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THE SPECTRUM OF ELEMENTS OF A COMMUTATIVE LMC-ALGEBRA RELATIVE TO A BANACH SUBALGEBRA

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ABSTRACT. Let $\mathcal{A}$ be a commutative LMC-algebra with unit, and let $\mathcal{B}$ be a Banach subalgebra of $\mathcal{A}$. For $a \in \mathcal{A}$ define $\sigma_{\mathcal{B}}(a) \equiv \{\lambda \in \mathbf{C} : (\lambda - a)^{-1} \notin \mathcal{B}\}$. In this paper the spectral theory corresponding to this spectrum is developed. Applications are given to concrete examples from analysis.

1. Introduction. Throughout, $\mathcal{A}$ is a complete commutative Hausdorff LMC-algebra. The topology of $\mathcal{A}$ is determined by a collection of algebra seminorms $\{P_\delta : \delta \in D\}$ (here D is an index set). In this paper we study the situation where $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach subalgebra of $\mathcal{A}$. Assume that $\mathcal{A}$ has a unit that is in $\mathcal{B}$. For $a \in \mathcal{A}$, define $\operatorname{res}_{\mathcal{B}}(a) \equiv \{\lambda \in \mathbf{C} : (\lambda - a)^{-1} \in \mathcal{B}\}$. This is the resolvent set of the element a relative to $\mathcal{B}$.

For some interesting choices of $\mathcal{A}$ and $\mathcal{B}$, classical theorems in analysis that hold for elements of $\mathcal{B}$ can be extended to those elements $a \in \mathcal{A}$ which have $\operatorname{res}_{\mathcal{B}}(a)$ nonempty. The purpose of this paper is to present the spectral theory involved, to point out interesting examples of pairs $(\mathcal{A}, \mathcal{B})$ that occur in analysis, and to extend some classical results that hold in $\mathcal{B}$ to the larger LMC-algebra $\mathcal{A}$.

There is a Gelfand representation theory that applies to $\mathcal{A}$ which is similar to the Gelfand theory of a commutative Banach algebra. Denote the space of all nonzero continuous multiplicative linear functionals on $\mathcal{A}$ by $\Omega_{\mathcal{A}}$. Equip $\Omega_{\mathcal{A}}$ with the $\mathcal{A}$-topology. Setting $\hat{a}(\varphi) = \varphi(a)$ when $a \in \mathcal{A}$, $\varphi \in \Omega_{\mathcal{A}}$, we have $a \to \hat{a}$ is a representation of $\mathcal{A}$ as an algebra of continuous functions on $\Omega_{\mathcal{A}}$. Then the spectrum of an element $a \in \mathcal{A}$,

$$\sigma(a) \equiv \{\lambda \in \mathbf{C} : (\lambda - a) \text{ is not invertible in } \mathcal{A}\},$$

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is the range of the function $\hat{a}$ on the space $\Omega_{\mathcal{A}}$ [**1**, Theorem (4.10–8)]. For $a \in \mathcal{A}$, define the spectrum of a relative to $\mathcal{B}$ by $\sigma_{\mathcal{B}}(a) \equiv \mathbf{C}\backslash \mathrm{res}_{\mathcal{B}}(a)$. For some pairs $(\mathcal{A}, \mathcal{B})$, this spectrum is useful in analysis. A natural question here is, how does one compute $\sigma_{\mathcal{B}}(a)$ in terms of properties of $\hat{a}$? A key result proved in Section 2 is that with reasonable assumptions, when $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty, then $\sigma_{\mathcal{B}}(a) = \overline{\sigma(a)}$. In terms of $\hat{a}$, under the same assumptions, a has an inverse in $\mathcal{B}$ exactly when $|\hat{a}|$ is bounded away from zero on $\Omega_{\mathcal{A}}$.

The relevant spectral theory of elements of $\mathcal{A}$ relative to $\mathcal{B}$ is developed in Sections 2 and 4. Section 3 is devoted to examples. In the last section the results in Sections 2 and 4 are applied to the examples. These applications often extend classical results in analysis.

The general situation, with $\mathcal{A}$ not necessarily commutative, is studied in [**2**].

2. Spectral theory. We adopt the notation introduced in the introduction. In addition, let

$$\begin{aligned} \mathrm{Inv}_{\mathcal{B}}(\mathcal{A}) &\equiv \{a \in \mathcal{A} : a \text{ is invertible in } \mathcal{A} \text{ and } a^{-1} \in \mathcal{B}\}; \\ r_{\mathcal{B}}(b) &\equiv \text{the spectral radius of } b \in \mathcal{B}. \end{aligned}$$

We do not assume in general that $\mathcal{A}$ has a unit. Thus we will have occasion to use quasi-inverses. When $a \in \mathcal{A}$ is quasi-invertible, we denote the quasi-inverse of a by a^q (recall, this means $aa^q = a^q a$ and $a + a^q - aa^q = 0$). Set

$$Q - \mathrm{inv}_{\mathcal{B}}(\mathcal{A}) \equiv \{a \in \mathcal{A} : a \text{ is a quasi-invertible in } \mathcal{A} \text{ and } a^q \in \mathcal{B}\}.$$

We make certain basic assumptions throughout this paper.

Standing assumptions. (1) $\mathcal{A}$ is a complete commutative Hausdorff LMC-algebra; (2) If $\mathcal{A}$ has a unit 1, then $1 \in \mathcal{B}$; (3) $\Omega_{\mathcal{A}}$ is dense in $\Omega_{\mathcal{B}}$ in the sense that $\{\varphi|_{\mathcal{B}} : \varphi \in \Omega_{\mathcal{A}}\}$ is dense in $\Omega_{\mathcal{B}}$ where $\varphi|_{\mathcal{B}}$ denotes the restriction of φ to $\mathcal{B}$. Assumption (3) is a major hypothesis. It holds for many interesting examples from analysis; see the examples in the next section.

Now we state without proof a useful result concerning spectral properties.

Note 1. (1) If $a,\ c \in \mathcal{A}$, $a \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$ and $a - c \in \mathcal{B}$ with $r_{\mathcal{B}}(a-c) < (r_{\mathcal{B}}(a^{-1}))^{-1}$, then $c \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$.

(2) For all $a \in \mathcal{A}$, $\mathrm{res}_{\mathcal{B}}(a)$ is open and $\sigma_{\mathcal{B}}(a)$ is closed.

Part (2) follows from (1), and the proof (1) uses standard Banach algebra facts.

Now we state another useful note (again, the elementary proof is omitted). For Γ a nonempty subset of $\mathbf{C}$, $\lambda \in \mathbf{C}$, we define $d(\lambda, \Gamma) = \inf\{|\lambda - \gamma| : \gamma \in \Gamma\}$.

Note 2. Assume $a \in \mathcal{A}$. Each of the equalities below follow from a straightforward computation.

(1) If $\lambda, \lambda_0 \notin \sigma(a)$, then $[(\lambda_0 - \lambda)^{-1} - (\lambda_0 - a)^{-1}]^{-1} = (\lambda_0 - \lambda) + (\lambda_0 - \lambda)^2(\lambda - a)^{-1}$.

(2) If $\lambda_0 \notin \sigma(a)$, then $\sigma_{\mathcal{B}}((\lambda_0 - a)^{-1}) \backslash \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma_{\mathcal{B}}(a)\}$.

