

THE DUAL OF BERGMAN METRIC VMO

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ABSTRACT. The space BMO_p denotes the variant of BMO based on balls of constant size in the Bergman metric on a strongly pseudoconvex domain, with the mean oscillation measured in an L^p sense. It is closely connected to the study of Hankel operators on the Bergman space which are bounded in the L^p norm. This paper presents a correct proof that the dual of the Bergman metric VMO_p is the space H_q^1 consisting of l^1 -sums of Bergman metric q -atoms, $1/p + 1/q = 1$; and that the second dual is BMO_p . In the course of the proof, it is shown that the linear functional corresponding to a sum of atoms is independent of this decomposition into atoms, and an intrinsic formula for the duality pairing (independent of the decomposition) is derived.

1. Introduction. I am writing this paper mainly to correct an oversight of mine in the paper, “BMO on strongly pseudoconvex domains: Hankel operators, duality and $\bar{\partial}$ -estimates,” [3], by Huiping Li and me. In that paper, in Theorem 4.5, is the claim that the dual of the space VMO_p , $p > 1$, is the space called H_q^1 , where $1/p + 1/q = 1$, and that the dual of H_q^1 is BMO_p (definitions in Section 2). These statements are indeed true, but the proof presented there (due entirely to me) is at best incomplete.

The proof presented in [3] makes the claim that the dual of VMO_p is entirely representable as l^1 sums of q -atoms, citing “standard functional analysis arguments” without actually exhibiting them. I did in fact have in mind a standard technique, but unfortunately it was one that did not apply to that situation! I had incorrectly reversed the roles of a Banach space and its dual. There were some less serious errors of omission as well: all essentially the omission of a verification that some mapping was well-defined. The erroneous proof and these omissions will be corrected here in Section 3.

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In addition, the association of an element of H_q^1 to a linear functional on VMO_p has always been somewhat unsatisfying. It goes as follows: an element $h \in H_q^1$ can be written (nonuniquely) as an l^1 sum $\sum_g \lambda_g g$, where the g are “ q -atoms.” The linear functional associated to h is defined by $L_h(f) = \sum_g \bar{\lambda}_g \int f \bar{g} dA$. What is missing is a method of calculating $L_h(f)$ directly from the values of the function h . Such a method is presented in Section 4. The obvious attempts: $\int f \bar{h} dA$ and $\lim_{r \rightarrow 1} \int_{|z| < r} f \bar{h} dA$ are shown not to work.

It was when attempting to present the proof of Theorem 4.5 of [3] in a graduate seminar that I realized it was in error. At about the same time I had occasion to read the paper [1] by Bierstedt and Summers, where it was shown that certain spaces defined by “big oh” conditions were the second dual of corresponding spaces defined by “little oh” conditions. It turns out that a correct proof of the duality can be given which has the same broad outline as the technique in that paper, but with the added task of identifying more or less explicitly the intermediate (first dual) space.

Remark 1.1. I have chosen to identify the dual space of VMO_p via a conjugate linear isomorphism. Thus, the pairings will be in the form of integrals like $\int f \bar{h} dA$, or their sums or limits.

2. The definitions. It will keep things relatively simple if I present the proof in the unit disk \mathbf{D} in the complex plane \mathbf{C} . The more general context in [3] (strongly pseudoconvex domains in \mathbf{C}^n) would require only a change of notation and a change in the justifications for some steps.

The unit disk is endowed with a *hyperbolic metric*

$$\beta(z, w) = \frac{1}{2} \log \left(\frac{1 + \psi(z, w)}{1 - \psi(z, w)} \right)$$

where

$$\psi(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

is the *pseudohyperbolic metric*. The Bergman metric, used in [3] in the context of a strongly pseudoconvex domain, coincides, in the case of the unit disk, with the hyperbolic metric. For $z \in \mathbf{D}$ and $r > 0$, the

disks $D(z, r) = \{w \in \mathbf{D} \mid \beta(z, w) < r\}$ are called *hyperbolic disks*, being balls in the hyperbolic metric as well as Euclidean disks. Fix some positive radius R once and for all, and let $D(z) = D(z, R)$. Denote area measure on \mathbf{D} by dA and the area of $D(z)$ by $|D(z)|$.

