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THE MAPPING PROPERTIES FOR A CLASS OF OSCILLATORY INTEGRALS

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ABSTRACT. In this paper we show that for $p = \frac{a+b}{b}$ that the operators given by

$$Tf(x) = \int_0^\infty e^{ig(x,y)} \varphi(x,y) f(y)\, dy,$$

map L^p into itself with $g(x,y) = x^b y^a + \gamma(x^{b/a})\gamma_2(y)$. The conditions on γ, γ_2 and φ as defined within.

0. Introduction. In this paper we show that, for $p = \frac{a+b}{b}$, a class of operators map L^p into itself, where the operators are given by

$$Tf(x) = \int_0^\infty K(x,y) f(y)\, dy, \quad x \geq 0, \tag{0.1}$$

and

$$K(x,y) = e^{ig(x,y)} \varphi(x,y), \tag{0.2}$$

$$g(x,y) = x^b \gamma_1(y) + \gamma(x^{b/a})\gamma_2(y), \tag{0.3}$$

$g(x,y)$ is real-valued and

$$\begin{cases} \text{(a)} \quad |\varphi(x,y)| \leq C, & \text{if } x, y \geq 0, \\ \text{(b)} \quad |D\varphi(x,y)| \leq C|x-y|^{-1}, & \text{if } |x-y| > 0. \end{cases} \tag{0.4}$$

We also suppose that $b \geq a \geq 2$ and we further impose conditions on γ, γ_j, $j = 1, 2$. These conditions appear in Section 1.

In case $\gamma_1 = y^a$ and $\gamma(x) = C$ with $a, b \geq 1$, we studied these operators in [**6**] and [**7**]. If case (0.4) holds, we proved in Theorem 3.1 and Corollary 3.2 of [**7**] that when $\gamma = C$ that these operators map L^p into itself if $p = \frac{a+b}{b}$ and if $q \neq p$ these operators do *not* map L^q

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into itself, if $|\varphi(x,y)| \geq C$. In [**6**], we studied the case where $\varphi(x,y)$ is of the form $|x-y|^{-r}$, $0 < r < 1$. In Theorem 1.6 we obtain for the more general phase functions in (0.3) and $\varphi(x,y)$ in (0.4), that these operators map L^p into itself if $p = \frac{a+b}{b}$, the same result as in [**7**] with $b \geq a \geq 2$. In the special case where $\gamma(x), \gamma_2(x)$ are power functions, we obtain this same result and that appears in Theorem 2.5. In [**11**] we discussed the L^2 result for a similar class of operators and we generalize that result to the operators in the Proposition in Section 2 in case $a, b \geq 2$. There is an extensive bibliography and results on oscillatory integrals in [**12**].

The letter C will stand for a positive constant that may change from line to line. We shall also find it convenient to employ subscripts, $C_1, C_2, C_3, \ldots$.

1. Admissible functions and preliminaries. We begin by stating our conditions on γ_j, γ that we use here. We should point out that for the most part, we take $\gamma_1(y) = y^a$, and the model case occurs when $\gamma_2(y)$ is given by (similarly for $\gamma(x)$)

$$\gamma_2(y) = \begin{cases} y^r & \text{if } 0 \leq y \leq \varepsilon \\ y^{m_2} & \text{if } y > \varepsilon, \end{cases}$$

with $r > a > m_2$ and $\varphi(x) = |x-y|^{i\tau}$, τ real.

We suppose for $u, v \geq \varepsilon$ and $m_1, m_2 > 0$ there is a real-valued function satisfying

$$\text{(1.1)} \qquad \begin{cases} (a) & |\lambda_1(u) - \lambda_1(v)| \geq C|u^{m_1} - v^{m_1}|, \quad \text{and} \\ (m_2) & |\lambda_2(u) - \lambda_2(v)| \leq C|u^{m_2} - v^{m_2}|. \end{cases}$$

Note that $C_1|u-v|(u+v)^{m-1} \leq |u^m - v^m| \leq C_2|u-v|(u+v)^{m-1}$ for $m > 0$, for two positive constants C_1, C_2. The cases worked on in [**7**] were when $r = 0$, $m_2 = 0$. We shall also use the convention that λ satisfies (1.1)(m), if we replace m_2 by m. We shall also think of $0 < \varepsilon \leq 1$.

In Theorem 1.7, we formulate conditions on γ, γ_j so that if $b \geq a \geq 2$, then

$$\text{(1.2)} \qquad \|Tf\|_p \leq C\|f\|_p, \quad \text{if } p = \frac{a+b}{b}.$$

We begin by showing for some $\varepsilon > 0$ that

$$\Big| \int_0^u e^{i(t^2\xi + \gamma(t)\eta)} dt \Big| \leq C|\xi|^{-1/2}, \quad \text{if } 0 < u \leq \varepsilon, \tag{1.3}$$

where C is independent of $\xi, \eta \in \mathbf{R}$ and u, for a class of admissible functions $\gamma(t)$ which will be defined below. This result is useful to us, since it enables us to handle the origin. But in doing Theorem 2.5, we find and use a different approach, see (2.9), which allows us to do the cases where $\gamma(t) = t^m$.

First set $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$, and begin with

Definition 1.1. We say that $f(x)$ is weakly monotonic (w.m.) on $[a, b]$ if f is continuous at b and

$$\begin{cases} \text{(i)} & [a,b) = \cup_{n=1}^\infty [a_n, b_n) = \cup_{n=1}^\infty I_n, \\ \text{(ii)} & I_n \cap I_k = \phi \quad \text{if } n \neq k, \\ \text{(iii)} & f'(x) \quad \text{does not change sign for } x \in I_n,\ n = 1, 2, 3, \ldots \\ \text{(iv)} & |f(x)| \leq C_n \|f\|_\infty \quad \text{for } a_n \leq x \leq b_n, \quad \text{and} \\ \text{(v)} & \sum_{n=1}^\infty C_n \leq M < \infty. \end{cases}$$

If $f(x)$ is differentiable and monotonic, then it is w.m. The function $\sin(x)/x^2$ is w.m. if $x \geq 1$, and $x^2 \sin(1/x)$ is w.m. if $0 \leq x \leq 1$.

