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OSCILLATION AND NONOSCILLATION THEOREMS FOR FOURTH ORDER DIFFERENCE EQUATIONS

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ABSTRACT. This paper is concerned with a class of fourth order nonlinear difference equations. Two types of nonoscillatory solutions will be considered. Relations between these types of solutions and their oscillatory behavior are the main purpose of this paper.

1. Introduction. In several papers the oscillatory and asymptotic behavior of solutions of third order difference equations have been discussed. For example, note the papers [**12**]–[**15**] and [**17**]. When compared to differential equations the study of difference equations has received little attention for orders greater than three. Fourth order linear difference equations are considered in [**4**], [**6**], [**16**], [**18**] and [**19**]. Fourth order nonlinear difference equations are studied in [**10**], [**11**] and [**20**].

In this paper we will study fourth order difference equations

$$\Delta^4 y_n = f(n, y_n), \quad n \in N = \{0, 1, 2, \dots\} \tag{E}$$

where Δ is the forward difference operator $\Delta y_n = y_{n+1} - y_n$ and $\Delta^k y_n = \Delta(\Delta^{k-1} y_n)$ for $k = 2, 3, \dots$. The sequence $y = \{y_n\}$ is the trivial sequence if there exists $n_0 \in N$ such that $y_n = 0$ for all $n \geq n_0$. By a solution of (E), we mean any nontrivial sequence $\{y_n\}$ satisfying equation (E), for all $n \in N$. In general, we will assume that the usual existence and uniqueness theorem for solutions of equation (E) holds. A solution is oscillatory if, for every $m \in N$, there exists $n \geq m$ such that $y_n y_{n+1} \leq 0$. Therefore a nonoscillatory solution is eventually positive or eventually negative. We assume that void sum is equal to zero. In the paper we assume that this function $f : N \times R \to R$ satisfies condition

$$x f(n, x) < 0 \quad \text{for } n \in N,\ x \in R \setminus \{0\}. \tag{$*$}$$

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We define operator F as

$$F(x_n) = x_{n-1}\Delta^3 x_n - \Delta x_{n-1}\Delta^2 x_n, \quad n \in N.$$

We use the F operator to classify solutions of equation (E).

Definition. If $F(y_n) \geq 0$ for all $n \in N$, then a solution y of equation (E) is called F_+-solution. If $F(y_n) < 0$ for some n, then a solution y of equation (E) is called F_--solution.

Operator F divides the set of solutions into two disjoint subsets F_+ and F_--solutions.

The following properties, see [**10**], of nonoscillatory solutions of equation (E) are known. Every nonoscillatory solution $\{y_n\}$ of equation (E) is one of four types:

$$\begin{array}{llllll}
(A4+) & y_n > 0, & \Delta y_n > 0, & \Delta^2 y_n > 0, & \Delta^3 y_n > 0, & \Delta^4 y_n < 0, \\
(A4-) & y_n < 0, & \Delta y_n < 0, & \Delta^2 y_n < 0, & \Delta^3 y_n < 0, & \Delta^4 y_n > 0, \\
(A2+) & y_n > 0, & \Delta y_n > 0, & \Delta^2 y_n < 0, & \Delta^3 y_n > 0, & \Delta^4 y_n < 0, \\
(A2-) & y_n < 0, & \Delta y_n < 0, & \Delta^2 y_n > 0, & \Delta^3 y_n < 0, & \Delta^4 y_n > 0,
\end{array}$$

eventually.

In the paper we consider the relationship between F_+, F_--solutions and (A4+), (A4-), (A2+), (A2-) types.

Lemma 1.

$$\Delta F(y_n) = y_n \Delta^4 y_n - \Delta^2 y_{n-1}\Delta^2 y_{n+1}. \tag{1}$$

The proof is evident.