(3) If $\lambda_0 \notin \sigma(a)$, then $\sigma_{\mathcal{B}}((a/\lambda_0)^q) \backslash \{1\} = \{\lambda(\lambda - \lambda_0)^{-1} : \lambda \in \sigma_{\mathcal{B}}(a)\}$.

(4) If $\lambda_0 \notin \sigma_{\mathcal{B}}(a)$, then $r_{\mathcal{B}}((\lambda_0 - a)^{-1}) = (d(\lambda_0, \sigma_{\mathcal{B}}(a)))^{-1}$ (this follows from (2)).

An element b of the commutative Banach algebra $\mathcal{B}$ (with unit) has the property: b is invertible in $\mathcal{B}$ if and only if $\hat{b}(\varphi) \neq 0$ for all $\varphi \in \Omega_{\mathcal{B}}$. This is a fundamental result in the Gelfand theory of commutative Banach algebras [**3**, Section 17]. Now we extend this result in a natural way to our situation.

Theorem 3. (1) *For $a \in \mathcal{A}$, either $\mathrm{res}_{\mathcal{B}}(a)$ is empty or $\sigma_{\mathcal{B}}(a) = \overline{\sigma(a)}$. In* (2) *and* (3) *assume $a \in \mathcal{A}$ has $\mathrm{res}_{\mathcal{B}}(a)$ nonempty.*

(2) *When $\mathcal{A}$ has a unit: $a \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$ if and only if $|\hat{a}|$ is bounded away from zero on $\Omega_{\mathcal{A}}$.*

(3) *In general: $a \in Q-\mathrm{inv}_{\mathcal{B}}(\mathcal{A})$ if and only if $|\hat{a} - 1|$ is bounded away from zero on $\Omega_{\mathcal{A}}$.*

(4) *Assume $\mathcal{A}$ has a unit. Let $\{a_1, a_2, \dots, a_m\} \subseteq \mathcal{A}$ have $\lambda_j \in$*

$\text{res}_{\mathcal{B}}(a_j)$ *for* $1 \leq j \leq m$. *Suppose that there exists* $\delta > 0$ *such that*

$$\sum_{k=1}^{m} |\hat{a}_k(\varphi)| \geq \delta \quad \textit{for all } \varphi \in \Omega_{\mathcal{A}}.$$

Then there exists $\{b_1, \ldots, b_m\} \subseteq \mathcal{B}$ *such that*

$$\sum_{k=1}^{m} b_k a_k = 1.$$

Proof. We assume in the proof that $\mathcal{A}$ has a unit, and we prove (4). (2) is a special case of (4), and (3) is an easy consequence of (2). At the end of the proof, we give a simple argument showing that (2) $\Rightarrow$ (1).

Let the collection $\{a_k\}$ and $\delta > 0$ be as in (4). We assume without loss of generality that $1 \in \text{res}_{\mathcal{B}}(a_j)$ for $1 \leq j \leq m$. Given any $\varphi \in \Omega_{\mathcal{A}}$, there exists a j such that $|\hat{a}_j(\varphi)| \geq \delta/m$. Choose $\varepsilon > 0$ such that

$$|z| \geq \delta/m,\ z \neq 1 \Longrightarrow |z(1-z)^{-1}| \geq \varepsilon.$$

This shows that

$$\sum_{k=1}^{m} |\hat{a}_k(\varphi)(1 - \hat{a}_k(\varphi))^{-1}| \geq \varepsilon \quad \text{for all } \varphi \in \Omega_{\mathcal{A}}. \tag{5}$$

Note that $a_k(1-a_k)^{-1} = -1 + (1-a_k)^{-1} \in \mathcal{B}$, $1 \leq k \leq m$. Since by assumption $\Omega_{\mathcal{A}}$ is dense in $\Omega_{\mathcal{B}}$, we have that (5) holds for all $\varphi \in \Omega_{\mathcal{B}}$. By a standard result in Banach algebra theory it follows that there exists a $d_k \in \mathcal{B}$, $1 \leq k \leq m$, such that

$$\sum_{k=1}^{m} (d_k(1-a_k)^{-1}) a_k = 1.$$

This proves (4).

(2) $\Rightarrow$ (1). Assume $a \in \mathcal{A}$ has $\text{res}_{\mathcal{B}}(a)$ nonempty. We have automatically that $\sigma(a) \subseteq \sigma_{\mathcal{B}}(a)$. Suppose $\lambda \notin \overline{\sigma(a)}$. Then there exists a $\delta > 0$

such that $|\lambda - \hat{a}(\varphi)| \geq \delta$ for all $\varphi \in \Omega_{\mathcal{A}}$. By (2), $(\lambda - a) \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$. Therefore, $\lambda \notin \sigma_{\mathcal{B}}(a)$. This argument proves $\overline{\sigma(a)} = \sigma_{\mathcal{B}}(a)$. □

The next result contains a spectral mapping property which holds for some rational functions.

Proposition 4. *Let $F(\lambda) = p(\lambda)/q(\lambda)$ where $p(\lambda), q(\lambda)$ are nonzero polynomials and $\deg(p) > \deg(q)$. Fix $a \in \mathcal{A}$ and assume that $q(\lambda)$ has no zeros on $\sigma_{\mathcal{B}}(a)$.*

(1) $\sigma_{\mathcal{B}}(F(a)) \subseteq F(\sigma_{\mathcal{B}}(a))$;

(2) *When $\overline{\mathrm{res}_{\mathcal{B}}(a)}$ is nonempty, then $\sigma_{\mathcal{B}}(F(a)) = F(\sigma_{\mathcal{B}}(a))$ and $\sigma_{\mathcal{B}}(F(a)) = \overline{\sigma(F(a))}$.*

Proof. We may assume p and q are monic. Let $\{\lambda_1, \ldots, \lambda_m\} \subseteq \mathbf{C}$ be such that $q(\lambda) = \prod_{k=1}^m (\lambda - \lambda_k)$. Assume $\mu \notin F(\sigma_{\mathcal{B}}(a))$ and $\{\mu_1, \ldots, \mu_n\} \subseteq \mathbf{C}$ with $n > m$, so that $(p(\lambda) - \mu q(\lambda)) = \prod_{k=1}^n (\lambda - \mu_k)$. Since $F(\lambda) - \mu = (p(\lambda) - \mu q(\lambda))/q(\lambda)$, it follows that $\mu_k \notin \sigma_{\mathcal{B}}(a)$ for $1 \leq k \leq n$. Therefore,

$$(F(a) - \mu)^{-1} = (a - \lambda_1) \cdots (a - \lambda_m)(a - \mu_1)^{-1} \cdots (a - \mu_n)^{-1}.$$

For $1 \leq j \leq m$,

$$\begin{aligned}(a - \lambda_j)(a - \mu_j)^{-1} &= [(a - \mu_j) + (\mu_j - \lambda_j)](a - \mu_j)^{-1} \\ &= 1 + (\mu_j - \lambda_j)(a - \mu_j)^{-1} \in \mathcal{B}.\end{aligned}$$

Therefore

$$(F(a) - \mu)^{-1} = \left(\prod_{j=1}^{m} (a - \lambda_j)(a - \mu_j)^{-1}\right)\left(\prod_{j=m+1}^{n} (a - \mu_j)^{-1}\right)$$

is in $\mathcal{B}$. Therefore $\mu \notin \sigma_{\mathcal{B}}(F(a))$, which proves (1).

