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OPERATING FUNCTIONS FOR BANACH FUNCTION SPACES

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ABSTRACT. Let B be an ultraseparating Banach function space on a compact Hausdorff space X. We prove that if there exists a continuous non-affine operating function for B, then there is a finite subset E of X, such that B contains every continuous function vanishing in a neighborhood of E.

A real-valued function h defined on some interval of the real line is said to *operate* on a vector space B of real-valued functions on a set X, if the composite function $h \circ b$ belongs to B whenever b belongs to B and $h \circ b$ is defined.

The only obvious operating functions for B are affine functions, i.e., functions of the type $h(t) = \alpha t$, or $h(t) = \alpha t + \beta$ if B contains the constant functions.

It turns out that if X is a compact Hausdorff space and B is a subspace of $C(X)$ containing the constant functions and separating the points of X then, unless B is uniformly dense in $C(X)$, the affine functions are the only continuous operating functions for B.

The Stone-Weierstrass theorem can be viewed as a partial result in this direction: If $h(t) = t^2$ operates on B, then B is uniformly dense in $C(X)$. Later K. de Leeuw and Y. Katznelson [**10**] proved that in this version of the Stone-Weierstrass theorem any continuous function, defined on some interval of the real line, which is not affine, can take the place of the function $h(t) = t^2$.

This result does not extend to arbitrary *Banach function spaces* on X, i.e., subspaces of $C(X)$ which separate the points of X, contain the constant functions and which are Banach spaces in some norm which dominates the sup norm. The space of continuously differentiable functions on the interval $[0, 1]$, where the norm is given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$, is a Banach function space. Obviously, $h(t) = t^2$ operates on this space.

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There is, however, an extension of the theorem of K. de Leeuw and Y. Katznelson to a certain type of a Banach function space. J. Wermer showed in [**12**] that if $h(t) = t^2$ operates on the real part B of a function algebra A on X, then $B = C(X)$ (and consequently $A = C_{\mathbf{C}}(X)$). The space B is a Banach function space in the norm $\|b\| = \inf\{\|b + ia\|_\infty : b + ia \in A\}$. Also, in this case, it has been proved (see [**3**] and [**7**]) that any non-affine function can take the place of the function $h(t) = t^2$. (It is known [**1**] that if a function, which is defined on an open interval, operates on the real part of a function algebra, then that function must be continuous.) The proofs are based on an ingenious idea of A. Bernard [**1**]. The idea is to apply the theorem of de Leeuw and Katznelson to the space $l^\infty(B)$ consisting of all bounded sequences of functions from B (bounded with respect to the Banach space norm $\|\cdot\|$ on B). The space $l^\infty(B)$ can be regarded as a subspace of $C(\beta(\mathbf{N} \times X))$, where $\beta(\mathbf{N} \times X)$ denotes the Stone-Čech compactification of $\mathbf{N} \times X$.

Bernard proved that if $l^\infty(B)$ is uniformly dense in $C(\beta(\mathbf{N} \times X))$, then $B = C(X)$. Thus, if we could apply the result of de Leeuw and Katznelson not to B, but rather to $l^\infty(B)$, we could conclude that $B = C(X)$. There are two difficulties, however: (i) we must know that $l^\infty(B)$ separates the points of $\beta(\mathbf{N} \times X)$ and (ii) the function h operating on B, must also operate on $l^\infty(B)$.

The first difficulty is easy to deal with. Since $B = C(X)$ only if $l^\infty(B)$ separates the points of $\beta(\mathbf{N} \times X)$, we may simply assume that B has this property. We call a Banach function space B on X containing the constant functions and having this property *ultraseparating* on X. For example, the real part B of a Dirichlet function algebra A is ultraseparating [**1**]; moreover, if there is a nonaffine operating function on B, then A is automatically Dirichlet.