The following definitions are required:

Definition 2.1. For a measurable function f on \mathbf{D} and $1 < p < \infty$, the p -mean oscillation of f at $z \in \mathbf{D}$ is denoted by $\text{MO}_p(f, z)$ and defined by

$$[\text{MO}_p(f, z)]^p = \inf_{c \in \mathbf{C}} \frac{1}{|D(z)|} \int_{D(z)} |f - c|^p dA.$$

We say f belongs to BMO_p if $\text{MO}_p(f, z)$ is a bounded function on \mathbf{D} . We say f belongs to VMO_p if $\text{MO}_p(f, z) \rightarrow 0$ as z tends to $\partial\mathbf{D}$.

The following makes it clear that a reasonable replacement for the best constant in the infimum that defines $\text{MO}_p(f, z)$ is the average $\tilde{f}(z)$ of f over $D(z)$. Let C be the constant achieving the infimum in the definition of $\text{MO}_p(f, z)$, then

$$\begin{aligned} |\tilde{f}(z) - C| &= \left| \frac{1}{|D(z)|} \int_{D(z)} f dA - C \right| \\ &\leq \frac{1}{|D(z)|} \int_{D(z)} |f - C| dA \\ &\leq \text{MO}_p(f, z) \end{aligned}$$

by Holder's inequality. Thus,

$$\text{MO}_p(f, z) \leq \left[\frac{1}{|D(z)|} \int_{D(z)} |f - \tilde{f}(z)|^p dA \right]^{1/p} \leq 2\text{MO}_p(f, z)$$

and replacing $\text{MO}_p(f, z)$ with the expression involving the average would produce an equivalent version of BMO_p .

The two spaces BMO_p and VMO_p are, modulo the constants, Banach spaces under the norm

$$\|f\|_{\text{BMO}, p} = \sup_{z \in \mathbf{D}} \text{MO}_p(f, z).$$

Definition 2.2. If $1 < q < \infty$, a locally integrable function g on the unit disk \mathbf{D} is said to be a q -atom if there is a disk $D(a)$ such that

- (i) g is supported in $D(a)$.
- (ii) The mean of g is zero: $\int g \, dA = 0$.
- (iii) g satisfies the following size condition:

$$\left[\frac{1}{|D(a)|} \int |g|^q \, dA \right]^{1/q} = \frac{1}{|D(a)|}.$$

Clearly, a q -atom in this sense is one in the “classical” sense, but an additional requirement here is that the supporting disk associated to g must have a fixed radius R in the hyperbolic metric.

Definition 2.3. For $1 < q < \infty$, the space H_q^1 consists of all measurable functions h that can be written in the form

$$h = \sum_{j=1}^{\infty} \lambda_j g_j$$

where each g_j is a q -atom and $\sum |\lambda_j| < \infty$. The space is normed by $\|h\|_{H_q^1} = \inf \sum |\lambda_j|$, with the infimum taken over all such representations of h .

By Holder’s inequality, an atom has L^1 -norm less than 1, so such a sum will always converge in L^1 . Moreover, if all the atoms in the sum are supported in a fixed compact subset of \mathbf{D} , then the sum converges in L^q .

If G is the collection of all q -atoms g , and if T is defined on an element $\lambda = (\lambda_g)$ of $l^1(G)$ by $T(\lambda) = \sum_{g \in G} \lambda_g g$, then T is continuous from $l^1(G)$ to L^1 . Its range is H_q^1 and the norm on H_q^1 makes the induced map from $l^1(G)/\ker T$ to H_q^1 an isometry. Thus H_q^1 is a Banach space. The first step of the duality proof will be to show that we can map H_q^1 into VMO_p^* . (This step was correct in [3] but will also be presented here for completeness.)

Let g be a q -atom and define a linear functional L_g on BMO_p by $L_g(f) = \int f \bar{g} \, dA$. Let D be the support disk associated to g in the

definition of q -atom, and let $f \in \text{BMO}_p$. Then, for any constant c ,

$$\begin{aligned} |L_g(f)| &= \left| \int (f - c)\bar{g} dA \right| \\ &\leq \left[\int |g|^q dA \right]^{1/q} \left[\int_D |f - c|^p dA \right]^{1/p} \\ &\leq |D|^{1/q-1} \left[\int_D |f - c|^p \right]^{1/p} \\ &\leq \left[\frac{1}{|D|} \int_D |f - c|^p \right]^{1/p}, \end{aligned}$$

and an infimum over c gives $|L_g(f)| \leq \|f\|_{\text{BMO}_p}$, so L_g is continuous and its norm is less than 1.