Now assume $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty. By Theorem 3, $\sigma_{\mathcal{B}}(a) = \overline{\sigma(a)}$. A straightforward algebraic argument establishes that $\sigma(F(a)) = F(\sigma(a))$. Then using (1) we have

$$\begin{aligned}\sigma_{\mathcal{B}}(F(a)) &\supseteq \overline{\sigma(F(a))} = \overline{F(\sigma(a))} \supseteq F(\overline{\sigma(a)}) \\ &= F(\sigma_{\mathcal{B}}(a)) \supseteq \sigma_{\mathcal{B}}(F(a)).\end{aligned}$$

Even in the case where $\sigma_{\mathcal{B}}(a) = \mathbf{C}$, the property that $\sigma_{\mathcal{B}}(a) = \overline{\sigma(a)}$ can be useful information. The next proposition shows that this property is preserved when taking a certain limit. $\square$

Proposition 5. *Assume* $\{a_n\} \subseteq \mathcal{A}$, $a \in \mathcal{A}$, $a_n - a \in \mathcal{B}$ *for all* n, *and* $\|a_n - a\|_{\mathcal{B}} \to 0$. *If* $\overline{\sigma(a_n)} = \sigma_{\mathcal{B}}(a_n)$ *for all* n, *then* $\overline{\sigma(a)} = \sigma_{\mathcal{B}}(a)$.

Proof. We may assume that $\mathcal{A}$ has a unit. It suffices to show that if $\mu \notin \overline{\sigma(a)}$, then $\mu \in \mathrm{res}_{\mathcal{B}}(a)$. Assuming $\mu \notin \overline{\sigma(a)}$, let $2\delta = d(\sigma(a), \mu) > 0$. Fix N such that $n \geq N \Rightarrow \|a_n - a\|_{\mathcal{B}} < \delta$. Assuming $n \geq N$, for all $\varphi \in \Omega_{\mathcal{A}}$,

$$\begin{aligned} |\hat{a}_n(\varphi) - \mu| &\geq |\hat{a}(\varphi) - \mu| - |\hat{a}_n(\varphi) - \hat{a}(\varphi)| \\ &\geq 2\delta - \|a_n - a\|_{\mathcal{B}} \geq \delta. \end{aligned}$$

Therefore $d(\sigma(a_n), \mu) \geq \delta$ for $n \geq N$. This implies by hypothesis that, when $n \geq N$, $(\mu - a_n) \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$, and $r_{\mathcal{B}}((\mu - a_n)^{-1}) \leq \delta^{-1}$ (see Note 2(4)). Thus,

$$r_{\mathcal{B}}((\mu - a_n) - (\mu - a))r_{\mathcal{B}}((\mu - a_n)^{-1}) < 1$$

for all n sufficiently large. It follows from Note 1 that $(\mu - a) \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$. $\square$

One useful consequence of the property, $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty, is the equality $\sigma_{\mathcal{B}}(a) = \overline{\sigma(a)}$. Now we note briefly some other consequences of this property.

First, when $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty, then there is a holomorphic functional calculus which applies to the element a. This functional calculus is exactly analogous to the functional calculus for a closed operator with nonempty resolvent; see [**2**] for details. We note that an application of this functional calculus extends Proposition 4. Specifically, if $F(\lambda) = p(\lambda)/q(\lambda)$ is a rational function with $\deg(q) \geq \deg(p)$ and $q(\lambda)$ has no zeros on $\sigma_{\mathcal{B}}(a)$, then $F(a) \in \mathcal{B}$. The spectral mapping theorem, [**2**, Theorem 4.7], implies that $\sigma_{\mathcal{B}}(F(a)) = F(\sigma_{\mathcal{B}}(a))$ in this case.

Secondly, assuming $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty implies the existence of certain boundary type limits of $\hat{a}$ at points in $\Omega_{\mathcal{B}} \backslash \Omega_{\mathcal{A}}$. This result is elementary, but it has interesting consequences; see Theorem 16.

Proposition 6. *Assume* $a \in Q - \mathrm{inv}_{\mathcal{B}}(A)$*, so* $a^q \in \mathcal{B}$*. Set* $b = a^q$*. Suppose* $\varphi_0 \in \Omega_{\mathcal{B}} \backslash \Omega_{\mathcal{A}}$ *with* $\hat{b}(\varphi_0) \neq 1$*. If* $\{\varphi_\lambda\}$ *is a net in* $\Omega_{\mathcal{A}}$ *such that* $\varphi_\lambda \to \varphi_0$ *in* $\Omega_{\mathcal{B}}$*, then the net* $\{\hat{a}(\varphi_\lambda)\}$ *has a limit.*

Proof. For all $\varphi \in \Omega_{\mathcal{A}}$, $\hat{a}(\varphi) = \hat{b}(\varphi)(\hat{b}(\varphi) - 1)^{-1}$. Therefore $\hat{a}(\varphi_\lambda) \to \hat{b}(\varphi_0)(\hat{b}(\varphi_0) - 1)^{-1}$. □

3. Examples. In this section we consider a number of interesting examples from analysis. The spectral theory developed in Section 2 and the results in Section 4 allow us to extend classical results involving certain well-known Banach algebras to a more general setting. As indicated previously, the algebras we consider will all be commutative, complete, Hausdorff LMC-algebras. Some of our results hold for even more general algebras.

One advantage of letting $\mathcal{A}$ be a commutative complete LMC-algebra is that such algebras have a Gelfand representation theory. Furthermore, the underlying space, $\Omega_{\mathcal{A}}$, can often be computed using information concerning the Gelfand space of familiar Banach algebras. Specifically, let $(\mathcal{A}, \{p_\delta : \delta \in D\})$ be the given algebra with algebra seminorms p_δ. For each $\delta \in D$, let N_δ be the ideal $N_\delta = \{a \in \mathcal{A} : p_\delta(a) = 0\}$. Let $\mathcal{A}_\delta$ be the factor Banach algebra obtained by completing the algebra $\mathcal{A}/N_\delta$ in the norm $\|a + N_\delta\|_\delta = p_\delta(a)$, $a \in A$. Let $\pi_\delta : \mathcal{A} \to \mathcal{A}_\delta$ be the continuous embedding of $\mathcal{A}$ into the factor algebra $\mathcal{A}_\delta$. Then $\Omega_{\mathcal{A}}$ can be computed by:

$$\Omega_{\mathcal{A}} = \cup\{\varphi_0\pi_\delta : \delta \in D, \varphi \in \Omega_{\mathcal{A}_\delta}\};$$

see [**1**, (4-10-7)].

In concrete examples the algebras $\mathcal{A}_\delta$ are often familiar Banach algebras, so in these cases $\Omega_{\mathcal{A}}$ is easily determined. This is illustrated in several of the examples that follow.