We need the following,

Lemma 1.2. *Let $h(t)$ be locally integrable on $[a, b]$ and $\|H\|_\infty = \sup_{a \leq x \leq b} |\int_a^x h(t)\, dt|$, and $f(t), g(t)$ be* w.m. *on $[a, b]$. Then*

$$\begin{cases} \text{(a)} & \Big| \displaystyle\int_a^b f(t)h(t)\, dt \Big| \leq C\|H\|_\infty \|f\|_\infty, \quad \textit{and} \\ \text{(b)} & \Big| \displaystyle\int_a^b f(t)g(t)h(t)\, dt \Big| \leq C\|H\|_\infty \|f\|_\infty \|g\|_\infty. \end{cases} \tag{1.4}$$

Proof. Let us first show (1.4)(a). Using i.b.p. we get that

$$\int_a^b f(t)h(t)\, dt = f(b) \int_a^b h(t)\, dt - \int_a^b f'(x) \Big(\int_a^x h(t)\, dt \Big)\, dx \tag{1.5}$$

but
$$\int_{a_n}^{b_n} |f'(x)| \left| \int_a^x h(t)\,dt \right| dx \le \|H\|_\infty \int_{a_n}^{b_n} |f'(x)|\,dx,$$
but $f'(x)$ stays one sign for $x \in I_n$. If we suppose that $f'(x) \ge 0$ for $x \in I_n$ then, and similarly for $f'(x) \le 0$,
$$\int_{a_n}^{b_n} |f'(x)|\,dx \le f(b_n) - f(a_n) \le 2C_n \|f\|_\infty,$$
therefore
$$\begin{aligned}\left| \int_a^b f'(x) \left(\int_a^x h(t)\,dt \right) dx \right| &\le \sum_{n=1}^{\infty} \int_{a_n}^{b_n} |f'(x)| \left| \int_a^x h(t)\,dt \right| dx \\ &\le 2 \sum_{n=1}^{\infty} C_n \|f\|_\infty \|H\|_\infty \le 2M \|f\|_\infty \|H\|_\infty.\end{aligned}$$
By (1.5) this completes the proof of (1.4)(a).

For (1.4)(b) we note that
$$\int_a^b f(t)g(t)h(t)\,dt = g(b) \int_a^b f(t)h(t)\,dt - \int_a^b g'(x) \left(\int_a^x f(t)h(t)\,dt \right) dx$$
but by (1.4)(a) we get that
$$\left| \int_a^x f(t)h(t)\,dt \right| \le C \|f\|_\infty \|H\|_\infty, \quad \text{for } a \le x \le b\,;$$
this completes our proof of (1.4)(b). □

Now let us return to the proof of (1.3). We begin with

Definition 1.3. For some $0 < \varepsilon (\le 1)$, we suppose that there are constants C, C_1 with $C_1 > 1$ so that
$$(1.6) \qquad \begin{cases} \text{(a)} \quad C \dfrac{\gamma'(t)}{t} \ge \gamma''(t) \ge C_1 \dfrac{\gamma'(t)}{t} \quad \text{for } 0 \le t \le \varepsilon, \quad \text{and} \\ \text{(b)} \quad \gamma'(0) = \gamma''(0) = 0. \end{cases}$$

Set $M(t) = \gamma''(t) - \frac{\gamma'(t)}{t}$, we further suppose that $M(t) > 0$ for $0 < t \leq \varepsilon$, and $\frac{M(t)}{t} \in L^1([0,\varepsilon])$. In this case we say that $\gamma(t)$ is an admissible function.

Note if $\frac{\gamma'(t)}{t} > 0$ for $0 < t \leq \varepsilon$, then $M(t) > 0$ follows from (1.6)(a). Also note that $\gamma(t) = t^2$ for $0 \leq t \leq 1$ just fails to be admissible, whereas $\gamma(t) = t^r$, $r > 2$, is admissible if $0 < t \leq 1$.

Next we consider

Proposition 1.4. *Let $\gamma(t)$ be an admissible function and let $M(t) = \gamma''(t) - \frac{\gamma'(t)}{t}$, then for some $\varepsilon > 0$ we get that*

$$(1.7)\quad \begin{cases} \text{(a)} \quad 0 \leq \dfrac{\gamma'(t)}{t} \quad \text{is strictly increasing for } 0 \leq t \leq \varepsilon, \text{ and} \\ \text{(b)} \quad \gamma''(t) - \dfrac{\gamma'(t)}{t} \geq C\gamma''(t_1) \quad \text{for } t_1 \leq t \leq \varepsilon, \end{cases}$$

where C does not depend on t.

Remark. Also notice that $1/t$, and $\frac{1}{u+v\frac{\gamma'(t)}{t}}$ are w.m., for any constants u, v and $0 \leq t \leq \varepsilon$.

