma = \{((1+\sqrt{1-z})/2)^n\}$. The convergence is locally uniform for z in the cut plane $D := \mathbf{C}\backslash[1,\infty)$.

A system $\{\mathcal{P}_n(x)\}$ of monic orthogonal polynomials are solutions of a three-term recurrence relation of the form

$$\mathcal{P}_n(x) = (x - c_n)\mathcal{P}_{n-1}(x) - \lambda_n \mathcal{P}_{n-2}(x) \quad n = 1, 2, 3, \ldots, \tag{6.7}$$

with initial values $\mathcal{P}_{-1}(x) \equiv 0$ and $\mathcal{P}_0(x) \equiv 1$. Hence the results in this paper give sufficient conditions for

$$\mathcal{P}_n(x) = \mathcal{O}(|b+x|^n) \quad \text{as } n \longrightarrow \infty. \tag{6.8}$$

Example 6.4. Let $c_n := (-1)^{n+1}d/n^\beta$ and $\lambda_n := -a + (-1)^n c/n^\alpha$ in (6.7), where $\alpha > 1/2$ and $\beta > 1/2$. Then it follows from Example 3.5 that $\mathcal{P}_n(x) = \mathcal{O}(|z|(\sqrt{1+4a/z^2}+1)/2|^n)$, locally uniformly in the cut plane where $\Re\sqrt{1+4a/z^2} > 0$.

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