Example 1. Let $\mathcal{D}$ be a commutative Banach algebra without unit. We denote the Gelfand space of $\mathcal{D}$ by Γ. Let $\mathcal{A} = \mathbf{C}(\Gamma)$, the algebra of all complex-valued continuous functions on Ω. The natural topology for $\mathcal{A}$ is defined by the collection of seminorms determined by nonempty compact subsets K of Γ:

$$p_K(f) \equiv \sup\{|f(\omega)| : \omega \in K\}, \quad f \in \mathcal{A}.$$

For $\mathcal{B}$ take the algebra $\mathcal{D}$ with identity adjoined. Then $\Omega_{\mathcal{A}} = \Gamma$, while $\Omega_{\mathcal{B}}$ is the one point compactification of Γ. The standing assumptions hold here.

Example 2. Let L^+ be the space of all measurable functions f on $(0, \infty)$ such that $f(t)e^{-ct} \in L^1(0, \infty)$ for all $c > 0$. L^+ is a convolution algebra with multiplication

$$f * g(x) \equiv \int_0^x f(x-t)g(t)\, dt, \quad x > 0.$$

For each $n \geq 1$, let ρ_n be the weight $\rho_n(t) = e^{-t/n}$. Set

$$p_n(f) = \int_0^\infty |f(t)|\rho_n(t)\, dt, \quad f \in L^+.$$

Then $(L^+, \{p_n : n \geq 1\})$ is a complete, Hausdorff LMC-algebra. For $z \in \mathbf{C}$ with $\mathrm{Re}\,(z) > 0$, define

$$\varphi_z(f) \equiv \int_0^\infty f(t)e^{-zt}\, dt, \quad f \in L^+.$$

Then $\Omega_{L^+} = \{\varphi_z : \mathrm{Re}\,(z) > 0\}$. Thus, Ω_{L^+} can be identified with the open half-plane $H_0 = \{z : \mathrm{Re}\,(z) > 0\}$. For $f \in L^+$, $z \in H_0$, let

$$\hat{f}(z) \equiv \hat{f}(\varphi_z) = \int_0^\infty f(t)e^{-zt}\, dt. \tag{1}$$

Therefore the Gelfand transform of $f \in L^+$ on Ω_{L^+} is identified with the usual Laplace transform of f on H_0. We have, for $f \in L^+$,

$$\sigma(f) = \{\hat{f}(z) : z \in H_0\} \cup \{0\}.$$

The factor algebras involved are the weighted convolution algebras $L^1((0, \infty), \rho_n)$. The Gelfand spaces of these algebras are well-known; see [**5**, Section 18]. Thus, Ω_{L^+} could be computed as the union of those Gelfand spaces (as remarked in the comments at the beginning of this section).

Now let $L_1^+ \equiv L^1([0, \infty))$. This Banach algebra is a subalgebra of L^+ and, as is well-known, its Gelfand space is naturally identified with $\overline{H_0}$, where for $f \in L_1^+$ and $z \in \overline{H_0}$, $\hat{f}(z)$ is defined as in (1). Throughout, when $\mathcal{A} = L^+$, we take the algebra $\mathcal{B}$ to be L_1^+. Note that the standing assumptions hold in this case.

Example 3. Let l_1^+ be the Banach space of all complex sequences, $a = \{a_k\}_{k\geq 0}$, with $\|a\|_1 = \sum_{k=0}^{\infty} |a_k| < \infty$. Define l^+ to be the space of all complex sequences $a = \{a_k\}_{k\geq 0}$ such that $\{a_k e^{-ck}\} \in l_1^+$ for all $c > 0$. Both of these spaces are convolution algebras with the multiplication of $a = \{a_k\}$ and $b = \{b_k\}$ given by

$$(a * b)_n \equiv \sum_{k=0}^{n} a_{n-k} b_k, \quad n \geq 0.$$

Now define weights ρ_n for $n \geq 1$ by

$$\rho_n(k) = e^{-k/n}, \quad k \geq 0,$$

and define corresponding algebra norms p_n by

$$p_n(a) \equiv \sum_{k=0}^{\infty} a_k p_n(k), \quad a \in l^+.$$

Then $(l^+, \{p_n : n \geq 1\})$ is a complete Hausdorff LMC-algebra.

Now we determine Ω_{l^+}. For $z \in \mathbf{C}$, $|z| < 1$, and $a = \{a_k\} \in l^+$, let

$$\varphi_z(a) = \sum_{k=0}^{\infty} a_k z^k. \tag{1}$$

It is straightforward to check that $\varphi_z \in \Omega_{l^+}$. Using results in [**5**, Section 19] applied to the factor algebras of l^+, it can be shown that $\Omega_{l^+} = \{\varphi_z : |z| < 1\}$. Set $U = \{z \in \mathbf{C} : |z| < 1\}$. For $a \in l^+$ and $z \in U$, let

$$\hat{a}(z) \equiv \hat{a}(\varphi_z) = \sum_{k=0}^{\infty} a_k z^k. \tag{2}$$

Thus Ω_{l^+} is identified with U, and $\hat{a}$ acting on U is a holomorphic function.

When $\mathcal{A} = l^+$, we take $\mathcal{B}$ to be the Banach algebra $(l_1^+\| \cdot \|_1)$. For $b = \{b_k\} \in l_1^+$, φ_z and $\hat{b}(z)$ are defined as in (1) and (2) above for all $z \in \bar{U}$. Note that the standing assumption holds.

One feature of Example 3 is that $a \to \hat{a}$, $a \in l^+$, maps l^+ into $\mathcal{H}(U)$, the algebra of all holomorphic functions on U. In fact the Gelfand map is onto, $\hat{l}^+ = \mathcal{H}(U)$. The algebra $\mathcal{H}(U)$ has other interesting Banach subalgebras besides $\hat{l}^+$. For example, with $\mathcal{A} = \mathcal{H}(U)$, one could consider spectral theory with respect to the disk algebra.

Example 4. As our last example, we consider a convolution algebra of sequences defined on the integers Z. Let $\mathcal{A}$ be the space of all complex sequences, $a = \{a_k\}_{k\in z}$, such that whenever $0 < r < R < 1$, then

$$\sum_{k=1}^{\infty} |a_{-k}|r^{-k} + \sum_{k=0}^{\infty} |a_k|R^k \quad \text{is finite.}$$

Multiplication of sequences is the usual convolution. Define weights $\rho_n(k)$, $n \geq 2$, by:

$$\rho_n(k) = \begin{cases} (n/(n+1))^k & \text{if } k \geq 0; \\ (1/n)^k & \text{if } k < 0. \end{cases}$$

Let $\{p_n : n \geq 2\}$ be the algebra norms on $\mathcal{A}$ defined by

$$p_n(a) = \sum_{k\in z} |a_k|\rho_n(k).$$

Then $(\mathcal{A}, \{p_n : n \geq 2\})$ is a complete Hausdorff LMC-algebra. Using [**5**, Section 19], it follows that the Gelfand space of the factor algebra determined by p_n is identified with the annulus

$$\Omega_n = \left\{ z \in \mathbf{C} : \frac{1}{n} \leq |z| \leq \frac{n}{n+1} \right\}.$$

Thus $\Omega_{\mathcal{A}} = \cup_{n=2}^{\infty} \Omega_n$ is identified with the punctured open unit disk $U \backslash \{0\}$. For $a \in \mathcal{A}$ and $z \in U\backslash\{0\}$, the Gelfand transform of a is given by

$$\hat{a}(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k} + \sum_{k=0}^{\infty} a_k z^k.$$

It is straightforward to check that $\{\hat{a} : a \in \mathcal{A}\}$ is the algebra of all holomorphic functions on $\bar{U}\backslash\{0\}$. A Banach subalgebra of $\mathcal{A}$ is $\mathcal{B} = l_1^+$. As noted before, $\Omega_{\mathcal{B}}$ is naturally identified with $\bar{U}$. The standing assumptions are satisfied for the pair $(\mathcal{A}, \mathcal{B})$.