The other difficulty concerning whether an operating function for B also operates on $l^\infty(B)$ is not easy to deal with. It is not the case that if $\sup \|b_n\| < \infty$, then necessarily $\sup \|h \circ b_n\| < \infty$, as the following example shows:

Example 1. Let $X = \mathbf{N} \cup \{\infty\}$, and put

$$A = \left\{ a = (a_n) : \|a\| := \sup |a_n| + \sum |a_{n+1} - a_n| < \infty \right\}.$$

Then A is a Banach function algebra on X. Let $h(t) = \cos(\pi/t)$ for $t \in (0,1)$. The function h operates on A because if $a \in A$ and $a(X) \subseteq (0,1)$, then $a(X) \subseteq [\varepsilon, 1-\varepsilon]$ for some positive number ε. Now, since h is Lipschitz on this interval with some constant M, we see that

$$\begin{aligned}\|h(a)\| &= \sup|h(a_n)| + \sum |h(a_{n+1}) - h(a_n)| \\ &\leq \sup|h(a_n)| + \sum M|a_{n+1} - a_n| < \infty.\end{aligned}$$

Fix $n_0 \in \mathbf{N}$ and put

$$a_n = \begin{cases} 1/n & \text{if } n \leq n_0 \\ 1/n_0 & \text{if } n \geq n_0 \end{cases}$$

and let $a = (a_n)$. Then $\|a\| = 1 + (1 - 1/n_0) < 2$, but

$$\|h(a)\| > \sum |\cos(n+1)\pi - \cos n\pi| = 2n_0.$$

We note that this algebra also has the property that there exists a constant $K > 0$ such that for all $x, y \in X$ there is a sequence $a \in A$ with $\|a\| < K$ such that $a(x) = 1$ and $a(y) = -1$.

In [**11**], S. Sidney used the Baire category theorem in a clever way to show that h operates on a part of $l^\infty(B)$, and this enabled him to extend the result of K. de Leeuw and Y. Katznelson to the real part of a function algebra, with some restrictions on the operating function. In [**7**], O. Hatori showed that these restrictions are unnecessary. (See also [**3**].)

Turning now to more general spaces, in [**4**] it is proved that if B is an ultraseparating Banach function space on X, and if there is a nonaffine continuous operating function for B, then B is locally $C(K_x)$ for all but finitely many $x \in X$, i.e., there is a finite subset E of X such that if x is not in E, then there is a compact neighborhood K_x of x such that $B|K_x = C(K_x)$.

There are other positive results in the case when one has some restrictions on the operating function h. In [**11**], S. Sidney proved that if B is an ultraseparating Banach function space on X for which there exists a continuous operating function which is totally nonaffine on some subinterval, i.e., not affine on any subinterval of some subinterval, then $B = C(X)$. Further, in [**5**], it was shown that the same conclusion holds if $|h(t)| \leq k|t|$ in some neighborhood of zero, and if h is not λ-homogeneous in any neighborhood of zero, i.e., for no neighborhood of zero does there exist a number $\lambda \neq 1$ such that $h(\lambda t) = \lambda h(t)$ for all t in that neighborhood. Then, recently O. Hatori [**9**] and A. Bernard [**2**] (in the metrizable case), have shown that the restriction that h is not λ-homogeneous suffices. In [**9**], O. Hatori also gives an example of an ultraseparating Banach function space on a compact Hausdorff space X, which is not $C(X)$, but for which the function $h(t) = |t|$ operates.

In [**4**], it is shown that if a nonaffine function operates on an ultraseparating Banach function space B on X, then B is locally a $C(K)$-space except for finitely many points in X, i.e., there is for all but finitely many x in X a compact neighborhood K_x of x such that $B|K_x = C(K_x)$. In this note we improve this result by showing that there is a finite subset E of X such that B contains every continuous function on X, vanishing in a neighborhood of E. Since the function $h(t) = |t|$ is λ-homogeneous for every positive number λ, the results in [**2, 4** and **9**] are just about the best results one can get for operating functions on ultraseparating Banach function spaces. (See also Theorem 3 in this note).

We also prove a general theorem for operating functions on subspaces of Banach function spaces and show how the result of A. Bernard and O. Hatori, concerning operating functions which are not λ-homogeneous, can be derived from this theorem. In light of S. Sidney's result in [**11**], we will only consider operating functions which are not totally nonaffine on any subinterval.