Lemma 2. *If $\{y_n\}$ is a nonoscillatory F_+-solution of equation* (E), *then*

$$\sum_{n=1}^{\infty}(\Delta^2 y_{n-1})(\Delta^2 y_{n+1}) < \infty, \tag{2}$$

and

$$\lim_{n\to\infty}(\Delta^2 y_{n-1})(\Delta^2 y_{n+1}) = 0. \tag{3}$$

Proof. Let $\{y_n\}$ be an F_+-solution of equation (E). From (1), we get

$$\Delta F(y_k) = y_k f(k, y_k) - \Delta^2 y_{k-1}\Delta^2 y_{k+1}.$$

Now by summation, we obtain

$$F(y_n) = F(y_1) + \sum_{k=1}^{n-1} y_k f(k, y_k) - \sum_{k=1}^{n-1} \Delta^2 y_{k-1}\Delta^2 y_{k+1}.$$

Since $F(y_n) \geq 0$, then

$$0 \leq F(y_1) + \sum_{k=1}^{n-1} y_k f(k, y_k) - \sum_{k=1}^{n-1} \Delta^2 y_{k-1}\Delta^2 y_{k+1},$$

and, from $(*)$,

$$F(y_1) \geq \sum_{k=1}^{n-1} \Delta^2 y_{k-1}\Delta^2 y_{k+1}.$$

Hence,

$$\sum_{k=1}^{\infty} \Delta^2 y_{k-1}\Delta^2 y_{k+1} < \infty.$$

Condition (3) follows directly from (2).

Main results.

Theorem 1. *If $\{y_n\}$ is a nonoscillatory solution of equation* (E), *then $F(y_n)$ is an eventually decreasing function on N.*

Proof. Let $\{y_n\}$ be an eventually positive solution of equation (E). (For an eventually negative solution, the proof is similar.) Then $\{y_n\}$ is an (A4+)-solution or an (A2+)-solution, implies $\Delta^2 y_{n-1}\Delta^2 y_{n+1} > 0$. From (1) and $(*)$, we have

$$\begin{aligned}\Delta F(y_n) &= y_n\Delta^4 y_n - \Delta^2 y_{n-1}\Delta^2 y_{n+1}\\ &= y_n f(n, y_n) - \Delta^2 y_{n-1}\Delta^2 y_{n+1} < 0.\end{aligned}$$

Hence the proof is complete. □

Remark 1. Assume $\{y_n\}$ is a positive or negative solution of equation (E); then $F(y_n)$ is a decreasing function on N.

Theorem 2. *If* $\{y_n\}$ *is a nonoscillatory* F_+*-solution of equation* (E), *then*

$$\sum_{n=1}^{\infty}(\Delta^{2+j} y_n)^2 < \infty, \tag{4}$$

and

$$\lim_{n\to\infty} \Delta^{2+j} y_n = 0 \quad \textit{for } j = 0, 1, 2, \dots . \tag{5}$$

Proof. We prove (4) for $j = 0$ first. Let $\{y_n\}$ is an eventually positive solution of equation (E). (For an eventually negative solution, the proof is similar.) Then $\{y_n\}$ is an (A4+)-solution or an (A2+)-solution. Let $\{y_n\}$ be an (A4+)-solution. Then $\Delta^2 y_{n+1} > \Delta^2 y_{n-1} > 0$. Hence $\Delta^2 y_{n-1}\Delta^2 y_{n+1} > (\Delta^2 y_{n-1})^2$, and

$$\sum_{n=1}^{\infty}\Delta^2 y_{n-1}\Delta^2 y_{n+1} > \sum_{n=1}^{\infty}(\Delta^2 y_{n-1})^2.$$

From Lemma 2,

$$\sum_{n=1}^{\infty}\Delta^2 y_{n-1}\Delta^2 y_{n+1} < \infty.$$

Then we get

$$\sum_{n=1}^{\infty}(\Delta^2 y_n)^2 < \infty.$$

If $\{y_n\}$ is an (A2+)-solution, the proof is analogous.

We will prove (4) by induction.