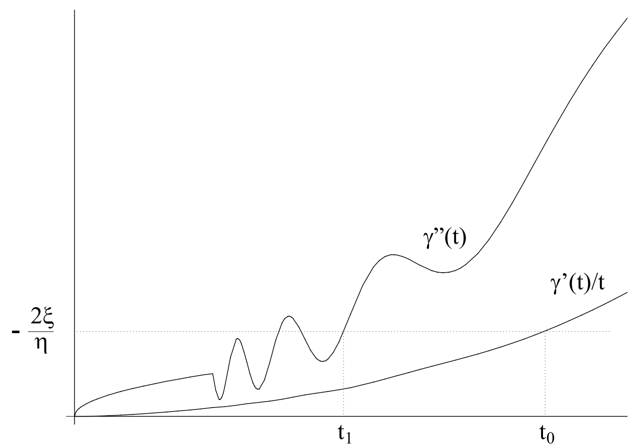
Proof. We notice that since $\frac{M(s)}{s} \in L^1([0,\varepsilon])$, then $\frac{\gamma'(t)}{t} = \int_0^t \frac{M(s)}{s}\, ds + C$, and as $t \to +0$ we get from (1.6)(b) that $\frac{\gamma'(t)}{t} = \int_0^t \frac{M(s)}{s}\, ds$, and since $M(s) > 0$ we get that (1.7)(a) holds.

From (1.6)(a) we get since $C_1 > 1$ that

$$\begin{aligned} \gamma''(t) - \frac{\gamma'(t)}{t} &\geq \gamma''(t)\left(1 - \frac{1}{C_1}\right) \quad \text{if } 0 \leq t \leq \varepsilon, \\ &\geq C\frac{\gamma'(t)}{t} \geq C\frac{\gamma'(t_1)}{t_1} \geq C\gamma''(t_1), \end{aligned}$$

where we used (1.6)(a) and (1.7)(a). This completes our proof. □

We are now in the position to show (1.3).



Proposition 1.5. *Let $\gamma(t)$ be an admissible function, then there exists an $\varepsilon > 0$, so that* (1.3) *holds.*

Proof. By (1.6)(a) and Proposition 1.4, there is a $C_1 > 1$ and $0 < \varepsilon \leq 1$, so that $\gamma''(t) \geq C_1 \frac{\gamma'(t)}{t}$ and $\frac{\gamma'(t)}{t}$ is strictly increasing for $0 \leq t \leq \varepsilon$. We argue the case where $\gamma''(t_1) = \frac{-2\xi}{\eta} = \frac{\gamma'(t_0)}{t_0}$ and if $\gamma''(t_2) = \frac{-2\xi}{\eta}$ then $t_2 \leq t_1 \leq \varepsilon$, all the remaining cases follow in a similar way.

Our purpose is to show (1.3), set $\psi(t) = t^2\xi + \gamma(t)\eta$ and so we have for $(0 \leq u \leq \varepsilon)$ that

$$\int_0^u e^{i\psi(t)}\, dt = \int_0^{t_1} + \int_{t_1}^{t_0} + \int_{t_0}^{u} = I + II + III.$$

Note it's possible that $u \leq t_0$ or $u \leq t_1$, and so in those cases disregard those integrals.

We begin with I.

If $t_1 \leq |\xi|^{-1/2}$, then we are finished and in a similar way we can suppose $t_1, u \geq |\xi|^{-1/2}$. Set $\delta = |\xi|^{-1/2}$, then

$$I = \int_0^\delta + \int_\delta^{t_1} = I_1 + I_2,$$

but

$$\frac{\gamma'(t)}{t} \leq \frac{\gamma'(t_1)}{t_1} \leq \frac{\gamma''(t_1)}{C_1} = \frac{2|\xi|}{C_1|\eta|}, \quad \text{for } 0 \leq t \leq t_1.$$

Thus,

$$\left|\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t}\right| \geq \frac{2|\xi|}{|\eta|} - \frac{2|\xi|}{C_1|\eta|} \geq 2\left(1 - \frac{1}{C_1}\right)\frac{|\xi|}{|\eta|} \quad \text{and} \quad C_1 > 1. \tag{1.8}$$

Hence

$$\begin{aligned}|\psi'(t)| = \left|\eta t\left(\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t}\right)\right| &\geq t|\eta|\left|\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t}\right| \\ &\geq Ct|\eta|\frac{|\xi|}{|\eta|} \geq Ct|\xi| \quad \text{for } 0 < t \leq t_1.\end{aligned}$$

From $|I_1| \leq C/|\xi|^{1/2}$ and

$$|I_2| = \left|\int_\delta^{t_1} \frac{e^{i\psi(t)}\psi'(t)\,dt}{\eta t(\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t})}\right| \leq \frac{C}{|\xi||\xi|^{-1/2}} \leq \frac{C}{|\xi|^{1/2}}$$

and this follows from (1.4)(b) of Lemma 1.2 with $f(t) = 1/t$ and $g(t) = \dfrac{1}{(\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t})}$.

Next we consider II.

For $t_1 \leq t \leq t_0$ we get by (1.7)(b) that

$$\gamma''(t) - \frac{\gamma'(t)}{t} \geq C_2\gamma''(t_1) = C_2\frac{2|\xi|}{|\eta|}. \tag{1.9}$$

It follows from (1.9) that $(t_1 \leq t \leq t_0)$ if

$$\left|\frac{2\xi}{\eta} + \frac{\gamma'(t)}{t}\right| \leq C_2\frac{|\xi|}{|\eta|}, \quad \text{then } \left|\frac{2\xi}{\eta} + \gamma''(t)\right| \geq C_2\frac{|\xi|}{|\eta|}$$

where C_2 comes from (1.9).