Thus, a sum like $\sum \lambda_g g$, where the g are q -atoms and $\sum |\lambda_g| < \infty$, can be used to define a continuous linear functional on VMO_p with norm at most $\sum |\lambda_g|$. Unfortunately, this merely associates a continuous linear functional to a *formal sum*, for it is not immediately obvious that the resulting linear functional is independent of the choice of representation. Thus, it is not clear that the functional $\sum \bar{\lambda}_g L_g$ must be the zero functional when the function $\sum \lambda_g g$ is zero almost everywhere. In [3] the proof that all linear functionals arise as such sums was incorrect, but also the proof that the $\sum \bar{\lambda}_g L_g$ associates a linear functional to the *function* $\sum \lambda_g g$ was omitted.

3. The proofs. We need the following key lemma. For $p = 2$, this appears in several places in the literature. Usually the proof involves the Bergman kernel, as in, for example, [5]. The proof here for all $1 < p < \infty$ will not involve the Bergman kernel.

Lemma 3.1. (a) *The $R > 0$ used in the definitions of BMO_p and VMO_p is arbitrary. That is, any choice of R will produce the same space with an equivalent norm.* (b) *If $f \in \text{BMO}_p$, then the averages \tilde{f} defined by*

$$\tilde{f}(z) = \frac{1}{|D(z)|} \int_{D(z)} f dA$$

satisfy

$$\begin{aligned} |\tilde{f}(z) - \tilde{f}(w)| &\leq \left[\frac{1}{|D(z)|} \int_{D(z)} |f(z) - \tilde{f}(w)|^p \right]^{1/p} \\ &\leq C(1 + \beta(z, w)) \|f\|_{\text{BMO}, p} \end{aligned}$$

with a constant C independent of f, z and w .

Proof. Let us temporarily include the dependence on R in notations like $\text{MO}_{p,R}$. Take any $r < R$, let $f \in \text{BMO}_{p,R}$, and let $a \in \mathbf{D}$. It will be convenient to establish the following estimates for any z such that $D(z, r) \subset D(a, R) = D(a)$. Let c denote any constant. Then

$$\begin{aligned} \text{MO}_{p,r}(f, z)^p &\leq \frac{1}{|D(z, r)|} \int_{D(z, r)} |f - c|^p dA \\ &\leq \frac{|D(a)|}{|D(z, r)|} \frac{1}{|D(a)|} \int_{D(a)} |f - c|^p dA. \end{aligned}$$

Let $C_{r,R} = \sup |D(a)|/|D(z, r)|$, with the supremum taken over all pairs a, z with $a \in \mathbf{D}$ and $D(z, r) \subset D(a)$. Then we get the following, after taking an infimum over all c :

$$(3.1) \quad \text{MO}_{p,r}(f, z) \leq C_{r,R}^{1/p} \text{MO}_{p,R}(f, a).$$

In particular, setting $z = a$ we see that the $\text{BMO}_{p,R}$ norm dominates the $\text{BMO}_{p,r}$ norm.

Before obtaining the reverse, we will obtain part (b). Fix any convenient value $r < R$, say $r = R/2$, and let $\tilde{f}_r(z)$ denote the average of f over $D(z, r)$. Again, let $D(z, r) \subset D(a)$. Then, by a similar argument,

$$\begin{aligned} |\tilde{f}_r(z) - \tilde{f}_r(a)| &= \left| \frac{1}{|D(z, r)|} \int_{D(z, r)} f - \tilde{f}_r(a) dA \right| \\ (3.2) \quad &\leq C \left[\frac{1}{|D(a)|} \int_{D(a)} |f - \tilde{f}_r(a)|^p dA \right]^{1/p} \\ &\leq C \text{MO}_{p,R}(f, a), \end{aligned}$$

by Holder’s inequality, the estimate (3.1) and the observation following the definition of $BMO_{p,R}$. Combining this estimate with the same estimate for some other point w with $D(w, r) \subset D(a)$, we get the following:

$$(3.3) \quad |\tilde{f}_r(z) - \tilde{f}_r(w)| \leq C \text{MO}_{p,R}(f, a) \leq C \|f\|_{BMO_{p,R}}.$$

For any pair $z, w \in \mathbf{D}$ there are finite sequences a_1, a_2, \dots, a_M and $z_0 = z, z_1, z_2, \dots, z_M = w$ with $M \leq C(1 + \beta(z, w))$ such that $D(z_{j-1}, r) \cup D(z_j, r) \subset D(a_j)$. Adding up M inequalities like (3.3) and two inequalities like (3.2) gives part (b).