The main results. We first present the main idea behind the proofs of our results:

Let ϕ be a C_0^∞-function with the support in a (small) neighborhood of zero. Let $h_\phi = h * \phi$, and look at the expression

$$\Lambda_{\phi,t,\delta}(b,c) = h_\phi \circ (b + (t+\delta)c) + h_\phi \circ (b + (t-\delta)c) - 2h_\phi \circ (b + tc)$$

this is

$$\Lambda_{\phi,t,\delta}(b,c) = \int (h \circ (b + (t+\delta)c - s) + h \circ (b + (t-\delta)c - s) \\ - 2h \circ (b + tc - s))\phi(s)\, ds,$$

where $b, c \in B$. Dividing by δ^2, letting $\delta \to 0$, and then putting $t = 0$, we obtain $c^2 \cdot h''_\phi \circ (b)$ as a limit. If h is not affine, we can choose ϕ and some constant function b such that $h''_\phi \circ (b) = k \neq 0$ and thus get $k \cdot c^2$ as a limit. Thus, if $\Lambda_{\phi,t,\delta} \circ (b, c) \in \overline{B}$, then also $c^2 \in \overline{B}$, where the bar denotes uniform closure.

For the argument above one needs B to contain the constant functions; otherwise, h_ϕ need not operate on B (or $\overline{B}$).

If B does not contain the constant functions, we could instead look at the following expression

$$\Lambda_{\phi,t,\delta}(b_0, b, c) = \int (h \circ (b + (t + \delta)c - sb_0) + h \circ (b + (t - \delta)c - sb_0) - 2h \circ (b + tc - sb_0))\phi(s)\, ds.$$

We observe that the expression reduces to zero at points $x \in X$ where $c(x) = 0$ and at points $x \in X$ where $b(x)$ belongs to some fixed open interval on which h is affine (if t, δ and the support of ϕ are sufficiently small). We also note that $\Lambda_{\phi,t,\delta}(b_0, b, c) = \Lambda_{\phi,t,\delta}(b, c)$ at those points $x \in X$, where $b_0(x) = 1$. Thus, if X can be written as a union of two sets $X = X_1 \cup X_2$ such that $b(X_1)$ is a subset of some interval on which h is affine and $b_0 = 1$ on $X_2 \cap \operatorname{supp}(c)$, then upon dividing by δ^2, letting $\delta \to 0$ and then putting $t = 0$, we get the limit $c^2 \cdot h''_\phi \circ (b)$ as before.

We will refer to the arguments above in the proofs.

Theorem 1. *Let B be an ultraseparating Banach function space on a compact Hausdorff space X. If there exists a continuous non-affine operating function for B, then there exists a finite subset E of X, such that B contains every continuous function vanishing in a neighborhood of E.*

Proof. Call the operating function h. By a result of S. Sidney [**11**] mentioned earlier, we may assume that every subinterval, of the interval on which h is defined, contains an interval on which h is affine. Composing h with a suitable affine function we may also assume that h is not affine in any neighborhood of zero, but that $h = 0$ on $[0, r]$ for some positive number r.

We already know that there is a finite subset E of X such that each x not in E has a neighborhood K_x for which $B|K_x = C(K_x)$.

Let $x_0 \notin E$. We first show that there is an open neighborhood V_0 of x_0 such that $B_0(V_0) = C_0(V_0)$, where

$$C_0(V) = \{f \in C(X) : f = 0 \text{ outside } V\}$$

and

$$B_0(V) = \{b \in B(X) : b = 0 \text{ outside } V\}.$$

Let U be a neighborhood of x_0 for which $B|U = C(U)$, and let V be an open neighborhood of x_0 such that $\overline{V} \subset U$. Put

$$B(U,V) = \{b \in B : b(x_0) = 0 \text{ and } b = 0 \text{ on } U \backslash V\}$$

and, for any $\lambda > 0$, let

$$B(U,V)_\lambda = \{b \in B(U,V) : \|b\| \leq \lambda\}.$$

By the theorem of K. de Leeuw and Y. Katznelson, B is dense in $C(X)$, and thus there is a function $b_1 \in B$ such that $b_1(x_0) = 0$ and such that $b_1(X \backslash U)$ is contained in the interval $(0, r)$. By the Baire category theorem, there is a function $b_2 \in B(U,V)_\lambda$ and positive numbers ε and M such that $\|h \circ (b_1 + b_2 + b)\| < M$ for all b in some dense subset of the ε-ball $B(U,V)_\varepsilon$.