It follows from this result that there is a number c^* so that

$$
(1.10)\qquad \begin{cases} \text{(a)} & \Big|\dfrac{2\xi}{\eta}+\dfrac{\gamma'(t)}{t}\Big| \ge C_2\dfrac{|\xi|}{|\eta|} \quad \text{if } t_1\le t\le c^* \\ \text{(b)} & \Big|\dfrac{2\xi}{\eta}+\gamma''(t)\Big| \ge C_2\dfrac{|\xi|}{|\eta|} \quad \text{if } c^*\le t\le t_0. \end{cases}
$$

Next note that $II \le |\int_{t_1}^{c^*}| + |\int_{c^*}^{t_0}| = II_1 + II_2$. Thus by (1.4)(b) as above we get that

$$
II_1 = \left|\int_{t_1}^{c^*} \frac{e^{i\psi(t)}\psi'(t)}{\psi'(t)}\,dt\right| \le \frac{C}{t_1|\xi|} \le C|\xi|^{-1/2}
$$

since $|\xi|^{-1/2}\le t_1\le t\le c^*$ and (1.10)(a).

For the term II_2, we get by (1.10)(b) that $|\psi''(t)| = |\eta||(2\xi/\eta)+\gamma''(t)| \ge C_2|\xi|$, and so Van der Corput applies and we get that $II_2 \le \dfrac{C}{|\xi|^{1/2}}$.

At last we consider III.

This time $t_0\le t\le\varepsilon$ and since $C_1>1$ and (1.7)(a) we get

$$
|\psi''(t)| = |\eta|\left|\frac{\gamma'(t_0)}{t_0}-\gamma''(t)\right| \ge |\eta|\left(\frac{C_1\gamma'(t)}{t}-\frac{\gamma'(t_0)}{t_0}\right) \ge C|\xi|
$$

and so again Van der Corput applies and we get that

$$
III \le C|\xi|^{-1/2}.
$$

This completes our result. □

Remark 1. We get from Proposition 1.5 that there is a C independent of ξ,η,u, $(\xi,\eta\in\mathbf{R})$ and $0\le u\le\varepsilon$ so that

$$
(1.11)\qquad \left|\int_0^u e^{i(t^a\xi+\gamma(t)\eta)}dt\right| \le C|\xi|^{-1/a}
$$

if $a\ge 2$ and $h(t)=\gamma(t^{2/a})$ is an admissible function. Just set $s^2=t^a$ and apply Proposition 1.5 to

$$
\left|\int_0^{u^{a/2}} \frac{e^{i(s^2\xi+\gamma(s^{2/a})\eta)}}{s^{1-2/a}}\,ds\right|.
$$

Note that we can extend Proposition 1.5 to a global result, if we assume that $\gamma(t)$ satisfies (1.6) for all t, i.e., letting $\varepsilon \to +\infty$ and $M(t)/t \in L_{\text{loc}}$. But we will not pursue that here.

We are now in a position to state our conditions on $\gamma(t)$. We shall state our conditions locally and so for some $0 < \varepsilon \leq 1$, we consider real-valued functions $\lambda(t)$ that satisfy

$$(1.12) \qquad \begin{cases} \text{(a)} \quad h(t) = \lambda(t^{2/a}) & \text{is admissible for } 0 \leq t \leq \varepsilon, \\ \text{(b)} \quad \lambda(t) & \text{satisfies (1.1)}(m) \text{ for } t > \varepsilon, \\ \text{(c)} \quad \lim_{t \to +\infty} \dfrac{|\lambda'(t)|}{t^{a-1}} = 0, & \text{and} \\ \text{(d)} \quad |\lambda'(t)| \leq C t^{a-1} & \text{if } t \geq \varepsilon. \end{cases}$$

Note we say that $\lambda(t)$ satisfies (1.12)(m_2) to mean that λ satisfies (1.12), but we replace (1.1)(m) by (1.1)(m_2) in (1.12)(b).

Note that from (0.2) that the operator with kernel $K(x,y)(1-\psi(x-y))$ where $\psi(x) \in C^\infty(\mathbf{R})$, $\psi(x) = 0$ for $|x| \leq 1$, $\psi(x) = 1$ for $|x| \geq 2$ and $0 \leq \psi(x) \leq 1$, maps L^p into itself for $1 \leq p \leq \infty$. We are left with the operator whose kernel is $K(x,y)\psi(x-y)$.

Our proof of (1.2), as we shall soon see, follows by proving that $\|S_1\|_{2,2} = \sup_{\|f\|_2 \leq 1} \|S_1 f\|_2 \leq C < \infty$, where

$$(1.13) \qquad S_1 f(x) = \int_0^\infty k_1(x,y) f(y)\, dy, \quad x \geq 0,$$

where $k_1(x,y) = e^{ig(x^{a/b},y)}\varphi_1(x^{a/b},y)$ and $g(x,y)$ is defined in (0.3) with $\varphi_1(x,y) = \varphi(x,y)\psi(x-y)$. Since this operator maps L^2 into itself it follows that the dual operator

$$S_1^* f(x) = \int_0^\infty k_1(y,x) f(y)\, dy, \quad x \geq 0$$

maps L^2 into itself. Let an operator associated to S_1^* be given by

$$\tilde{T}_1 f(x) = \int_0^\infty k_1(y, x^{b/a}) f(y)\, dy, \quad x \geq 0.$$

We get that this operator $\tilde{T}_1$ maps L^p into itself for $p = \dfrac{a+b}{a}$.

Theorem 1.6. *Let* $\gamma_1(y) = y^a$, $b \geq a \geq 2$ *and* $0 < m,\ m_2 \leq a$. *Suppose that* (0.4) *holds,* $\dfrac{\gamma'(x)}{x^{a-1}}$ *and* $\dfrac{\gamma_2'(x)}{x^{a-1}}$ *are monotonic for* $x \geq \varepsilon$ *and* $\gamma(x)$ *satisfies* (1.12) *and* $\gamma_2(x)$ *satisfies* (1.12)(m_2). *Then* Tf *defined in* (0.1) *satisfies* (1.2).