4. Perturbation results. In this section we consider questions of how $\sigma_{\mathcal{B}}(t)$ and $\mathrm{res}_{\mathcal{B}}(t)$ change when an element $b \in \mathcal{B}$ is added to t. Of special interest here is the question:

If $\mathrm{res}_{\mathcal{B}}(t)$ is nonempty, under what conditions is $\mathrm{res}_{\mathcal{B}}(t+b)$ nonempty?

Part (2) of the next proposition provides one answer to this question.

In what follows we use the notation

$$d(\lambda, \Gamma) = \inf\{|\lambda - \gamma| : \gamma \in \Gamma\} \quad \text{where } \lambda \in \mathbf{C} \text{ and } \Gamma \text{ is a subset of } \mathbf{C}.$$

Proposition 7. *Assume $a, t \in \mathcal{A}$.*

(1) *If $t \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$, $t^{-1}a \in \mathcal{B}$ and $r_{\mathcal{B}}(t^{-1}a) < 1$, then $t + a \in \mathrm{Inv}_{\mathcal{B}}(A)$.*

(2) *If $\sup\{d(\lambda, \sigma_{\mathcal{B}}(t)) : \lambda \in \mathrm{res}_{\mathcal{B}}(t)\} = \infty$, then $\mathrm{res}_{\mathcal{B}}(t+b)$ is nonempty for all $b \in \mathcal{B}$.*

Proof. Assume as in (1) that $r_{\mathcal{B}}(t^{-1}a) < 1$. Then $1+t^{-1}a$ is invertible in $\mathcal{B}$. Setting $s \equiv (1 + t^{-1}a)^{-1}$, we have $st^{-1}(t + a) = s(1 + t^{-1}a) = 1$. This verifies (1).

To prove (2), note that for $\lambda \in \mathrm{res}_{\mathcal{B}}(t)$, $r_{\mathcal{B}}((\lambda - t)^{-1}) = d(\lambda, \sigma_{\mathcal{B}}(t))^{-1}$ (see Note 2(4)). Therefore, by hypothesis for any $b \in \mathcal{B}$, there exists some $\lambda \in \mathrm{res}_{\mathcal{B}}(t)$ such that $r_{\mathcal{B}}((-\lambda + t)^{-1}b) < 1$. Applying (1), it follows that $-\lambda + t + b \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$. □

Next we consider a case of special importance: when $\sigma_{\mathcal{B}}(t)$ is contained in a half-plane. In this case a strong result holds concerning the $\mathcal{B}$-resolvent set of perturbations of t. We begin with a preliminary lemma.

Lemma 8. *Let $t \in \mathcal{A}$, $t \notin \mathcal{B}$, with $\mathrm{res}_{\mathcal{B}}(t)$ nonempty. For $\lambda \in \mathrm{res}_{\mathcal{B}}(t)$, let*

$$\gamma(\lambda, t) \equiv \sup\{|\mu(\mu - \lambda)^{-1}| : \mu \in \sigma_{\mathcal{B}}(t)\}.$$

Assume there exists an $\eta \geq 1$ *such that*

$$\sup\{d(\lambda, \sigma_{\mathcal{B}}(t)) : \lambda \in \mathrm{res}_{\mathcal{B}}(t), \gamma(\lambda, t) \leq \eta\} = \infty.$$

If $b \in \mathcal{B}$ *has* $r_{\mathcal{B}}(b) < \eta^{-1}$ *and* $|\xi| \geq 1$*, then*

$$\sup\{d(\lambda, \sigma_{\mathcal{B}}(\xi t + tb)) : \lambda \in \mathrm{res}_{\mathcal{B}}(\xi t + tb)\} = \infty.$$

Proof. We do the proof for the case where $\mathcal{A}$ has a unit. Set $\Gamma = \{\lambda \in \mathrm{res}_{\mathcal{B}}(t) : \gamma(\lambda, t) \leq \eta\}$. For each $\lambda \in \Gamma$, $\lambda \neq 0$, choose $s_\lambda = (\lambda^{-1}t)^q$. Then $(\lambda - t)\lambda^{-1}(1 - s_\lambda) = 1$. By Note 2,

$$\sigma_{\mathcal{B}}(s_\lambda) = \{\mu(\mu - \lambda)^{-1} : \mu \in \sigma_{\mathcal{B}}(t)\} \cup \{1\}.$$

Since $\lambda \in \Gamma$, we have by definition that $r_{\mathcal{B}}(s_\lambda) \leq \eta$. Now fix $\lambda \in \Gamma$, and assume $r_{\mathcal{B}}(b) < \eta^{-1}$ and $|\xi| \geq 1$. Then $(\lambda - t)^{-1}t\xi^{-1}b = -s_\lambda \xi^{-1}b$, and therefore,

$$r_{\mathcal{B}}((\lambda - t)^{-1}t\xi^{-1}b) \leq r_{\mathcal{B}}(s_\lambda)|\xi^{-1}|r_{\mathcal{B}}(b) < 1.$$

By Proposition 7, $(\lambda - t) - t\xi^{-1}b \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$, so $\xi\lambda - (\xi t + tb) \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$. It follows that

$$\sup\{d(\lambda, \sigma_{\mathcal{B}}(\xi t + tb)) : \lambda \in \mathrm{res}_{\mathcal{B}}(\xi t + tb)\} = \infty. \qquad \square$$

Now we prove the main result of this section. An application of this result is given in Theorem 14.

Theorem 9. *Assume* $t \in \mathcal{A}$ *with* $\sigma_{\mathcal{B}}(t)$ *contained in a half-plane.*

(1) *If* $b \in \mathcal{B}$ *and* $\xi \in \mathbf{C}$*, then* $\mathrm{res}_{\mathcal{B}}(\xi t + b)$ *is nonempty.*

(2) *If* $b, c \in \mathcal{B}$, $r_{\mathcal{B}}(b) \leq 1$ *and* $|\xi| \geq 1$*, then* $\mathrm{res}_{\mathcal{B}}(\xi t + tb + c)$ *is nonempty.*

Proof. (1) follows immediately from Proposition 7 (2). To prove (2), note that by rotating and translating, if necessary, we may assume

$\sigma_{\mathcal{B}}(t) \subseteq \bar{H}_0$. Let $\varepsilon > 0$ be fixed. We prove that $\eta = 1 + \varepsilon$ works in the hypothesis of Lemma 8. Consider the cone

$$C_\varepsilon = \{\lambda \in \mathbf{C} : \mathrm{Re}\,(\lambda) < 0, |\mathrm{Im}\,(\lambda)| \leq |\mathrm{Re}\,(\lambda)|\varepsilon\}.$$