Let $b_0 = b_1 + b_2$. Taking λ and ε sufficiently small, we may assume that $(b_0 + b)(X \backslash U) \subseteq (0, r)$ for any $\|b\| \leq \varepsilon$. Since $h = 0$ on $[0, r]$, we have

$$h \circ (b_0 + c_1 - b) + h \circ (b_0 + c_2 - b) - 2h \circ (b_0 + c_3 - b) \in B_0(V)$$

if the c_i's and b belong to $B(U,V)$ and have sufficiently small norms. It follows that if $\{b_n\}$ and $\{c_n\}$ belong to $l^\infty(B(U,V))$ and if s, t and δ are sufficiently small real numbers, then

$$h \circ \{b_0 + (t+\delta)c_n - sb_n\} + h \circ \{b_0 + (t-\delta)c_n - sb_n\} \\ - 2h \circ \{b_0 + tc_n - sb_n\}$$

belongs to $\overline{l^\infty(B_0(V))}$ and thus

$$\int (h \circ \{b_0 + (t+\delta)c_n - sb_n\} + h \circ \{b_0 + (t-\delta)c_n - sb_n\} \\ - 2h \circ \{b_0 + tc_n - sb_n\})\phi(s)\, ds$$

belongs to $\overline{l^\infty(B_0(V))}$, if t and δ are sufficiently small and if $\phi \in C_0^\infty(\mathbf{R})$ has support in a sufficiently small neighborhood of zero.

Let us take an element $\{c_n\}$ in $l^\infty(B(U,V))$ where each c_n is zero near x_0. Since $B|\overline{U} = C(\overline{U})$ we can for each n choose b_n in $B(U,V)$ with the property that $b_n = 1$ on the set $\{x \in V : c_n(x) \neq 0\}$. Using the open mapping theorem we can also assume that the sequence $\{\|b_n\|\}$ is bounded so that $\{b_n\} \in l^\infty B(U,V)$. With this choice of $\{c_n\}$ and $\{b_n\}$ the expression above takes the form

$$h_\phi \circ \{b_0 + (t+\delta)c_n\} + h_\phi \circ \{b_0 + (t-\delta)c_n\} - 2h_\phi \circ \{b_0 + tc_n\}$$

where $h_\phi = h * \phi$. Dividing by δ^2, letting δ tend to 0 and then putting $t = 0$, we deduce that $\{c_n\}^2 \cdot h''_\phi \circ \{b_0\} \in \overline{l^\infty(B_0(V))}$, where $\{b_0\} = \{b_0, b_0, \dots\}$. Approximating an arbitrary element in $l^\infty(B(U,V))$ with a sequence $\{c_n\}$ as above, we find that $\{c_n\}^2 \cdot h''_\phi \circ \{b_0\} \in \overline{l^\infty(B_0(V))}$ for all $\{c_n\}$ in $l^\infty(B(U,V))$. Since $h''_\phi \circ b_0 = 0$ on $X \backslash U$ and since $B|\overline{U} = C(\overline{U})$, it follows that

$$\{c_n\}^2 \cdot h''_\phi \circ \{b_0\} \in \overline{l^\infty(B_0(V))}$$

if $\{c_n\} \in l^\infty(C_0(V))$ and $c_n(x_0) = 0$ for all n.

Since B is dense in $C(X)$, we can find a function $b \in B$ such that if $a = h \circ b$, then $a(x_0) \neq 0$ and $a = 0$ outside some given neighborhood V_1 of x_0 with $\overline{V}_1 \subset V$. Thus, $\{a\} + \{c_n\}^2 \cdot h''_\phi \circ \{b_0\} \in \overline{l^\infty(B_0(V))}$ for all $\{c_n\} \in l^\infty(C_0(V))$ with $c_n(x_0) = 0$ for all n. We now use the fact that h is not affine in any neighborhood of 0 to choose ϕ with the property that $h''_\phi(0) \neq 0$. It follows that there is a neighborhood V_0 of x_0, with $\overline{V}_0 \subset V$, such that $l^\infty(C_0(V_0)) \subset \overline{l^\infty(B_0(V))}$. Hence $l^\infty(B_0(V_0))$ is dense in $l^\infty(C_0(V_0))$ and, by the result of Bernard mentioned earlier, we have $B_0(V_0) = C_0(V_0)$.