To prove Theorem 1.6 we show that the operator S_1 defined in (1.13) maps L^2 into itself. More precisely we show,

Theorem 1.7. *Let* $a \geq 2$, $0 < m,\ m_2 \leq a$ *and* $\gamma_1(y) = y^a$. *Suppose that* (0.4) *holds,* $\dfrac{\gamma'(x)}{x^{a-1}}$ *and* $\dfrac{\gamma_2'(x)}{x^{a-1}}$ *are monotonic for* $x \geq \varepsilon$, $\gamma(x)$ *satisfies* (1.12) *and* $\gamma_2(x)$ *satisfies* (1.12)(m_2). *Then* S_1 *defined in* (1.13) *maps* L^2 *into itself.*

We now show that Theorem 1.6 follows directly from Theorem 1.7.

Proof of Theorem 1.6. Assume that Theorem 1.7 holds. We then get that

$$\text{(i)} \quad \|S_1 f\|_2 \leq C\|f\|_2$$

and

$$\text{(ii)} \quad \|S_1 f\|_\infty \leq C\|f\|_1.$$

Notice that $T_1 f(x) = S_1 f(x^{b/a})$ and $\tilde{T}_1 f(x) = S_1^* f(x^{b/a})$ and so it follows from (i) and (ii) that

$$\int_0^\infty |T_1 f(x)|^p\, dx = \frac{a}{b}\int_0^\infty x^{(a/b)-1}|S_1 f(x)|^p dx \leq C\|f\|_p^p$$

as long as $p - 2 = (a/b) - 1$ and $a \leq b$. A similar argument holds for the operator $\tilde{T}_1$. This completes our proof of Theorem 1.6. □

2. Proof of Theorem 1.7. We begin with the following and we suppose that $\dfrac{\gamma'(t)}{t^{a-1}}$ is monotonic for $t \geq \varepsilon$, but we really only need the monotonicity condition for relevant t, $r_1 \leq t \leq r_2$,

Proposition 2.1. *Let $\varepsilon > 0$ be given with $u, v, t, \geq \varepsilon$. Suppose that $\gamma(t)$ satisfies* (1.12)(c) *and* (d), $0 < m_2 \leq m_1 \leq a$. *If γ_1 satisfies* (1.1)(a) *and γ_2 satisfies* (1.1)(m_2)*, then with*

$$\alpha(t) = t^a(\gamma_1(u) - \gamma_1(v)) + \gamma(t)(\gamma_2(u) - \gamma_2(v))$$

we get that

$$|\alpha'(t)| \geq Ct^{a-1}|u^{m_1} - v^{m_1}|. \tag{2.1}$$

If in addition $\dfrac{\gamma'(t)}{t^{a-1}}$ is monotonic for $t \geq \varepsilon$, then

$$\left|\int_{r_1}^{r_2} e^{i\alpha(t)}\, dt\right| \leq \frac{C}{r_1^{a-1}|u^{m_1} - v^{m_1}|}, \tag{2.2}$$

if either $r_2 \geq r_1 \geq N_\varepsilon$ for N_ε sufficiently large or $u + v$ is large and $m_2 < m_1$.

Proof. We note that

$$\alpha'(t) = at^{a-1}\Big[\gamma_1(u) - \gamma_1(v) + \frac{\gamma'(t)}{at^{a-1}}(\gamma_2(u) - \gamma_2(v))\Big].$$

By (1.1) applied to γ_j, $j = 1, 2$, we get that

$$|\alpha'(t)| \geq at^{a-1}|u - v|\Big[C_1(u + v)^{m_1-1} - C_2\Big|\frac{\gamma'(t)}{t^{a-1}}\Big|(u + v)^{m_2-1}\Big].$$

To complete our proof of (2.1) it suffices to show that

$$(u + v)^{m_1-1} \geq C\frac{|\gamma'(t)|}{t^{a-1}}(u + v)^{m_2-1} \tag{2.3}$$

for some C large enough. But if t is large enough, $t \geq N_\varepsilon$, then by (1.12)(c) we get that

$$(u + v)^{m_1-m_2} \geq C\frac{|\gamma'(t)|}{t^{a-1}}.$$

While the last inequality holds even if $\underline{t}$ is not large, but then $u + v$ must be sufficiently large and $m_2 < m_1$, that follows from (1.12)(d).

To see (2.2) with $\varepsilon < r_1 \le r_2$, we apply Lemma 1.2 with $h(t) = e^{i\alpha(t)}\alpha'(t)$

$$\left|\int_{r_1}^{r_2} \frac{e^{i\alpha(t)}\alpha'(t)}{\alpha'(t)}\,dt\right|, \quad \text{where } f(t) = t^{1-a}$$
$$\text{and} \quad g(t) = \Big[\gamma_1(u) - \gamma_1(v) + \frac{\gamma'(t)}{t^{a-1}}(\gamma_2(u) - \gamma_2(v))\Big]^{-1}$$

This completes our proof. □

For the operator S_1 defined in (1.13) our (2,2) problem is reduced to estimating the terms

$$I_{ij} = \int_{E_i} \Big| \int_{E_j} k_1(x,y)f(y)\,dy\Big|^2\,dx, \quad i,j = 1,2,$$

if $i + j > 2$, $E_1 = [0,N]$, $E_2 = [N,\infty)$, N from Proposition 2.1 and $k_1(x,y)$ is defined below (1.13).

We begin with the following result.