Now suppose f belongs to $BMO_{p,r}$; then (b) is satisfied for the averages $\tilde{f}_r(z)$. Let $D(a, R)$ be given; there are K points $z_j \in D(a, R)$, K depending only on R and r , such that the disks $D(z_j, r)$ cover $D(a, R)$. By (b) and the fact that $\beta(z_j, z_k) < 2R$, we have $|\tilde{f}_r(z_j) - \tilde{f}_r(z_k)| \leq C \|f\|_{BMO_{p,r}}$. Let c be any constant with $|c - \tilde{f}_r(z_j)| \leq C \|f\|_{BMO_{p,r}}$ for all j . Then

$$\begin{aligned} \frac{1}{|D(a)|} \int_{D(a)} |f - c|^p dA &\leq \frac{1}{|D(a)|} \sum_{j=1}^K \int_{D(z_j, r)} |f - c|^p dA \\ &\leq C \sum_{j=1}^K \frac{1}{|D(z_j, r)|} \int_{D(z_j, r)} |f - c|^p dA \\ &\leq C \|f\|_{BMO_{p,r}}^p \\ &\quad + C \sum_{j=1}^K \frac{1}{|D(z_j, r)|} \int_{D(z_j, r)} |f - \tilde{f}_r(z_j)|^p dA \\ &\leq C \|f\|_{BMO_{p,r}}^p. \end{aligned}$$

This shows that the $BMO_{p,r}$ norm dominates the $BMO_{p,R}$ norm. □

If Lemma 3.1 (b) is applied with $w = 0$, we get the following corollary.

Corollary 3.2. *There is a constant C such that, for any function $f \in BMO_p$ we have*

$$\left[\frac{1}{|D(z)|} \int_{D(z)} |f(z) - \tilde{f}(0)|^p dA \right]^{1/p} \leq C \left[1 + \log \left(\frac{1}{1 - |z|} \right) \right] \|f\|_{BMO_p}.$$

Since functions in BMO_p are really only defined up to a constant, we can say that any element of BMO_p may be identified with a function such that the means $[\int_{D(z)} |f(z)|^p dA/|D(z)|]^{1/p}$ grow at worst like $\log[1/(1-|z|)]$.

Theorem 3.3. *Let $1 < p < \infty$, and let q be the conjugate exponent, satisfying $1/p + 1/q = 1$. Let G be the collection of q -atoms, L_g the corresponding functionals on VMO_p and (λ_g) an element of $l^1(G)$. If $\sum_{g \in G} \lambda_g g = 0$ almost everywhere, then the linear functional $\sum \bar{\lambda}_g L_g$ on VMO_p is zero. Furthermore, all bounded linear functionals on VMO_p arise in this manner from an l^1 sum of q -atoms.*

Remark 3.4. The proof will show that $\sum \bar{\lambda}_g L_g = 0$ even when considered as a linear functional on BMO_p .

Proof. Assume that $\sum \lambda_g g = 0$ almost everywhere. We need to show that if f is any function in BMO_p then the sum $\sum \bar{\lambda}_g L_g(f)$ is zero. Since this sum converges absolutely, it suffices to show that some rearrangement of the terms gives a sum of zero. Let $D(a_g)$ denote the support disk for g and let $\{t_n\}$ denote the increasing sequence of nonnegative numbers defined by $t_0 = 0$ and $\beta(t_n, t_{n+1}) = R$, $n \geq 0$. If we let $\rho = \tanh R$, then $t_{n+1} = (t_n + \rho)/(1 + t_n \rho)$. From this, it follows easily that $1 - t_{n+1} = (1 - \rho)(1 - t_n)/(1 + t_n \rho)$, whence $2^{-n}(1 - \rho)^n < 1 - t_n < (1 - \rho)^n$.

For integers $n \geq 0$, let D_n denote the disk $|z| < t_n$, so $D_0 = \emptyset$, and A_n the annulus $D_n \setminus D_{n-1}$, and let

$$S_n = \sum_{a_g \in D_n} \bar{\lambda}_g L_g(f).$$

The proof will be complete if we show that some subsequence S_{n_j} converges to zero, because this corresponds to some rearrangement of the original sum. I claim that for $f \in BMO_p$,

$$(3.4) \quad S_n = \int f \left(\sum_{a_g \in D_n} \bar{\lambda}_g \bar{g} \right) dA.$$

Indeed, it suffices to notice that all the g in the sum have their support within D_{n+1} , so the sum converges in L^q , while $f|_{D_{n+1}}$ is in L^p .