To end the proof of Theorem 1, let $f \in C(X)$ and assume that $K = \operatorname{supp}(f)$ does not intersect E. Let $\{V_\gamma : \gamma \in \Gamma\}$ be a finite open cover of K such that $B_0(V_\gamma) = C_0(V_\gamma)$ for all γ. Let $\{f_\gamma\}$ be a finite partition of unity on K, subordinated to the open cover $\{V_\gamma : \gamma \in \Gamma\}$. We can extend each function from the family f_γ to a continuous function on X. Since $f = \sum_\gamma f_\gamma f$ and $f_\gamma f \in B$ we get $f \in B$. □

Corollary. *Let B be a Banach function space on X with the property that each x in X has a compact neighborhood K_x for which $B|K_x = C(K_x)$. If there is a continuous nonaffine operating function for B, then $B = C(X)$.*

The next result concerns operating functions and subspaces which do not contain the constant functions. We say that h operates *boundedly* on a Banach function space A if h operates on A and there exist numbers ε and $M > 0$ such that $\|h \circ b\| < M$ for all b in some dense subset of the ε-ball of A.

Theorem 2. *Let A be a Banach space of continuous functions on a locally compact Hausdorff space Y, where the norm dominates the sup-norm. Assume that $l^\infty(A)$ separates the points of $\overline{\mathbf{N} \times K}$ for each compact subset K of Y, where the bar denotes closure in $\beta(\mathbf{N} \times Y)$. Suppose further that h is a continuous function with $h(0) = 0$ operating boundedly on A, which is not λ-homogeneous for any $\lambda \neq 1$, and with the property that every neighborhood of zero contains a subinterval on which h is affine. Then $A|K = C(K)$ for every compact subset K of Y.*

Proof. We begin by proving that if K is a compact subset of Y for which there is a function b_0 in A, with $b_0 = 1$ on K, then there exists a neighborhood U of K such that $A|\overline{U} = C(\overline{U})$.

Let $\{b_0\} = \{b_0, b_0, \dots\}$ and let $\mathcal{K} = \{\eta \in \beta(\mathbf{N} \times Y) : \{b_0\}(\eta) = 1\}$. The space $l^\infty(A)|\mathcal{K}$ separates the points of $\mathcal{K}$ and contains the constant functions. Further, $h \circ \{b_n\} \in \overline{l^\infty(A)|\mathcal{K}}$ if $\{b_n\} \in l^\infty(A)|\mathcal{K}$ and $\|\{b_n\}\| < \varepsilon$. It follows from the theorem of K. de Leeuw and Y. Katznelson that $l^\infty(A)|\mathcal{K}$ is dense in $C(\mathcal{K})$. A local version [**3**, Theorem 1], of Bernard's theorem yields the existence of the desired neighborhood U.

We show next that if $K = K_1 \cup K_2$, where $A|K_i = C(K_i)$ for $i = 1, 2$, then $A|K = C(K)$. We know that $h \circ \{b_n\} \in l^\infty(A|K)$, if the b_n's belong to some dense subset of the ε-ball A_ε of A, and hence $h \circ \{b_n\} \in \overline{l^\infty(A|K)}$, if $\{b_n\} \in l^\infty(A|K)_\varepsilon$.

To show that $A|K = C(K)$ we show that $l^\infty(A|K)$ is dense in $C(\overline{\mathbf{N} \times K})$. Since $A|K$ is a Banach space in the quotient norm, and since $\overline{\mathbf{N} \times K} = \beta(\mathbf{N} \times K)$, Bernard's theorem then shows that

$A|K = C(K)$.