Proposition 2.2. *Assume $a > 1$ and that $0 < m_2 \le a$. Also suppose that* (0.4) *holds, $\gamma(x)$ satisfies* (1.12)(c) *and $\gamma_2(x)$ satisfies* (1.1)(m_2). *If $\dfrac{\gamma'(x)}{x^{a-1}}$ is monotonic for $x \ge N$, then*

$$I_{22} \le C\int_0^\infty |f|^2\,dy.$$

Proof. We notice that

$$I_{22} = \sum_{j=1}^\infty \int_N^\infty \chi_j(x)\Big|\sum_{l=1}^\infty \int_N^\infty k_1(x,y)\chi_l(|x-y|)f(y)\,dy\Big|^2\,dx,$$

where $\chi_l(y) = \chi(2^{l-1} \le y \le 2^l)$, $l = 1,2,3,\dots$. Thus

$$I_{22}^{1/2} \le \sum_{l=1}^\infty \bigg(\sum_{j=1}^\infty \int_N^\infty \chi_j(x)\Big|\int_N^\infty k_1(x,y)\chi_l(|x-y|)f(y)\,dy\Big|^2 dx\bigg)^{1/2}.$$

We analyze the integrand,

$$\begin{aligned}\tilde{I}_{jl} &= \int_N^\infty \chi_j(x)\Big|\int_N^\infty k_1(x,y)\chi_l(|x-y|)f(y)\,dy\Big|^2\,dx \\ &= \int_N^\infty f(u)\bigg(\int_N^\infty \bar{f}(v)A_{jl}(u,v)\,dv\bigg)\,du,\end{aligned}$$

with

$$A_{jl}(u,v) = \int_N^\infty \chi_j(x)k_1(x,u)\bar{k}_1(x,v)\chi_l(|x-u|)\chi_l(|x-v|)\,dx.$$

We focus on the term

$$\sup_w \int_{S_{jl}} |\partial_x(\varphi(x^{a/b},u)\bar{\varphi}(x^{a/b},v))|\,dx = b_{jl},$$

where w stands for u or v and the set $S_{jl} = \{x \geq N : |x-w| \in [2^{l-1}, 2^l]$, and $x \in [2^{j-1}, 2^j]\}$.

If the set $S_{jl} \neq \phi$ then $|S_{jl}| \leq C2^{j\wedge l}$ and u, v must satisfy

$$\text{(2.4)} \qquad \begin{cases} \text{(a)} & |u-v| \leq 2^{l+2}, \\ \text{(b)} & 2^{(j\vee l)-1} \leq u,\ v \leq 2^{j+1}+2^l, \quad \text{if } j \neq l, l+1, \text{ and} \\ \text{(c)} & u, v \geq N \qquad\qquad\qquad\qquad \text{if } j = l \text{ or } l+1. \end{cases}$$

We set $S_{jl} = [x_1, x_2]$ and by the trivial estimate we get that $|A_{jl}| \leq C2^{j\wedge l}\chi_{jl}(u)\chi_{jl}(v)$, where $\chi_{jl} = \chi(u \geq 2^{j\vee l})$ if $j \neq l, l+1$ and $\chi_{jl}(u) = \chi(u \geq N)$ if $j = l$ or $l+1$.

Integrating by parts we get two terms, namely

$$\begin{aligned}A_{jl} &= \varphi(x_2^{a/b},u)\bar{\varphi}(x_2^{a/b},v)\int_{x_1}^{x_2} e^{i\alpha(t)}\,dt \\ &\quad + \int_{x_1}^{x_2}\Big(\partial_x\Big(\varphi(x^{a/b},u)\bar{\varphi}(x^{a/b},v)\Big)\Big(\int_{x_1}^{x} e^{i\alpha(t)}\,dt\Big)\Big)\,dx\end{aligned}$$

where $\alpha(x) = x^a(u^a - v^a) + \gamma(x)(\gamma_2(u) - \gamma_2(v))$. By (2.2) of Proposition 2.1 it follows that

(2.5)

$$\Big|\int_{x_1}^{x} e^{i\alpha(t)}\,dt\Big| \leq \frac{C}{|u-v|2^{j(a-1)}}\begin{cases} \text{(a)} & \dfrac{1}{2^{(j\vee l)(a-1)}}, \quad \text{if } j \neq l, l+1 \\ \text{(b)} & 1, \qquad\qquad\quad \text{if } j = l \text{ or } l+1. \end{cases}$$

Then by (2.5) we get that,

$$|A_{jl}(u,v)| \le C \begin{cases} \text{(a)} \quad \dfrac{1+b_{jl}}{|u-v|2^{j(a-1)}2^{(j\vee l)(a-1)}} & \text{if } j \ne l, l+1 \\ \text{(b)} \quad \dfrac{1+b_{jl}}{|u-v|2^{j(a-1)}} & \text{if } j = l, l+1. \end{cases}$$

We estimate b_{jl} and notice from (0.4) that

$$\begin{aligned} &\int_{x_1}^{x_2} x^{(a/b)-1}(|\varphi'(x^{a/b},u)\varphi(x^{a/b},v)| + |\varphi(x^{a/b},u)\varphi'(x^{a/b},v)|)\,dx \\ &\qquad \le C\int_{x_1^{a/b}}^{x_2^{a/b}} [(1+|x-u|)^{-1} + (1+|x-v|)^{-1}]\,dx \le C(j\vee l). \end{aligned}$$

Thus we get that

$$|A_{jl}(u,v)| \le C\frac{(j\vee l)\chi_{[0,2^{l+2}]}(|u-v|)}{|u-v|2^{j(a-1)}} \begin{cases} \text{(a)} \quad 2^{-(j\vee l)(a-1)} & \text{if } j \ne l, l+1 \\ \text{(b)} \quad 1 & \text{if } j = l, l+1. \end{cases}$$

Thus if we set

$$\begin{aligned} d_{jl}(u,v) &= (j\vee l)\chi_{jl}(u)\chi_{jl}(v) \\ &\quad \times \begin{cases} \text{(a)} \quad \dfrac{1}{1+2^{\rho_{jl}}|u-v|2^{(j\vee l)(a-1)}} & \text{if } j \ne l, l+1 \\ \text{(b)} \quad \dfrac{1}{1+2^{\rho_{jl}}|u-v|} & \text{if } j = l, l+1, \end{cases} \end{aligned}$$

and also employing the trivial estimate, we get that

$$|A_{jl}| \le C2^{j\wedge l}\chi_{[0,2^{l+2}]}(|u-v|)d_{jl}(u,v),$$

where $\rho_{jl} = (j\wedge l) + j(a-1)$. We get that these terms sum and thus

$$I_{22}^{1/2} \le \sum_{l=1}^{\infty}\left(\sum_{j=1}^{\infty}\tilde{I}_{jl}\right)^{1/2} \le C\|f\|_2. \qquad \square$$

Next we estimate the remaining terms I_{12} and I_{21}.