Thus, the righthand side is unchanged if we multiply through by f and integrate term by term, which gives S_n .

Note that, for any a_g not in D_n , the hyperbolic disk $D(a_g)$ is disjoint from D_{n-1} so, for almost every $z \in D_{n-1}$, $0 = \sum \lambda_g g(z) = \sum_{a_g \in D_n} \lambda_g g(z)$. Moreover, if $a_g \in D_n$, then the support of g lies in D_{n+1} . Thus, the sum $\sum_{a_g \in D_n} \lambda_g g(z)$ appearing in (3.4) is actually supported in $A_n \cup A_{n+1}$. Thus we obtain

$$S_n = \int_{A_n \cup A_{n+1}} f \left(\sum_{a_g \in D_n} \bar{\lambda}_g \bar{g} \right) dA = \sum_{a_g \in D_n} \bar{\lambda}_g \int_{A_n \cup A_{n+1}} f \bar{g} dA.$$

Also, if $a_g \in D_{n-2}$, then $g = 0$ on $A_n \cup A_{n+1}$, so the sum can be taken over only those a_g in $A_{n-1} \cup A_n$. Therefore,

$$\begin{aligned} |S_n| &\leq \sum_{a_g \in A_{n-1} \cup A_n} |\lambda_g| \left[\int_{D(a_g)} |f|^p dA \right]^{1/p} \left[\int_{D(a_g)} |g|^q dA \right]^{1/q} \\ &\leq \sum_{a_g \in A_{n-1} \cup A_n} |\lambda_g| \left[\frac{1}{|D(a_g)|} \int_{D(a_g)} |f|^p dA \right]^{1/p} \\ &\quad \cdot \left[|D(a_g)|^{q-1} \int_{D(a_g)} |g|^q dA \right]^{1/q} \\ &\leq C \|f\|_{\text{BMO},p} \sum_{a_g \in A_{n-1} \cup A_n} |\lambda_g| \log \left(\frac{1}{1 - |a_g|} \right). \end{aligned}$$

For $a_g \in A_{n-1} \cup A_n$, we have $\log[1/(1 - |a_g|)] \leq \log[1/(1 - t_n)] \leq n \log[2/(1 - \rho)]$. This gives

$$\begin{aligned} |S_n| &\leq Cn \|f\|_{\text{BMO},p} \sum_{a_g \in A_{n-1} \cup A_n} |\lambda_g| \\ &\leq Cn \mu_n, \end{aligned}$$

where $\mu_n = \sum_{a_g \in A_{n-1} \cup A_n} |\lambda_g|$. Since $\sum_n \mu_n = 2 \sum_g |\lambda_g|$, the sequence μ_n is summable. It follows therefore that some subsequence $n_j \mu_{n_j}$ tends to zero; otherwise we would have $\mu_n > \varepsilon/n$ for all but finitely many n .

We have thus far shown that $\sum \lambda_g g \mapsto \sum \bar{\lambda}_g L_g$ is a well-defined linear map from H_q^1 to VMO_p^* . It is relatively clear that the map is one-to-one. For, if $f \in H_q^1$ and the corresponding functional L_f is zero, then $\int h \bar{f} dA = 0$ for every continuous function h with compact support in \mathbf{D} . Thus, $f = 0$ almost everywhere.

It remains to show that every continuous linear functional on VMO_p has the form $\sum_{g \in \mathcal{G}} \bar{\lambda}_g L_g$. To this end, we will embed VMO_p as a subspace of a space with a known dual, therefore identifying VMO_p^* with a quotient space of that dual.

Let $\{a_j\}$ be a sequence of points in \mathbf{D} such that the disks $D(a_j, R/2)$ cover \mathbf{D} , and such that $\beta(a_j, a_k) \geq R/2$ whenever $j \neq k$. Every function $f \in \text{VMO}_p$ defines a sequence of functions $f_j = f|_{D(a_j)}$. These functions belong to $L^p(D(a_j), dA/|D(a_j)|)$. Necessary and sufficient conditions on a sequence $\{f_j\}$ that it arise in this manner, by restriction of a function from VMO_p , are that

1) the distance from f_j to the constants in the norm of $L^p(D(a_j), dA/|D(a_j)|)$ tends to zero as $j \rightarrow \infty$ (as a_j tends to $\partial\mathbf{D}$) and that

2) $f_j = f_k$ on the intersection $D(a_j) \cap D(a_k)$.