Let

$$\sum_{n=1}^{\infty}(\Delta^{2+j}y_n)^2 < \infty.$$

We will prove that $\sum_{n=1}^{\infty}(\Delta^{2+j+1}y_n)^2 < \infty$. Using inequality $-2ab \leq a^2+b^2$, we have

$$\begin{aligned}(\Delta^{2+j+1}y_n)^2 &= (\Delta^{2+j}y_{n+1})^2 - 2\Delta^{2+j}y_{n+1}\Delta^{2+j}y_n + (\Delta^{2+j}y_n)^2 \\ &\leq 2(\Delta^{2+j}y_{n+1})^2 + 2(\Delta^{2+j}y_n)^2.\end{aligned}$$

So

$$\begin{aligned}\sum_{n=1}^{\infty}(\Delta^{2+j+1}y_n)^2 &\leq 2\sum_{n=1}^{\infty}(\Delta^{2+j}y_{n+1})^2 + 2\sum_{n=1}^{\infty}(\Delta^{2+j}y_n)^2 \\ &= 2\sum_{n=2}^{\infty}(\Delta^{2+j}y_n)^2 + 2\sum_{n=1}^{\infty}(\Delta^{2+j}y_n)^2 < \infty.\end{aligned}$$

Hence (4) holds for $j = 0, 1, 2, \dots$.

Condition (5) follows directly from (4).

Theorem 3. *If there exists $\varepsilon > 0$ such that*

$$|f(n,x)| \geq \varepsilon \quad \textit{for } (n,x) \in N \times \{R \setminus \{0\}\}, \tag{6}$$

then equation (E) *does not have a nonoscillatory F_+-solution.*

Proof. Suppose that there exists a nonoscillatory F_+-solution $\{y_n\}$ of equation (E). By Theorem 2, we have

$$\lim_{n\to\infty}\Delta^4 y_n = 0.$$

Then, from equation (E), $\lim_{n\to\infty} f(n,y_n) = 0$, so $\lim_{n\to\infty}|f(n,y_n)| = 0$. On the other hand, by (6), if there exists $\lim_{n\to\infty}|f(n,y_n)|$, then $\lim_{n\to\infty}|f(n,y_n)| \geq \varepsilon$. The proof is completed by contradiction.

Solution $\{y_n\}$ of equation (E) is called an F-solution if $\{y_n\}$ is an F_+-solution or an F_--solution. Solution $\{y_n\}$ of equation (E) is called an (A2)-solution if $\{y_n\}$ is an (A2+)-solution or (A2)-solution. Solution $\{y_n\}$ of equation (E) is called an (A4)-solution if $\{y_n\}$ is an (A4+)-solution or an (A4-)-solution. Solution $\{y_n\}$ of equation (E) is called an A-solution if $\{y_n\}$ is an (A2)-solution or (A4)-solution.

Theorem 4. *Each nonoscillatory F_+-solution of equation* (E) *is an* (A2)*-solution. Each* (A2)*-solution is an F_+-solution.*

Proof. Let $\{y_n\}$ be any eventually positive solution. (The proof for eventually negative solution is similar.)

Assume that $\{y_n\}$ is an F_+-solution. Suppose to the contrary that $\{y_n\}$ is an (A4+)-solution. Then $\Delta^3 y_n > 0$ and $\Delta^2 y_n > 0$ so $\{\Delta^2 y_n\}$ is an eventually increasing positive sequence. This contradicts condition (5). Then $\{y_n\}$ is an (A2+)-solution.

Assume that $\{y_n\}$ is an (A2+)-solution. Then we have: $y_n > 0$, $\Delta^3 y_n > 0$, $\Delta y_n > 0$ and $\Delta^2 y_n < 0$, so $F(y_n) > 0$ eventually.

Theorem 5. *Let $\{y_n\}$ be a positive solution of equation* (E). *If*

$$\sum_{n=1}^{\infty} (\Delta^2 y_n)^2 < \infty, \tag{7}$$

then $\{y_n\}$ is an F_+-solution.

Proof. Let $\{y_n\}$ be a positive solution of equation (E), for which condition (4) holds. Suppose that $\{y_n\}$ is an F_--solution. Then, for some $m \in N$, we have $F(y_m) < 0$. So, by Remark 1, $F(y_n) < F(y_m) < 0$ for $n \geq m$. Since $\{y_n\}$ is a positive solution of equation (E) then $\{y_n\}$ is an (A4+)-solution or (A2+)-solution. We exclude both of the cases. For the (A4+)-solution we have $\sum_{n=1}^{\infty} \Delta^2 y_n = \infty$, so $\sum_{n=1}^{\infty} (\Delta^2 y_n)^2 = \infty$. By Theorem 4, the (A2+)-solution is an F_+-solution. Thus $\{y_n\}$ is an F_+-solution.