By assumption, $C_\varepsilon \subseteq \mathrm{res}_{\mathcal{B}}(t)$. Assume $\lambda \in C_\varepsilon$, and set $a = \mathrm{Re}\,(\lambda)$. For $\mu \in \bar{H}_0$, $|\mu - \lambda| \geq |a|$. Also, $|\mu| \leq |\mu - a|$. Therefore

$$\begin{aligned}|\mu| &\leq |\mu - (a + i\mathrm{Im}\,(\lambda))| + |\mathrm{Im}\,(\lambda)| \\ &\leq |\mu - \lambda| + \varepsilon|a| \leq (1+\varepsilon)|\mu - \lambda|.\end{aligned}$$

This proves

$$C_\varepsilon \subseteq \{\lambda \in \mathrm{res}_{\mathcal{B}}(t) : \gamma(\lambda, t) \leq \eta = 1 + \varepsilon\}.$$

Now if b, c and ξ are as in the statement of (2), there exists $\varepsilon > 0$ such that $r_{\mathcal{B}}(b) < (1+\varepsilon)^{-1}$. By the argument above, Lemma 8 implies

$$\sup\{d(\lambda, \sigma_{\mathcal{B}}(\xi t + tb)) : \lambda \in \mathrm{res}_{\mathcal{B}}(\xi_t + tb)\} = \infty.$$

Therefore, the result follows from Proposition 7. $\square$

The final result of this section involves a continuity result for $\sigma_{\mathcal{B}}(\cdot)$.

Theorem 10. *Let $t \in \mathcal{A}$ and $b \in \mathcal{B}$. When $\lambda \in \sigma_{\mathcal{B}}(t+b)$, then $d(\lambda, \sigma_{\mathcal{B}}(t)) \leq r_{\mathcal{B}}(b)$. When $\lambda \in \sigma_{\mathcal{B}}(t)$, then $d(\lambda, \sigma_{\mathcal{B}}(t+b)) \leq r_{\mathcal{B}}(b)$.*

Proof. Assume $r_{\mathcal{B}}(b) < d(\lambda, \sigma_{\mathcal{B}}(t))$. Set $c = (\lambda - t)^{-1} \in \mathcal{B}$. By Note 2 (4), $r_{\mathcal{B}}(b) < r_{\mathcal{B}}(c)^{-1}$, and so $r_{\mathcal{B}}(bc) < 1$. Thus, $(\lambda - (t+b))c = 1 - bc \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$. It follows that $\lambda - (t+b) \in \mathrm{Inv}_{\mathcal{B}}(\mathcal{A})$, so $\lambda \notin \sigma_{\mathcal{B}}(t+b)$. This proves the first assertion.

To verify the second assertion, in the first statement replace t by $(t+b)$ and b by $-b$. $\square$

Corollary 11. *For $\delta > 0$ let $U_\delta = \{\lambda \in \mathbf{C} : |\lambda| < \delta\}$. Assume $t \in \mathcal{A}$ and $b \in \mathcal{B}$ with $r_{\mathcal{B}}(b) < \delta$. Then*

$$\sigma_{\mathcal{B}}(t+b) \subseteq \sigma_{\mathcal{B}}(t) + U_\delta.$$

Proof. Suppose $\lambda \in \sigma_{\mathcal{B}}(t+b)$. By Theorem 10 we have $d(\lambda, \sigma_{\mathcal{B}}(t)) \leq r_{\mathcal{B}}(b) < \delta$. Therefore there exists $\mu \in \sigma_{\mathcal{B}}(t)$ such that $|\lambda - \mu| < \delta$. Thus,

$$\lambda = \mu + (\lambda - \mu) \in \sigma_{\mathcal{B}}(t) + U_\delta. \qquad \square$$

5. Some applications. In this section we apply the results of Sections 2 and 4 to the spectral theory of L^+ and l^+. Some of these applications are extensions of classical theorems. Throughout this section, when $\mathcal{A} = L^+$, then $\mathcal{B} = L_1^+$, and when $\mathcal{A} = l^+$, then $\mathcal{B} = l_1^+$.

First we construct an example to show that it is not always the case that $\overline{\sigma(a)} = \sigma_{\mathcal{B}}(a)$. There exists a sequence $a = \{a_k\}$ such that $\sum_{k=0}^{\infty} a_k e^{ikt}$ is the Fourier series of a continuous function, but $\{a_k\} \notin l_1^+$; see [**8**, p. 32] for example. In this case $a \in l^+$ and

$$\hat{a}(z) = \sum_{k=0}^{\infty} a_k z^k$$

is bounded on U by [**10**, Theorem 11.8].

Now set

$$f(t) = \sum_{k=0}^{\infty} a_k \chi_{[k,k+1)(t)}, \quad t \geq 0.$$

Then $f \in L^+$, but $f \notin L_1^+$. For $\mathrm{Re}\,(s) > 0$,

$$\begin{aligned} \hat{f}(s) &= \sum_{k=0}^{\infty} a_k \int_k^{k+1} e^{-st}\,dt \\ &= \sum_{k=0}^{\infty} a_k s^{-1} [e^{-sk} - e^{-s(k+1)}] \\ &= s^{-1}(1 - e^{-s}) \sum_{k=0}^{\infty} a_k (e^{-s})^k. \end{aligned}$$

The integration term by term is justified by the Lebesgue dominated convergence theorem. Since $e^{-s} \in U$ when $\mathrm{Re}\,(s) > 0$, $\hat{f}(s)$ is bounded on H_0.

We have proved the following result.

Proposition 12. *For either $\mathcal{A} = L^+$ or $\mathcal{A} = l^+$, there exists $a \in \mathcal{A}$ such that $\sigma(a)$ is bounded, but $\sigma_{\mathcal{B}}(a) = \mathbf{C}$.*

Now we identify a large class of functions in L^+ for which $\sigma_{\mathcal{B}}(f) = \overline{\sigma(f)}$. Throughout this discussion 1 denotes the constant function on $[0, \infty)$. We need several elementary results:

(i) $\hat{1}(s) = 1/s$ for $s \in H_0$;

(ii) For $\mathrm{Re}\,(\lambda) < 0$,

$$(\lambda^{-1}1)^q(t) = -\lambda^{-1} e^{\lambda^{-1}t} \in L^1[0, \infty);$$

(iii)

$$(1)^{(*n)}(t) = \frac{1}{(n-1)!} t^{n-1}, \quad n \geq 1;$$

(iv) If $G(z) = z/(1 - \mu z)$ with $\mathrm{Re}\,(\mu) < 0$, then evaluating G at 1 with respect to the operational calculus, we have $G(1)(t) = e^{\mu t}$.

