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UNIQUE SOLVABILITY OF AN ORDINARY FREE BOUNDARY PROBLEM

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ABSTRACT. In this paper an existence theorem for an ordinary free boundary problem is given. The result is based on differential inequalities and completes the uniqueness theorem by R.C. Thompson in 1982.

1. Introduction. In the present paper we consider the ordinary free boundary problem:

Find $s > 0$ and $u(x) : [0, \infty) \to \mathbf{R}$ such that

$$\begin{cases} u''(x) = f(x, u(x), u'(x)) & \text{for } x \in [0, s], \\ u(0) = \alpha, \quad u'(s) = 0, & \\ u(x) = 0, & \text{for } x \in [s, \infty), \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $f(x, z, p) : [0, \infty) \times \mathbf{R}^2 \to \mathbf{R}$ are given. For an application we refer to [**1**] and [**2**]. Thompson has given the following result in 1982.

Theorem 1.1. *Let α satisfy the inequality $\alpha > 0$ and suppose that $f(x, z, p)$: $[0, \infty) \times \mathbf{R}^2 \to \mathbf{R}$ satisfies the following conditions*:

1. $f(x, 0, 0) > 0$;

2. *$f(x, z, p)$ is increasing with respect to z, i.e., the inequality $z \leq \tilde{z}$ implies $f(x, z, p) \leq f(x, \tilde{z}, p)$*;

3. *$f(x, z, p)$ satisfies a Lipschitz condition in p on bounded subsets of its domain in $\mathbf{R}^3$.*

Then solutions to problem (1.1) *are unique when they exist.*

For the proof we refer to Corollary 1 in [**3**]. In this paper we will prove a theorem that guarantees the existence of a solution of (1.1).

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The proof is motivated by [**4**].

2. Preliminaries. The results of this section are used to prove the main theorem in Section 3.

Theorem 2.1. *Let* $s > 0$. *Assume that* $f = f(x, z, p)$, $(x, z, p) \in [0, \infty) \times \mathbf{R}^2$ *is continuous and increasing with respect to* z, *i.e., the inequality* $z \leq \tilde{z}$ *implies* $f(x, z, p) \leq z(x, \tilde{z}, p)$. *If* $v, w \in C^2[0, s]$ *satisfy*

$$v'' - f(x, v, v') < 0 = w'' - f(x, w, w') \quad \textit{for } x \in [0, s],$$
$$v(x) = w(x) = 0, \quad \textit{for } x \in [s, \infty),$$
$$v'(s) = w'(s) = 0,$$

then

$$v(x) < w(x) \quad \textit{for } x \in [0, s)$$

follows.

Proof. We assume that there exists $x_0 \in [0, s)$ such that $v(x_0) \geq w(x_0)$.

Case 1. Suppose $v(x) \geq w(x)$ holds for all $x \in [x_0, s]$. Then we define $\Phi(x) = w(x) - v(x)$, $x \in [0, \infty)$. On the one hand, this implies

$$\Phi(x) \leq 0 = \Phi(s) \quad \text{for } x \in [x_0, s];$$

on the other hand, we will show that for some $\varepsilon_0 > 0$ it is

$$\Phi(s - \varepsilon) > 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{2.1}$$

This is a contradiction. To show (2.1) we consider

$$\begin{aligned} \lim_{\varepsilon \to 0+} \Phi''(s - \varepsilon) &= \lim_{\varepsilon \to 0+} w''(s - \varepsilon) - \lim_{\varepsilon \to 0+} v''(s - \varepsilon) \\ &= f(s, 0, 0) - \lim_{\varepsilon \to 0+} v''(s - \varepsilon) \\ &> f(s, 0, 0) - f(s, 0, 0) = 0. \end{aligned}$$

In other words, there exists $\varepsilon_0 > 0$ such that $\Phi'(x)$ is strictly increasing on $[s - \varepsilon_0, s]$. Since $\Phi'(s) = 0$ we can conclude $\Phi'(x) < 0$, $x \in [s - \varepsilon_0, s)$. So Φ is strictly decreasing on $[s - \varepsilon_0, s]$. Due to $\Phi(s) = 0$, (2.1) follows.

Case 2. There exists $x_1 \in (x_0, s)$ such that $w(x_1) > v(x_1)$. Then we can conclude that there are $x_2, x_3 \in [x_0, s]$ such that

$$w(x) > v(x) \quad \text{for } x \in (x_2, x_3)$$

and

$$w(x_2) = v(x_2) \quad \text{and} \quad w(x_3) = v(x_3).$$

Again we define $\Phi(x) = w(x) - v(x)$, $x \in [0, \infty)$. It is $\Phi(x_2) = 0$, $\Phi(x_3) = 0$ and $\Phi(x) > 0$ for $x \in (x_2, x_3)$. Hence, there exists $\bar{x} \in (x_2, x_3)$ such that

$$\Phi(\bar{x}) = \max_{x \in [x_2, x_3]} \Phi(x) > 0.$$

Since $\Phi \in C^2[x_2, x_3]$, it follows that $\Phi''(\bar{x}) \leq 0$ and $\Phi'(\bar{x}) = 0$, which implies $w'(\bar{x}) = v'(\bar{x})$. But it also holds that

$$\begin{aligned}\Phi''(\bar{x}) &= w''(\bar{x}) - v''(\bar{x}) = f(\bar{x}, w(\bar{x}), w'(\bar{x})) - v''(\bar{x}) \\ &> f(\bar{x}, w(\bar{x}), w'(\bar{x})) - f(\bar{x}, v(\bar{x}), v'(\bar{x})) \geq 0,\end{aligned}$$

since $w(\bar{x}) > v(\bar{x})$. This is again a contradiction. □

Lemma 2.1. *Let $c, s > 0$, $r \in \mathbf{R}$ and $b \geq 0$, and let $v(x; s)$ be the unique solution of the initial value problem*

$$\begin{gathered} u''(x) = c + b \cdot u(x) + r \cdot u'(x), \quad x \in [0, s], \\ u(s) = 0, \quad u'(s) = 0. \end{gathered}$$

Then $\lim_{s \to \infty} v(0; s) = \infty$.