Let μ be a measure on $\overline{\mathbf{N}\times K}$ which annihilates $l^\infty(A|K)$, and suppose that μ is not zero around a point $\eta \in \overline{\mathbf{N}\times K_2}\backslash\overline{\mathbf{N}\times K_1}$, by this we mean that $|\mu|(\mathcal{V}) > 0$ for any neighborhood $\mathcal{V}$ of η. For each $f \in C(K)$, let $\{f\} = \{f, f, f, \dots\}$. Since the map $f \mapsto \{f\}(\eta)$ is a multiplicative linear functional on $C(K)$, there is a point $y_0 \in K$ such that $\{f\}(\eta) = f(y_0)$ for all $f \in C(K)$. Clearly, $y_0 \in K_2\backslash K_1$.

We claim that there is a function $\{b_0\} \in l^\infty(A|K)$ such that $\{b_0\} = 1$ on $\overline{\mathbf{N}\times K_1}$ and $b_0(y_0) = 0$. Since $A|K_1 = C(K_1)$ there is a $b \in A$ such that $b = 1$ on K_1. If $b(y_0) = 1$, then $A|K_1 \cup \{y_0\} = C(K_1 \cup \{y_0\})$ as we saw above, implying the existence of the desired function b_0. If $b(y_0) = \chi \neq 1$, then we look at the functions $h \circ (tb)$ where t is a real number with $|t| \leq 1$. Since h is not λ-homogeneous, we can choose t such that $h(t\chi) \neq \chi h(t)$. A suitable linear combination of b and $h \circ (tb)$ gives the desired function b_0.

Let r be a real number such that if $\{b_n\}$, $\{c_n\} \in l^\infty(A|K)$ then $\{rb_0 + tc_n - sb_n\} \in l^\infty(A|K)_\varepsilon$ and $\{rb_0 + tc_n - sb_n\}(\overline{\mathbf{N}\times K_1}) \subseteq J$ for all $|s|$ and $|t|$ sufficiently small, where J is an open interval on which h is affine.

Since $A|K_2 = C(K_2)$ we can choose $\{b_n\}$ such that $\{b_n\} = 1$ on $\overline{\mathbf{N}\times K_2}$. As we saw in the discussion preceding Theorem 1 (with $\{b_0\}$, $\{c_n\}$ and $\{b_n\}$ in place of b, c and b_0), we can conclude that $\{c_n\}^2 \cdot h''_\phi \circ \{rb_0\} \in \overline{l^\infty(A|K)}$. Also, since h is not affine in any neighborhood of zero, we can choose ϕ such that $h''_\phi(0) \neq 0$. Further, since $l^\infty(A)|\overline{\mathbf{N}\times K_2} = C(\overline{\mathbf{N}\times K_2})$ and since $h''_\phi \circ \{rb_0\} = 0$ on $\overline{\mathbf{N}\times K_1}$, we can replace $\{c_n\}$ by any element of $C(\overline{\mathbf{N}\times K})$. Hence, there is a neighborhood $\mathcal{W}$ of η relative to $\overline{\mathbf{N}\times K}$ such that

$$\{f \in C(\overline{\mathbf{N}\times K}) : f = 0 \text{ outside } \mathcal{W}\} \subseteq l^\infty(A)|\overline{\mathbf{N}\times K}.$$

It follows that μ is zero around η. This contradiction together with the regularity of μ shows that μ must have all of its mass on $\overline{\mathbf{N}\times K_1}$. But $l^\infty(A)|\overline{\mathbf{N}\times K_1} = C(\overline{\mathbf{N}\times K_1})$, and hence $\mu = 0$.

To finish the proof of the theorem, we note that if $y \in Y$ then $l^\infty(A)$ separates the points of $\overline{\mathbf{N}\times\{y\}}$ and thus there is a function $b \in A$ with $b(y) = 1$. It follows that there is a neighborhood U of y such that $A|\overline{U} = C(\overline{U})$. $\square$

We end this note by showing how the last theorem can be used to give a short proof of the following theorem of A. Bernard and O. Hatori (cf. [**2**] and [**9**]):

Theorem 3. *Let B be an ultraseparating Banach function space on a compact Hausdorff space X, and suppose there is a continuous operating function h for B, defined in a neighborhood of zero and with $h(0) = 0$, which is not λ-homogeneous in any neighborhood of zero. Then $B = C(X)$.*

Proof. As mentioned earlier, we may assume, by a result of S. Sidney [**11**], that every neighborhood of zero contains an interval on which h is affine. The function h is, of course, not affine in any neighborhood of zero since it is not λ-homogeneous. By the corollary to Theorem 1, it suffices to show that each $x \in X$ has a compact neighborhood K_x for which $B|K_x = C(K_x)$.