Theorem 13. *Assume $f(t) = p(t) + g(t)$ where p is a polynomial and $g \in L^1[0, \infty)$. Then*

$$\sigma_{\mathcal{B}}(f) = \{\hat{f}(s) : \mathrm{Re}\,(s) > 0\}^-.$$

Proof. First note that by (i) and (ii), $\sigma(1) = H_0$ and $\sigma_{\mathcal{B}}(1) = \bar{H}_0$. Assume $p(t) = \sum_{k=0}^m b_k t^k$ and $\{a_1, \ldots, a_n\} \subseteq \mathbf{C}$, $\{\mu_1, \ldots, \mu_n\} \subseteq \mathbf{C}$ with $\mathrm{Re}\,(\mu_k) < 0$ for $1 \leq k \leq n$. Set

$$F(z) = \sum_{k=1}^{m+1} \frac{b_{k-1}}{(k-1)!} z^k + \sum_{k=1}^{n} \frac{a_k z}{1 - \mu_k z}.$$

If $b_j \neq 0$ for some j, then the rational function $F(z)$ satisfies the hypothesis of Proposition 4. Therefore, in this case, $\sigma_{\mathcal{B}}(F(1)) = \overline{\sigma(F(1))}$. If $b_k = 0$ for all k, then $F(1) \in L^1[0, \infty)$, so again, $\sigma_{\mathcal{B}}(F(1)) = \overline{\sigma(F(1))}$. Using (iii) and (iv), we have

$$F(1)(t) = \sum_{k=0}^{m} b_k t^k + \sum_{k=1}^{n} a_k e^{\mu_k t}.$$

Now sums of the form

$$\sum_{k=1}^{n} a_k e^{\mu_k t}, \quad \mathrm{Re}\,(\mu_k) < 0$$

are dense in $L^1[0,\infty)$. Therefore, by Proposition 5, $\sigma_{\mathcal{B}}(f) = \overline{\sigma(f)}$. $\square$

As indicated before, in Theorem 13, $\sigma_{\mathcal{B}}(1)$ is the closed half-plane $\bar{H}_0$. Therefore, Theorem 9 applies. We derive from this a result of the type discussed in [**11**, pp. 313–314].

Theorem 14. *Assume $b(x) \in L^+$ has the properties that there exist a $D > 0$ and an f such that $f \le 0$ almost everywhere on (D,∞) and $f \in L^1([d,\infty))$ whenever $d > D$, such that:*

(i) $b(x) = \int_d^x f(t)\,dt + b(d)$ *for* $d > D$;

(ii) $b(d) < 1$ *for* $d > D$;

(iii) $\lim_{x\to\infty} b(x) = 0$.

If $|\xi| \ge 1$, $d > D$, and $c(x) \in L^1[0,\infty)$, then

$$\mathrm{res}_{\mathcal{B}}((\xi - b(d))1 + b(x) + c(x)) \quad \textit{is nonempty.}$$

Proof. Fix $d > D$. Note that

$$\begin{aligned} 1 * (\chi_{[d,\infty)} f)(x) &= \chi_{[d,\infty)}(x) \int_d^x f(t)\,dt \\ &= \chi_{[d,\infty)}(x)[b(x) - b(d)]. \end{aligned}$$

Also,

$$\|\chi_{[d,\infty)} f\|_1 = \int_d^\infty |f(t)|\,dt = \left| \int_d^\infty f(t)\,dt \right| = |b(d)| < 1.$$

Now $\chi_{[0,d)}(x)(b(x) - b(d)) + c(x) \in L^1[0,\infty)$. Finally, for $|\xi| \ge 1$,

$$\begin{aligned} (\xi - b(d))1 + b(x) + c(x) = \xi 1 + 1 * (\chi_{[d,\infty)} f)(x) \\ + [\chi_{[0,d)}(x)(b(x) - b(d)) + c(x)]. \end{aligned}$$

Therefore the result follows from Theorem 9. □

It is straightforward to check that for the sequence $1 = (1, 1, 1, \dots)$ in l^+, $\sigma_{\mathcal{B}}(1)$ is a closed half-plane. Thus, a result analogous to Theorem 14 can be proved in l^+.

We give a specific example where Theorem 14 applies.

Example 15. Fix $\beta \geq 0$, $0 < \gamma \leq 1$. Let $b(x) = (x+\beta)^{-\gamma}$. If $\beta \geq 1$, then set $D = 0$. If $0 \leq \beta < 1$, set $D = 1-\beta$. Let $f(t) = -\gamma(t+\beta)^{-(\gamma+1)}$. Applying Theorem 14, we have: for $|\xi| \geq 1$, and $0 < \mu < 1$ when $\beta < 1$, or $0 < \mu < \beta^{-\gamma}$ when $\beta \geq 1$, and $c(x) \in L^1[0, \infty)$,

$$\mathrm{res}_{\mathcal{B}}((\xi - \mu)1 + (x+\beta)^{-\gamma} + c(x)) \quad \text{is nonempty.}$$

When $\beta > 0$, $\gamma > 1$, then $(x+\beta)^{-\gamma} \in L^1[0, \infty)$. It follows from Theorem 9 that for any $\xi \in \mathbf{C}$ and any $c(x) \in L^1[0, \infty)$,

$$\mathrm{res}_{\mathcal{B}}(\xi 1 + (x+\beta)^{-\gamma} + c(x)) \quad \text{is nonempty.}$$

In the next two results we consider some consequences of the situation where $\mathrm{res}_{\mathcal{B}}(f)$ is nonempty. First we show that limiting boundary values must exist almost everywhere.

Theorem 16. (1) *Assume $a = \{a_k\} \in l^+$, and $\mathrm{res}_{\mathcal{B}}(a)$ is nonempty. Then $\lim_{z \to e^{it}, |z|<1} \hat{a}(z)$ exists for all $t \in [-\pi, \pi]$ with the exception of a set $E \subseteq [-\pi, \pi]$ of Lebesgue measure zero.*

(2) *Assume $f \in L^+$, with $\mathrm{res}_{\mathcal{B}}(f)$ nonempty. Then $\lim_{z \to iy, \mathrm{Re}\,(z)>0} \hat{f}(z)$ exists for all $y \in \mathbf{R}$ with the exception of a set $F \subseteq \mathbf{R}$ of Lebesgue measure zero.*

Proof. Let $a = l^+$ be as in (1). We may assume $b = a^q$ exists and is in l_1^+. Then $\hat{b}(z)$ is a continuous function on the closed unit disk $\bar{U}$ which is holomorphic on U. It follows from [**6**, Corollary, p. 52] that $E = \{t \in [-\pi, \pi] : \hat{b}(e^{it}) = 1\}$ is a set of Lebesgue measure zero. Therefore (1) follows immediately from Proposition 6.

Now let $f \in L^+$ be as in (2). We may assume $g = f^q \in L_1^+$. Then $\hat{g}(s)$ is a continuous function on $\overline{H_0}$ and is holomorphic in H_0. Set

$$\varphi(z) = (1+z)(1-z)^{-1}$$

and note that φ maps $\bar{U}\backslash\{1\}$ onto $\overline{H_0}$ with the boundary of $\bar{U}$ mapping onto the line $\{iy : y \in \mathbf{R}\}$. Let $F = \{iy : \hat{g}(iy) = 1\}$. Set $E = \varphi^{-1}(F)$. It is not hard to verify that if $\{t \in [-\pi, \pi] : e^{it} \in E\}$ has Lebesgue measure zero, then so does F. Consider

$$k(z) = \hat{g}(\varphi(z)), \quad z \neq 1, \quad k(1) = 0.$$

Then $k(z)$ is a continuous function on $\bar{U}$ and is holomorphic on U. Again, as in (1), it follows that E has Lebesgue measure zero. □

The next result extends a classical theorem of Paley and Wiener concerning functions in L_1^+ [**9**, Theorem 17]. The proof given here essentially follows their proof.