It is clear that this second condition allows one to construct a function f whose restriction to each $D(a_j)$ is f_j , and that the first implies the sequence $\text{MO}_p(f, a_j)$ tends to zero. In fact, the function f must be in VMO_p . For, if z is any point of \mathbf{D} , there exists some $D(a_j, R/2)$ containing it, whence $D(z, R/2) \subset D(a_j, R)$. By the proof of Lemma 3.1, this implies that $\text{MO}_{p,R/2}(f, z) \leq C \text{MO}_{p,R}(f, a_j)$. If a sequence $\{z_n\}$ tends to $\partial\mathbf{D}$, so do the corresponding a_{j_n} ; thus $\text{MO}_{p,R/2}(f, z_n) \rightarrow 0$, and so $f \in \text{VMO}_p$.

For each j , let X_j denote the Banach space $L^p(D(a_j), dA/|D(a_j)|)/\mathbb{C}$ with the quotient norm. Then the above restriction mapping embeds VMO_p (modulo constants) into the c_0 -direct sum $X = \oplus_0 X_j = \{(f_j)_j \mid \|f_j\|_{X_j} \rightarrow 0 \text{ as } j \rightarrow \infty\}$ with norm $\|(f_j)_j\|_X = \sup_j \|f_j\|_{X_j}$. This mapping is obviously bounded, and the argument above shows that it is bounded below.

The dual of X is easily determined to be the l^1 sum $\oplus_1 X_j^*$, and each X_j^* can be identified with the subset of $L^q(D(a_j), |D(a_j)|^{q-1} dA)$ of functions with mean zero. The unusual multiple of dA in the measure on this space is so that we may use the duality pairing

$$\langle (f_j) \cdot (g_j) \rangle = \sum \int_{D(a_j)} f_j \bar{g}_j dA \text{ for } f_j \in X_j \text{ and } g_j \in X_j^*.$$

Now, given a continuous linear functional L on VMO_p , we may extend it to a linear functional on $\oplus_0 X_j$ and therefore identify it with a sequence in $\oplus_1 X_j^*$ which may be written $(\lambda_j g_j)$ where the λ_j satisfy $\sum |\lambda_j| < \infty$ and the g_j are elements of X_j^* with unit norm. This identification satisfies, for every $f \in \text{VMO}_p$, $L(f) = \sum_j \bar{\lambda}_j \int_{D(a_j)} f_j \bar{g}_j$, where f_j is the restriction of f to $D(a_j)$. The condition that g_j have unit norm in X_j^* is equivalent to g_j being a q -atom. Thus, the continuous linear functional L on VMO_p gives rise to an element $h = \sum \lambda_j g_j$ in H_q^1 . Its identification with a linear functional L_h on VMO_p is the one defined by $L_h(f) = \sum \bar{\lambda}_j \int_{D(a_j)} f \bar{g}_j dA$, and, therefore, $L_h = L$. This completes the identification of VMO_p^* with H_q^1 . \square

Remark 3.5. Since it has already been shown that $|\int f \bar{g} dA| \leq \|f\|_{\text{BMO}_p}$ for any q -atom g , it follows that any function f in BMO_p defines a continuous linear functional Φ_f on H_q^1 via $\Phi_f(h) = \Phi_f(\sum \lambda_g g) = \sum \lambda_g \int g \bar{f} dA$. (This also equals $\sum \lambda_g \overline{L_g(f)}$, which has already been shown to be independent of the representation of $h = \sum \lambda_g g$.) The remainder of the argument in [3] is correct and now shows that any continuous linear functional on H_q^1 arises in this manner from some function in BMO_p .

Remark 3.6. The continuous functions on $\mathbf{D} \cup \partial \mathbf{D}$ are dense in VMO_p . A proof that $h \mapsto L_h$ is well defined could have been based on this fact. This would not, however, have been enough to prove that it is well defined as a map from H_q^1 to linear functionals on BMO_p . We needed this stronger result for the previous remark. Moreover, the main result of the next section might not have been discovered using such an approach.