Remark 2. Let $\{y_n\}$ be a negative solution of equation (E). If condition (7) holds, then $\{y_n\}$ is an F_+-solution.

Remark 3. Let $\{y_n\}$ be a negative or positive solution of equation (E). Then $\{y_n\}$ is an F_+-solution if and only if

$$\sum_{n=1}^{\infty}(\Delta^2 y_n)^2 < \infty.$$

Theorem 6. *If $\{y_n\}$ is a nonoscillatory F_--solution of equation* (E)*, then $\{y_n\}$ has an unbounded first difference.*

Proof. By Theorem 4, the nonoscillatory F_--solution has to be an (A4)-solution. Then, for an eventually positive solution, we have

$$\Delta^2 y_n > \Delta^2 y_m > 0 \quad \text{for } n > m,$$

so by summation we get

$$\Delta y_n \geq \Delta y_m + (n-m)\Delta^2 y_m, \quad n \geq m.$$

Hence $\lim_{n\to\infty} \Delta y_n = \infty$.

Remark 4. Every nonoscillatory F_--solution of equation (E) is unbounded.

Proof. Suppose that $\{y_n\}$ is an F_--solution of equation (E) and $\{y_n\}$ is bounded. Then there exists C such that

$$|y_n| \leq C \quad \text{for } n \in N.$$

So $|\Delta y_n| \leq |y_{n+1}| + |y_n| \leq 2C$. It is impossible by Theorem 4.

Remark 5. Every nonoscillatory bounded solution of equation (E) is an F_+-solution.

Remark 6. Every nonoscillatory F_+-solution of equation (E) has a bounded first difference.

Theorem 7. *If there exists a positive constant δ such that*

$$xf(n,x) \le -\frac{\delta}{n}, \quad \textit{for } (n,x) \in N \times \{R \setminus \{0\}\}. \tag{8}$$

then equation (E) *does not have a nonoscillatory F_+-solution.*

Proof. Suppose that there exists an F_+-solution of equation (E). Let $\{y_n\}$ be such a solution. Then

$$0 < F(y_m) = F(y_1) + \sum_{j=1}^{m-1} y_j f(j, y_j) - \sum_{j=1}^{m-1} \Delta^2 y_{j-1} \Delta^2 y_{j+1}.$$

Hence

$$-\sum_{j=1}^{m-1} y_j f(j, y_j) < F(y_1) - \sum_{j=1}^{m-1} \Delta^2 y_{j-1} \Delta^2 y_{j+1}.$$

From Lemma 2

$$-\sum_{j=1}^{\infty} y_j f(j, y_j) < F(y_1) - \sum_{j=1}^{\infty} \Delta^2 y_{j-1} \Delta^2 y_{j+1} < \infty.$$

On the other hand, by (8),

$$-y_j f(j, y_j) \ge \frac{\delta}{j}.$$

Then

$$-\sum_{j=1}^{\infty} y_j f(j, y_j) \ge \delta \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

The proof is complete by contradiction.

Theorem 8. *If for arbitrary positive constant ε there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(n,x)| > \frac{\delta}{n^2}, \quad \textit{for } n \in N \textit{ and } |x| > \varepsilon, \tag{9}$$

then every F_+-solution of equation (E) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory F_+-solution of equation (E). Then, from Theorem 2,

$$\lim_{n\to\infty} \Delta^2 y_n = \lim_{n\to\infty} \Delta^3 y_n = 0.$$

Suppose that $\{y_n\}$ is an eventually positive solution of equation (E). (For an eventually negative solution the proof is similar.) Then there exists $m \in N$ such that $y_n > 0$ for $n > m$. Summing equation (E), we obtain

$$\Delta^3 y_n - \Delta^3 y_k = \sum_{j=k}^{n-1} f(j, y_j).$$

Then

$$-\Delta^3 y_k = \sum_{j=k}^{\infty} f(j, y_j). \tag{10}$$

Therefore series $\sum_{j=k}^{\infty} f(i, y_j)$ converges. From (10), we have

$$-\Delta^2 y_n + \Delta^2 y_m = \sum_{k=m}^{n-1} \sum_{j=k}^{\infty} f(j, y_j). \tag{11}$$