Proof. Case 1. Let $b > 0$. Then it is an elementary result (see [**5**]) that

$$v(x; s) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} - \frac{c}{b}$$

with

$$\lambda_1 = \frac{r}{2} + \sqrt{\frac{r^2}{4} + b} > 0, \quad \lambda_2 = \frac{r}{2} - \sqrt{\frac{r^2}{4} + b} < 0$$

and

$$A = -\frac{\lambda_2}{e^{\lambda_1 s}(\lambda_1 - \lambda_2)} \cdot \frac{c}{b}, \quad B = \frac{\lambda_1}{e^{\lambda_2 s}(\lambda_1 - \lambda_2)} \cdot \frac{c}{b}.$$

Hence,

$$\lim_{s\to\infty} v(0;s) = \lim_{s\to\infty} \frac{c}{b}\left(\frac{\lambda_1}{e^{\lambda_2 s}(\lambda_1-\lambda_2)} - \frac{\lambda_2}{e^{\lambda_1 s}(\lambda_1-\lambda_2)} - 1\right) = \infty.$$

Case 2. Let $b = 0$. Suppose $r \neq 0$, then it is

$$v(x;s) = Ae^{rx} + B - \frac{c}{r}x$$

with

$$A = \frac{c}{r^2}e^{-rs} \quad \text{and} \quad B = \frac{c}{r}s - \frac{c}{r^2}.$$

Hence,

$$\lim_{s\to\infty} v(0;s) = \lim_{s\to\infty}\left(\frac{c}{r^2}e^{-rs} + \frac{c}{r}s - \frac{c}{r^2}\right) = \infty.$$

Suppose $r = 0$; then it is $v(x;s) = (c/2)(x-s)^2$ and $\lim_{s\to\infty} v(0;s) = \infty$.
□

3. The main theorem.

Theorem 3.1. *Let $\alpha > 0$ and suppose that f satisfies the following assumptions*:

(V1) *$f = f(x,z,p)$ is continuous in $[0,\infty) \times \mathbf{R}^2$;*

(V2) *For every $s > 0$ there exists $L(s) > 0$ such that f is Lipschitz continuous in z and p with Lipschitz constant $L(s)$, i.e.,*

$$|f(x,z,p) - f(x,\bar{z},\bar{p})| \leq L(s)|z-\bar{z}| + L(s)|p-\bar{p}|;$$

for every $(x,z,p), (x,\bar{z},\bar{p}) \in [0,s] \times \mathbf{R}^2$;

(V3) *$f = f(x,z,p)$ is increasing with respect to z, i.e., the inequality $z \leq \tilde{z}$ implies $f(x,z,p) \leq f(x,\tilde{z},p)$;*

(V4) *There exist $r \in \mathbf{R}$, $b \geq 0$ and $c > 0$ such that*

$$f(x,z,p) > c + b\cdot z + r\cdot p \quad \text{for } (x,z,p) \in [0,\infty) \times \mathbf{R}^2.$$

Then the ordinary free boundary problem (1.1) *has a unique solution $\{s, w(x;s)\}$. Moreover, $w(x;s) > 0$ for $x \in [0,s)$.*

Proof. It is well known (see, for example, Section 1.6 in [**5**]) that for fixed $s > 0$ the initial value problem

$$(3.1) \qquad \begin{gathered} 0 = u''(x) - f(x, u(x), u'(x)), \quad x \in [-1, s] \\ u(s) = u'(s) = 0 \end{gathered}$$

has a unique solution $w(x; s)$ which is also continuous in s. Since $w(s; s) = 0$, we have

$$\lim_{s \to 0+} w(0; s) = 0.$$

Next we will show that

$$(3.2) \qquad \lim_{s \to \infty} w(0; s) = \infty.$$

From this we conclude that, for every $\alpha > 0$, there is at least one $s > 0$ such that $w(0; s) = \alpha$, i.e., $\{s, w(x; s)\}$ solves (1.1).

We consider the function $v(x; s)$ defined in Lemma 2.1 where r, b, c come from (V4), i.e.,

$$\begin{aligned} &v''(x; s) - f(x, v(x; s), v'(x; s)) \\ &\quad < v''(x; s) - (c + b \cdot v(x; s) + r \cdot v'(x; s)) = 0, \\ &v(s; s) = v'(s; s) = 0. \end{aligned}$$

Since $w(x; s)$ solves (3.1), we get

$$v(x; s) < w(x; s) \quad \text{for } x \in [0, s)$$

due to Theorem 2.1. Due to Lemma 2.1, (3.2) follows.

The solution is unique, since the assumptions (V1)–(V4) imply the assumptions of Theorem 1.1. Finally, in Lemma 2 of [**3**] it has already been mentioned that $w(x; s) > 0$ for $x \in [0, s)$. □

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