Remark 3.7. In [4, 2] and [3] it was shown that a function f in L^p , $p > 1$, defines a bounded Hankel operator H_f on the Bergman space A^p if and only if there is an analytic function $\phi \in A_p$ such that $f - \phi \in \text{BMO}_p$. If $f \in \text{BMO}_p$, h_1 is a polynomial and $h_2 \in (A^p)^\perp \subset L^q$,

then we have the estimate

$$\begin{aligned} \left| \int_{\mathbf{D}} f h_1 \bar{h}_2 dA \right| &\leq \|H_f\| \cdot \|h_1\|_{A^p} \|h_2\|_{L^q} \\ &\leq C \|f\|_{\text{BMO},p} \|h_1\|_{A^p} \|h_2\|_{L^q}. \end{aligned}$$

This implies that $\bar{h}_1 h_2$ defines a continuous linear functional on VMO_p , a fact that extends to any $h_1 \in A^p$ by taking limits of polynomials (though it need not always be given by the above integral unless f is bounded). That linear functional must be $L_h(f)$ for some $h \in H_q^1$. If we apply this to f varying over the set of compactly supported continuous functions (where the integrals do give the correct values), we deduce that h and $\bar{h}_1 h_2$ differ by a constant. Since both have mean zero, they are equal. Thus $\bar{h}_1 h_2$ belongs to H_q^1 . A similar argument shows that any function in A_0^1 belongs to H_q^1 . In fact, the argument in [3] can now correctly be invoked to show that H_q^1 is precisely the collection of all sums

$$h_0 + \sum_{j=1}^{\infty} \bar{h}_j k_j$$

with $h_0 \in A_0^1$, $h_j \in A^p$, $k_j \in (A^p)^\perp$ and $\sum \|h_j\|_{L^p} \|k_j\|_{L^q} < \infty$.

Remark 3.8. In [3], the following intrinsic characterization of H_q^1 was obtained: A function h belongs to H_q^1 if and only if h , Ph and Th all belong to L_q^1 , where P is the Bergman projection, having kernel $(1-\bar{w}z)^{-2}$, and T is the integral operator with kernel $[(1-\bar{w}z)(w-z)]^{-1}$. The space L_q^1 is the subset of $g \in L^1$ such that $[\int_{D(z)} |g|^q dA/|D(z)|]^{1/q}$ belongs to L^1 . Since this result relied on the duality between VMO_p and H_q^1 , its proof is only now complete.

Remark 3.9. If there is given a sequence a_j in \mathbf{D} , and numbers $r < R$ such that the disks $D(a_j, r)$ cover \mathbf{D} , then the proof of Theorem 3.3 could be slightly generalized to show that elements of H_q^1 can be represented by sums of atoms, each having its support in some disk of the *specified* sequence $D(a_j, R)$.

4. An intrinsic duality pairing. If $h \in H_q^1$, $h = \sum \lambda_g g$, then the corresponding linear functional L_h cannot necessarily be represented in

the form $L_h(f) = \int_{\mathbf{D}} f \bar{h} dA$ for all $f \in \text{VMO}_p$, since the product $|fh|$ need not be integrable. Certainly the integral on any proper subdisk of \mathbf{D} is defined, but we will show that the limit

$$\lim_{r \rightarrow 1^-} \int_{|z| < r} f \bar{h} dA$$

need not exist, although we will see that these integrals converge to the “correct” value for some sequence $r_n \rightarrow 0$. In the sum $h = \sum_{g \in G} \lambda_g g$, let G' denote the collection of q -atoms whose supporting disk lies in $|z| < r$, and let G'' denote those with supporting disk that meets $|z| < r$ but is not contained in it. Then

$$(4.1) \quad \int_{|z| < r} f \bar{h} dA = \sum_{g \in G'} \bar{\lambda}_g L_g(f) + \sum_{g \in G''} \bar{\lambda}_g \int_{r' < |z| < r} f \bar{g}$$

where $0 < r' < r$ satisfies $\beta(r', r) = 2R$. The first sum tends, as $r \rightarrow 1^-$, to the appropriate value, but it is relatively simple to construct examples where the second sum does not tend to zero. For example, choose a sequence t_n as in Section 3 and choose atoms with their centers on $|z| = t_n$, having a single positive value on the portion of their supports lying in $|z| < t_n$ and a negative value on $|z| > t_n$. Taking $r = t_n$ in (4.1) and $f = \log[1/(1 - |z|)]$ in BMO_p , the second sum is on the order of

$$Cn \sum_{|a_g| = t_n} |\lambda_g|$$

and it is trivial to construct an l^1 sequence λ_g so that this does not tend to zero. (An example with $f \in \text{VMO}_p$ is then easily obtained on multiplying $\log[1/(1 - |z|)]$ by a suitable function tending slowly to zero at the boundary.) This situation is analogous to the situation described in the previous section where $n\mu_n$ does not tend to zero. The argument there shows that some subsequence (corresponding to some $r_j = t_{n_j}$ tending to 1) can be chosen that tends to zero.