Proposition 2.3. *Let $a \geq 2$ and assume that* (0.4) *holds, and* $0 < m_2 < a$. *If $\gamma(x)$ satisfies* (1.12)(*a*) *and* (*d*) *and $\gamma_2(x)$ satisfies* (1.1)(m_2), *then if $\dfrac{\gamma'(x)}{x^{a-1}}$ is monotonic for $x \geq \varepsilon$, then*

$$I_{12} \leq C\|f\|_2^2.$$

Proof. And we notice that

$$I_{12} = \int_N^\infty f(u)\Big(\int_N^\infty \bar{f}(v)A(u,v)\,dv\Big)\,du,$$

with

$$A(u,v) = \int_0^N k_1(x,u)\bar{k}_1(x,v)\,dx.$$

Using integration by parts we get that

(2.6)

$$\begin{aligned}|A(u,v)| \leq {}& \Big|\varphi_1(N^{a/b},u)\bar{\varphi_1}(N^{a/b},v)\int_0^N e^{i\alpha(t)}\,dt\Big| \\ &+ \int_0^N \Big|\Big(\partial_x(\varphi_1(x^{a/b},u)\varphi_1(x^{a/b},v))\Big(\int_0^x e^{i\alpha(t)}\,dt\Big)\Big)\Big|\,dx\end{aligned}$$

where $\alpha(x) = x^a(u^a - v^a) + \gamma(x)(\gamma_2(u) - \gamma_2(v))$. Note from (2.2) we get that

$$\Big|\int_\varepsilon^x e^{i\alpha(t)}\,dt\Big| \leq \frac{C}{|u^a - v^a|} \quad \text{for } x \geq \varepsilon,$$

since $u + v \geq 2N$ and N is sufficiently large, but still fixed. Also from (1.11) we get that

$$\Big|\int_0^x e^{i\alpha(t)}\,dt\Big| \leq \frac{C}{|u^a - v^a|^{1/a}} \quad \text{if } 0 \leq x \leq \varepsilon.$$

Putting these two estimates together, we get that for $\varepsilon \leq x \leq N$

$$\begin{aligned}\Big|\int_0^x e^{i\alpha(t)}\,dt\Big| \leq {}& \Big|\int_0^\varepsilon e^{i\alpha(t)}\,dt\Big| \\ &+ \Big|\int_\varepsilon^x e^{i\alpha(t)}\,dt\Big|^{1/a} \cdot \Big|\int_\varepsilon^x e^{i\alpha(t)}\,dt\Big|^{1-\frac{1}{a}} = I + II.\end{aligned}$$

And $II \le \dfrac{CN^{\frac{a-1}{a}}}{|u^a - v^a|^{1/a}}$ and so this implies (note N is fixed but large) by (0.4) and (2.6) that

$$|A(u,v)| \le \frac{C}{|u^a - v^a|^{1/a}}.$$

But, since $a > 1$, $A(u,v)$ is the kernel of an operator that maps L^2 into itself by Schur's lemma. This completes our estimates of I_{12}. Hence this completes our proof. □

At last we show

Proposition 2.4. *Let $a \ge 2$, $0 < m, m_2 \le a$ and (0.4) hold. If $\gamma_2(x)$ satisfies (1.12)(a) and (d) and $\gamma(x)$ satisfies (1.1)(m), then if $\dfrac{\gamma_2'(x)}{x^{a-1}}$ is monotonic for $x \ge \varepsilon$, then we get that*

$$I_{21} \le C\|f\|_2^2.$$

Proof. Note that

$$I_{21} = \int_N^\infty \left| \int_0^N k_1(x,y) f(y)\, dy \right|^2 dx,$$

but by duality it suffices to prove that

$$\int_0^N \left| \int_N^\infty k_1(y,x) f(y)\, dy \right|^2 dx \le C\|f\|_2^2 \tag{2.7}$$

this time

$$A(u,v) = \int_0^N k_1(u,x)\bar{k}_1(v,x)\, dx, \quad \text{and}$$
$$\alpha(x) = x^a(u^a - v^a) + \gamma_2(x)(\gamma(u) - \gamma(v)).$$

Once again the argument follows the approach below (2.6) in Proposition 2.3, but here we utilize our hypothesis on $\gamma(x)$ and $\gamma_2(x)$. □

Now we are in a position to prove Theorem 1.7.

Proof of Theorem 1.7. We note that

$$\|Sf\|_2^2 = \sum_{i,j=1}^{2} I_{ij}$$

and so our proof follows from Propositions 2.2, 2.3, and 2.4. □

Let $b > a > 1$ and let $\varphi(x, y)$ satisfy

$$\begin{cases} \text{(i)} \quad |\varphi(x,y)| \le C|x-y|^{-\frac{b-a}{2b}} & \text{if } |x-y| > 0, \text{ and} \\ \text{(ii)} \quad |D\varphi(x,y)| \le C|x-y|^{-\frac{b-a}{2b}-1}, & \text{if } |x-y| > 0. \end{cases}$$

Set

$$Uf(x) = \int_0^\infty \varphi(x,y) e^{ig(x,y)} f(y)\, dy.$$

Here $g(x, y)$ is defined in (0.3) and $\varphi(x, y)$ satisfies (i) and (ii). With the usual regularity conditions, as defined below, on $\gamma(x), \gamma_2(y)$ we get that $\|Uf\|_2 \le C\|f\|_2$.