Then $\Delta^2 y_m = \sum_{k=m}^{\infty} \sum_{j=k}^{\infty} f(j, y_j)$. Therefore, series $\sum_{k=m}^{\infty} \sum_{j=k}^{\infty} f(j, y_j)$ converges. Also

$$\begin{aligned}\sum_{k=m}^{n-1} \sum_{j=k}^{\infty} f(j, y_j) &= \sum_{k=m}^{n-2} (k+1-m) f(k, y_k) + (n-m) \sum_{k=n+1}^{\infty} f(k, y_k) \\ &\leq \sum_{k=m}^{n-2} (k+1-m) f(k, y_k),\end{aligned}$$

for $n > m$. From (11), we get

$$\Delta^2 y_n - \Delta^2 y_m \geq - \sum_{k=m}^{n-2} (k+1-m) f(k, y_k).$$

So,

$$\begin{aligned}-\Delta^2 y_m &= \lim_{n\to\infty}[\Delta^2 y_n - \Delta^2 y_m]\\ &\geq \sum_{k=m}^{\infty}(k+1-m)[-f(k,y_k)]\\ &\geq \sum_{k=m}^{\infty}(k+1-m)\,\frac{\delta}{k^2}=\infty.\end{aligned}$$

Because $-\Delta^2 y_m$ is finite, the obtained contradiction proves our theorem.

Theorem 9. *Assume that f is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$. If*

$$\sum_{j=1}^{\infty} j^3|f(j,C)|=\infty, \tag{12}$$

for every constant $C\neq 0$, then equation (E) *does not possess a nonoscillatory bounded solution.*

Proof. Let $\{y_n\}$ be a nonoscillatory bounded solution of equation (E). Then, by Remark 5, $\{y_n\}$ is an F_+-solution and, by Theorem 4, $\{y_n\}$ is an (A2)-solution. Let $\{y_n\}$ be an (A2+)-solution. (For the (A2-)-solution the proof is similar.) Let us denote

$$v_n=\sum_{k=0}^{3}\frac{(-1)^k}{(3-k)!}\,(n+2-k)^{(3-k)}\Delta^{3-k}y_n,\quad n\in N, \tag{13}$$

then

$$\Delta v_n=\frac{1}{6}(n+3)^{(3)}\Delta^4 y_n,$$

hence

$$\Delta v_n=\frac{1}{6}(n+3)^{(3)}f(n,y_n).$$

Summing the equality, we have

$$v_n=v_{n_0}+\frac{1}{6}\sum_{j=n_0}^{n-1}(j+3)^{(3)}f(j,y_j).$$

Let $v_{n_0} = C$. From (13) and by definition of an (A2+)-solution, we get

$$-y_n - \frac{1}{6}\sum_{j=n_0}^{n-1}(j+3)^{(3)}f(j,y_j) < C.$$

Since $\{y_n\}$ is increasing and bounded, there exists a constant C_1 such that $y_n \le C_1$ and $f(j,y_j) \le f(j,C_1)$. Therefore,

$$\begin{aligned} -\sum_{j=n_0}^{n-1}(j+3)^{(3)}f(j,C_1) &\le -\sum_{j=n_0}^{n-1}(j+3)^{(3)}f(j,y_j) \\ &< 6C_1 + 6C = C_2. \end{aligned}$$

Hence

$$\sum_{j=n_0}^{n-1}(j+3)^{(3)}|f(j,C_1)| < C_2 \quad \text{for } n > n_0.$$

Letting n go to infinity, we obtain

$$\sum_{j=n_0}^{\infty}(j+3)^{(3)}|f(j,C_1)| \le C_2$$

but $(j+3)^{(3)} = (j+3)(j+2)(j+1) > j^3$, so we obtain contradiction with (12).

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