In order to get an intrinsic formula for the functional L_h , independent of the representation of h as a sum of q -atoms, it might be expected that some sort of summability method would be needed. This turns out to be the case, and we have the following theorem.

Theorem 4.1. *Let ω_n be a sequence of nonnegative measurable functions on \mathbf{D} satisfying:*

1. The functions $\omega_n(z) \log[1/(1-|z|^2)]$ are bounded (but not uniformly bounded because of the following condition).
2. $\omega_n(z)$ increases to 1 uniformly on compact sets in \mathbf{D} .
3. There is a constant C (independent of n and z) such that

$$\sup_{\zeta \in D(z)} |\omega_n(\zeta) - \omega_n(z)| \log \left(\frac{1}{1-|z|^2} \right) \leq C.$$

Then if $h = \sum \lambda_g g \in H_q^1$ and $f \in \text{BMO}_p$, $1/p + 1/q = 1$, the corresponding linear functional $L_h = \sum \bar{\lambda}_g L_g$ satisfies

$$L_h(f) = \lim_{n \rightarrow \infty} \int_{\mathbf{D}} \omega_n f \bar{h} dA.$$

In particular,

$$(4.2) \quad L_h(f) = \lim_{r \rightarrow 1^-} \int_{|z| < r} \left(1 - \frac{\log(1/(1-|z|^2))}{\log(1/(1-r^2))} \right) f \bar{h} dA$$

and

$$(4.3) \quad L_h(f) = \lim_{\alpha \rightarrow 0} \int_{\mathbf{D}} (1-|z|^2)^\alpha f \bar{h} dA.$$

Proof. The two examples correspond respectively to

$$\omega_n(z) = \left(1 - \frac{\log(1/(1-|z|^2))}{\log(1/(1-r_n^2))} \right) \chi_{r_n \mathbf{D}}(z)$$

for sequences r_n increasing to 1 and

$$\omega_n(z) = (1-|z|^2)^{\alpha_n}$$

for sequences α_n decreasing to zero. The first two conditions are trivial for both, while the third is a routine estimate. The formula (4.2) was discovered by applying a Cesàro mean to equation (4.1) with $r = t_n$ (t_n as in Theorem 3.3). The simpler formula (4.3) came from applying a Poisson mean.

Because of condition (1) and the growth estimate on functions f in BMO_p , Corollary 3.2, we can calculate

$$\int_{\mathbf{D}} \omega_n f \bar{h} dA = \int_{\mathbf{D}} \omega_n f \sum_g \bar{\lambda}_g \bar{g} dA,$$

integrating term by term to get

$$\begin{aligned} \int_{\mathbf{D}} \omega_n f \bar{h} dA &= \sum_g \bar{\lambda}_g \int_{D(a_g)} \omega_n f \bar{g} dA \\ &= \sum_g \bar{\lambda}_g \omega_n(a_g) \int_{D(a_g)} f \bar{g} dA \\ &\quad + \sum_g \bar{\lambda}_g \int_{D(a_g)} (\omega_n - \omega_n(a_g)) f \bar{g} dA \\ &= \sum_g \bar{\lambda}_g \omega_n(a_g) L_g(f) + \sum_g \bar{\lambda}_g \int_{D(a_g)} (\omega_n - \omega_n(a_g)) f \bar{g} dA. \end{aligned}$$

As $n \rightarrow \infty$, the first sum converges to $L_h(f)$ by dominated convergence (the terms of the sum have absolute values less than $|\lambda_g| \cdot \|f\|_{BMO,p}$), so we need only show that the second sum tends to zero as $n \rightarrow \infty$. Using condition (3), Holder's inequality, and then Corollary 3.2, the integrals in the last sum are bounded functions of g . Also, for fixed g , they tend to zero as $n \rightarrow \infty$ by condition (2). Thus, the second sum above tends to zero, by dominated convergence. \square

The intrinsic representation of the duality pairing from Theorem 4.1 allows one to write the pairing between the result of a Hankel operator $H_f h \in L^p$ and $k \in (A^p)^\perp$, see Remark 3.7, in an explicit form, even if f is unbounded:

$$\langle H_f h, k \rangle = \lim_{\alpha \rightarrow 0} \int_{\mathbf{D}} f(z) h(z) \overline{k(z)} (1 - |z|^2)^\alpha dA(z).$$

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