Let $x \in X$ and put $B(x) = \{b \in B : b(x) = 0\}$. By the Baire category theorem, there is a function $b_0 \in B(x)$ and positive numbers ε and M such that $h \circ (b_0 + b) \in (B(x))_M$ for all b in a dense subset of an ε-ball $B(x)_\varepsilon$ of $B(x)$. It follows that $h \circ \{b_0 + b_n\} \in l^\infty(B(x))_M$ for $\{b_n\}$ in a dense subset of $l^\infty(B(x))_\varepsilon$.

Put $\mathcal{X} = \{\eta \in \beta(\mathbf{N} \times X) : \{b_0\}(\eta) = 0\}$. If we can show that $l^\infty(B)|\mathcal{X}$ is dense in $C(\mathcal{X})$, then by the local version of Bernard's theorem [**3**], there exists a compact neighborhood K_x of x for which $B|K_x = C(K_x)$.

The space $l^\infty(B(x))|\mathcal{X}$ is a Banach space of continuous functions on $\mathcal{X}$ in the quotient norm. Since $\{b_0 + b_n\}|\mathcal{X} = \{b_n\}|\mathcal{X}$ for $\{b_n\} \in l^\infty(B)$, we have $h \circ \{b_n\}|\mathcal{X} \in (l^\infty(B(x)))|\mathcal{X}_M$ for all $\{b_n\}|\mathcal{X}$ in a dense subset of $(l^\infty(B(x))|\mathcal{X})_\varepsilon$.

Put $A = l^\infty(B(x))|\mathcal{X}$ and $Y = \mathcal{X} \backslash \overline{\mathbf{N} \times \{x\}}$. We proved that $h \circ a \in A_M$ for all a in some dense subset of A_ε. By an argument similar to one in the proof of the previous theorem, we see that if K is a compact subset of Y, then $l^\infty(A)$ separates the points of $\overline{\mathbf{N} \times K}$.

Let us now turn to showing that $l^\infty(B)|\mathcal{X}$ is dense in $C(\mathcal{X})$. To do so it suffices to show that $l^\infty(B(x))|\mathcal{X}$ is dense in $C_0(\mathcal{X})$, the space of all continuous functions on $\mathcal{X}$, which vanish on $\overline{\mathbf{N} \times \{x\}}$. Let μ be an annihilating measure for $l^\infty(B(x))|\mathcal{X}$ with no mass on $\overline{\mathbf{N} \times \{x\}}$. We choose compact subsets $\mathcal{K}_n$ of $\mathcal{X}\backslash\overline{\mathbf{N}\times\{x\}}$ and functions $f_n \in C_0(\mathcal{X})_1$ such that $\int_{\mathcal{K}_n} f_n d\mu \to \|\mu\|$.

Put $\tilde{\mathcal{K}} = \overline{\cup\{n\}\times\mathcal{K}_n}$. Now $\tilde{\mathcal{K}}$ is contained in

$$\beta(\mathbf{N}\times\mathcal{X})\backslash\overline{(\mathbf{N}\times(\overline{\mathbf{N}\times\{x\}}))}$$

and thus there is an element $\{b_n\} \in l^\infty(l^\infty(B(x))|\mathcal{X})$ such that $\{b_n\}|\tilde{\mathcal{K}} = \{f_n\}|\tilde{\mathcal{K}}$ and hence $b_n|\mathcal{K}_n = f_n|\mathcal{K}_n$ for all n. It follows that $\int_{\mathcal{K}_n} b_n\, d\mu \to \|\mu\|$. Since μ annihilates each b_n, and since $\{b_n\}$ is a bounded sequence, we conclude that $\mu = 0$. This finishes the proof of Theorem 3. □

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