This (2,2) result essentially follows from Theorem 0.1 of [**11**]. But here we employ a more general phase function $g(x, y)$ and thus estimate a more general operator U. Also we need only estimate the operator U_1, defined by replacing $\varphi(x, y)$ in U by $\varphi_1(x, y) = \varphi(x, y)\psi(x - y)$. We are able to prove that

Proposition. *Let $b > a$ and suppose $\varphi(x, y)$ satisfies* (i) *and* (ii)*. Also $g(x, y)$ as in* (0.3) *where $a \ge 2$, $0 \le m < b$ and $0 \le m_2 < a$. If $\gamma(x)$ satisfies* (1.12)*, $\gamma_2(x)$ satisfies* (1.12)(m_2) *and $\dfrac{\gamma'(x)}{x^{a-1}}, \dfrac{\gamma_2'(x)}{x^{a-1}}$ are monotonic for $x \ge \varepsilon$. Then,*

$$\|Uf\|_2 \le C\|f\|_2.$$

Proof. We shall be brief here. Also note that $m = 0$ or $m_2 = 0$ was done in [**7**].

We begin with the term I_{12} and follow closely the proof of Proposition 2.3. This time we get, as in (2.6), that

$$|A(u,v)| \leq \left|\varphi_1(N,u)\varphi_1(N,v)\int_0^N e^{i\alpha(t)}\,dt\right| + \int_0^N \left|\partial_x(\varphi_1(x,u)\varphi_1(x,v))\left(\int_0^x e^{i\alpha(t)}\,dt\right)\right| dx$$

but here $\alpha(t) = t^b(u^a - v^a) + \gamma(t^{b/a})(\gamma_2(u) - \gamma_2(v))$. Again (2.2) applies here and we get that

$$\left|\int_\varepsilon^x e^{i\alpha(t)}\,dt\right| \leq \frac{C}{|u^a - v^a|} \text{ for } x \geq \varepsilon.$$

And from (1.11) we get that

$$\left|\int_0^x e^{i\alpha(t)}\,dt\right| \leq \frac{C}{|u^a - v^a|^{1/b}} \text{ for } 0 \leq x \leq \varepsilon.$$

And it follows from (i) and (ii), using our estimates from Proposition 2.3, that

$$|A(u,v)| \leq \frac{C}{|u^a - v^a|^{1/a}} + \frac{C\chi(u \geq N)\chi(v \geq N)}{|u^a - v^a|^{1/b}[(1+(u-N))(1+(v-N))]^{\frac{b-a}{2b}}},$$

but by Schur's lemma, it follows that $A(u,v)$ is a kernel that maps L^2 into itself. This completes our estimates of I_{12}.

Arguing as we did in Proposition 2.4, we estimate the term I_{21}, and employing our hypothesis on $\gamma_2(y)$ and $\gamma(x)$, it follows that

$$|A(u,v)| \leq \frac{C}{|u^b - v^b|^{1/b}} + \frac{C\chi(u \geq N)\chi(v \geq N)}{|u^b - v^b|^{1/a}[(1+(u-N))(1+(v-N))]^{\frac{b-a}{2b}}}.$$

Once again this is the kernel of an operator that maps L^2 into itself by Schur's lemma. This completes our estimates of I_{21}.

We complete our argument once we notice (as before) we can apply the proof of Proposition 2.2 to estimate the term I_{22}. This now completes our proof. $\square$

Note we also get the following result in the special cases where $\gamma(x) = x^m$ and $\gamma_2(x) = x^{m_2}$ and $g(x, y)$ is defined in (0.3) ($\gamma_1(y) = y^a$). Thus $K(x, y)$ (the kernel of T) is defined in (0.2) and $K(x^{a/b}, y)$ is the kernel of the operator S, see also (1.13). We employ Lemma 2.1 of [**11**] as our main tool.

Theorem 2.5. *Let* $b \geq a \geq 2, \gamma(x) = x^m$*,* $\gamma_2(x) = x^{m_2}$ *and* $0 < m, m_2 < a$*. If* (0.4) *holds, then*

(i) $\|Sf\|_2 \leq C\|f\|_2$ *and*

(ii) $\|Tf\|_p \leq C\|f\|_p$ *for* $p = \dfrac{a+b}{b}$.

Proof. Note that since $\|Sf\|_\infty \leq C\|f\|_1$, (ii) follows from (i). Thus it suffices to show (i). We need a replacement for Proposition 1.5, which does not apply here.

The following is proved in Lemma 2.1 of [**11**].

$$\left| \int_0^T e^{i\psi(t)}\, dt \right| \leq C|\xi|^{-1/a} \quad \text{if } \psi(t) = t^a \xi + t^r \eta, \tag{2.8}$$

$r \neq a$, $a \geq 2$ and C does not depend upon ξ, η, or T.

We first note that $\gamma'(x) = x^{m-1}$ and since $m < a$, we get that (1.12)(c) and (d) are satisfied and $\gamma_2(x)$ satisfies (1.1)(m_2). Therefore Proposition 2.2 applies. Now we get that the kernel in (2.6) satisfies

$$|A(u, v)| \leq \frac{C}{|u^a - v^a|^{1/a}}\,; \tag{2.9}$$

this follows from (2.8) and gets us the term I_{12}. The estimate of I_{21} follows in a similar way. Schur's lemma completes our proof. $\square$

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