Theorem 17. *Assume $f \in L^+$ has $\operatorname{res}_{\mathcal{B}}(f)$ nonempty. Suppose h is a measurable function on $[0, \infty)$ and h is bounded on each interval $[0, t]$, $t \geq 0$. Assume $\hat{f}(s) - 1$ is bounded away from zero on H_0. If $\lim_{x\to+\infty}[h(x) - \int_0^x f(x-t)h(t)\,dt] = \gamma$, then $\lim_{x\to\infty} h(x)$ exists. If $\lim_{x\to\infty} h(x) \neq 0$, then $\lim_{s\in H_0, s\to 0} \hat{f}(s) = \beta$ exists and $\lim_{x\to\infty} h(x) = \gamma(1-\beta)^{-1}$.*

Proof. By Theorem 3, $g = f^q \in L_1^+$. Recall that $f + g - f * g = 0$. Set $w = h - f * h$. By hypothesis, $\lim_{x\to+\infty} w(x) = \gamma$. Since g is integrable, by the dominated convergence theorem,

$$\begin{aligned}\lim_{x\to+\infty} \int_0^x w(x-t)g(t)\,dt &= \lim_{x\to+\infty} \int_0^{+\infty} \chi_{[0,x]}(t)w(x-t)g(t)\,dt \\ &= \gamma \int_0^\infty g(t)\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to+\infty} h(x) &= \lim_{x\to+\infty} (w(x) - w * g(x)) \\ &= \gamma - \gamma \int_0^{+\infty} g(t)\,dt = \gamma(1 - \hat{g}(0)).\end{aligned}$$

If $\lim_{x\to+\infty} h(x) \neq 0$, then $1 - \hat{g}(0) \neq 0$. Then for $\operatorname{Re}(s) > 0$, $\hat{f}(s) + \hat{g}(s) - \hat{f}(s)\hat{g}(s) = 0$, so that

$$\lim_{s\in H_0, s\to 0} \hat{f}(s) = \frac{-\hat{g}(0)}{1 - \hat{g}(0)} = \beta.$$

Therefore,

$$\lim_{x\to+\infty} h(x) = \gamma(1-\hat{g}(0)) = \gamma(1-\beta)^{-1}. \qquad \square$$

Now we consider briefly some applications to convolution operators. Fix p with $1 \le p \le +\infty$. Let p' be the conjugate index of p $(p^{-1}+(p')^{-1}=1)$. Denote the usual p-norm on $L^p(0,+\infty)$ by $|||\cdot|||_p$. First note that if $g \in L^p(0,\infty)$, then

$$p_m(g) = \int_0^{+\infty} |g(x)|e^{-(x/m)}\,dx \le |||g|||_p|||e^{-(x/m)}|||_{p'} < +\infty.$$

It follows from this inequality that

$$(\#) \qquad \begin{aligned} &L^p(0,+\infty) \subseteq L^+; \\ &\qquad \text{and if } \{g_n\} \subseteq L^p, g \in L^p \\ &\qquad \text{and } |||g_n - g|||_p \to 0, \text{ then } g_n \to g \text{ in } L^+. \end{aligned}$$

Fix $f \in L^+$. Define $\operatorname{dom}(T_f) = \{g \in L^p : f * g \in L^p\}$, and

$$T_f(g)(x) = f * g(x) = \int_0^x f(x-t)g(t)\,dt, \quad g \in \operatorname{dom}(T_f).$$

Making use of $(\#)$, it is easy to show that T_f is a closed operator. Note that when $f \in L^1(0,+\infty)$, then $T_f \in B(L^p)$.

Theorem 18. *Assume $f \in L^+$ with $\overline{\sigma(f)} = \sigma_{\mathcal{B}}(f)$. Let T_f be the closed convolution operator on L^p defined above. Then $\sigma_{\mathcal{B}}(f) = \sigma(T_f)$ (the spectrum of T_f), and when $\lambda I - T_f$ has a bounded inverse in $B(L^p)$, then it has the form $(1/\lambda)I - T_g$ where $g \in L^1$.*

Proof. Suppose $\lambda \notin \sigma_{\mathcal{B}}(f)$. Then $g = (f/\lambda)^q$ is in L_1^+, and $g + (f/\lambda) - (f/\lambda) * g = 0$. It follows immediately that $(\lambda I - Tf)((1/\lambda)I - T_g) = I$. Thus, $\lambda \notin \sigma(T_f)$.

Now assume that $\lambda I - T_f$ has a bounded inverse on L^p. Suppose that there exists a $z \in \mathbf{C}$ with $\operatorname{Re}(z) > 0$ such that $\hat{f}(z) = \lambda$. Choose

$h \in L^p$ with $\hat{h}(z) \neq 0$ (it is easy to see that such an h exists). Now $\mathcal{R}(\lambda I - T_f) = L^p$, so there exists a $g \in L^p$ with $(\lambda - f) * g = h$. Therefore

$$0 = \lambda \hat{g}(z) - \hat{f}(z)\hat{g}(z) = \hat{h}(z),$$

a contradiction. This proves that $\hat{f}(z) \neq \lambda$ for all z with $\mathrm{Re}\,(z) > 0$, so $\lambda \notin \sigma(f)$. Thus

$$\sigma(f) \subseteq \sigma(T_f) \subseteq \sigma_{\mathcal{B}}(f).$$

Since $\overline{\sigma(f)} = \sigma_{\mathcal{B}}(f)$, taking closures in the inclusions above we have $\sigma(T_f) = \sigma_{\mathcal{B}}(f)$. $\square$

For $f \notin L^+$ one can consider the operator C_f given by convolution by f acting on L^p_{LOC}, the space of all measurable functions on $[0, \infty)$ which are in $L^p[0, a]$ for all $a > 0$. For $g \in L^p_{\mathrm{LOC}}$,

$$C_f(g) = f * g \in L^p_{\mathrm{LOC}}.$$

When $\mathrm{res}_{\mathcal{B}}(f)$ is nonempty, the equation $(\lambda - Cf)(g) = h$, $h \in L^p_{\mathrm{LOC}}$, will have a solution $g \in L^p$ whenever $h \in L^p$, exactly when $\lambda \notin \sigma_{\mathcal{B}}(f)$. This follows just as in the proof of Theorem 18.

There are other sequences of seminorms which determine LMC-algebras of interest in Laplace transform theory. Some of these are considered by G. Jordan, O. Staffans and R. Wheeler in [**7**]. We consider briefly one such sequence. Fix $\beta > 0$. For each $n \geq 1$, let

$$\rho_n(t) = (1+t)^{\beta} e^{-(1/n)t}, \quad t \geq 0.$$

Define algebra seminorms by

$$\|f\|_n = \int_0^{\infty} |f(t)| \rho_n(t)\, dt, \quad f \text{ measurable on } [0, \infty).$$

Let $\mathcal{A}$ be the complete LMC-algebra of all measurable functions on $[0, \infty)$ such that $\|f\|_n < \infty$ for all $n \geq 1$. It is straightforward to verify that $\mathcal{A} = L^+$. Let $\mathcal{B} = \{f \in L^+ : \int_0^{\infty} |f(t)|(1+t)^{\beta}\, dt < \infty\}$. In this case, just as before, $\Omega_{\mathcal{A}} = H_0$ and $\Omega_{\mathcal{B}} = \bar{H}_0$. This leads to interesting applications of the sort derived in this section. In particular, Theorems 13 and 14 hold if, in their hypotheses, $L^1[0, \infty)$ is replaced by $\mathcal{B}$.

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