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ALMOST SKEW-SYMMETRIC MATRICES

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ABSTRACT. Almost skew-symmetric matrices are real matrices whose symmetric parts have rank one. Using the notion of the numerical range, we obtain eigenvalue inequalities and a localization of the spectrum of an almost skew-symmetric matrix. We show that almost skew-symmetry is invariant under principal pivot transformation and inversion, and that the symmetric parts of Schur complements in almost skew-symmetric matrices have rank at most one. We also use affine combinations of A and A^t to gain further insight into eigenvalue location and the numerical range of an almost skew-symmetric matrix.

1. Introduction. Let $\mathcal{M}_n(\mathbf{R})$ ($\mathcal{M}_n(\mathbf{C})$) be the algebra of all $n \times n$ real (complex) matrices. In this article, we consider matrices $A \in \mathcal{M}_n(\mathbf{R})$, $n \geq 2$, whose symmetric parts have rank one. This means that the spectrum of the symmetric part of such a matrix A consists of the eigenvalue 0 with multiplicity $n-1$ and a simple nonzero eigenvalue, which will be assumed to be positive for simplicity. We shall then refer to A as an *almost skew-symmetric* matrix.

Tournament matrices and their generalizations [**11, 9**] are closely related to almost skew-symmetric matrices and have provided the motivation for this subject. Indeed, $T \in \mathcal{M}_n(\mathbf{R})$ is a *pseudo-tournament* (i.e., rank $(T+T^t+I) = 1$) if and only if $T+\frac{1}{2}I$ or its negative is almost skew-symmetric. If $T \in \mathcal{M}_n(\mathbf{R})$ is a *hypertournament* (i.e., T has zero diagonal entries and $T + T^t = ww^t - I$ for some nonzero $w \in \mathbf{R}^n$) then $T + \frac{1}{2}I$ is almost skew-symmetric.

The spectra of pseudo-tournaments were initially studied in [**11**]. The spectra of almost skew-symmetric matrices and compact operators were considered in [**4,5**]. In [**4**] it was shown that the real eigenvalues of an almost skew-symmetric matrix satisfy certain interesting inequalities.

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In [**5**], two half-planes were provided, one containing exactly one eigenvalue and the other containing all remaining eigenvalues of an almost skew-symmetric matrix. More recently, a new approach was used in [**10**] to study the spectra of hypertournaments, which involved Schur complementation and the numerical range.

Our plan to study almost skew-symmetric matrices unfolds as follows. Firstly, we will provide tight regions that contain the eigenvalues of an almost skew-symmetric matrix by adapting and generalizing the results in [**10**]. The eigenvalue results herein can be viewed as a refinement and an extension of those in [**4,5**]. Secondly, we will study the inverses, principal pivot transforms and Schur complements of almost skew-symmetric matrices. Thirdly, we will consider affine combinations of A and A^t (Levinger's transformation), leading to further information about the spectrum and the numerical range of an almost skew-symmetric matrix A. Some applications to tournament matrices will also be presented.

2. Preliminaries and notation.

For any $A \in \mathcal{M}_n(\mathbf{C})$, the *spectrum* of A is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. The *numerical range* (also known as the *field of values*) of A is the set

$$F(A) = \{v^* A v \in \mathbf{C} : v \in \mathbf{C}^n \text{ with } v^* v = 1\},$$

which is a compact and convex subset of $\mathbf{C}$ that contains the spectrum of A, see [**8**]. The *numerical radius* of A is defined and denoted by $r(A) = \max\{|\lambda| : \lambda \in F(A)\}$. Recall that A is Hermitian if and only if $F(A) \subset \mathbf{R}$, and that if A is normal, then $F(A)$ coincides with the convex hull of $\sigma(A)$. It is also well known that

$$\operatorname{Re} F(A) = F\left(\frac{A + A^*}{2}\right) \quad \text{and} \quad i \operatorname{Im} F(A) = F\left(\frac{A - A^*}{2}\right).$$

When $A \in \mathcal{M}_n(\mathbf{R})$, then $F(A)$ is symmetric with respect to the real axis. Also, any eigenvalue $\lambda \in \sigma(A)$ that belongs to the boundary of the numerical range, $\partial F(A)$, is a *normal eigenvalue* of A; namely, there exists a unitary matrix $U \in \mathcal{M}_n(\mathbf{C})$ such that

$$U^* A U = \lambda I_k \oplus B,$$

where k is the (algebraic) multiplicity of λ and $\lambda \notin \sigma(B)$.

For any $A \in \mathcal{M}_n(\mathbf{R})$, we write $A = S(A) + K(A)$, where

$$S(A) = \frac{A + A^t}{2} \quad \text{and} \quad K(A) = \frac{A - A^t}{2}$$

are the *symmetric part* and the *skew-symmetric part* of A, respectively. We also define *Levinger's transformation* of $A \in \mathcal{M}_n(\mathbf{R})$ as (the parametrized family of matrices)

$$\mathcal{L}(A, a) = (1 - a)A + aA^t, \quad a \in \mathbf{R}.$$

When A is entrywise nonnegative, the spectral radius and the numerical range of this transformation were considered in [**2, 3**].

An *almost skew-symmetric* matrix $A \in \mathcal{M}_n(\mathbf{R})$, $n \geq 2$, is a matrix whose symmetric part $S(A)$ has rank one. The sole nonzero eigenvalue of $S(A)$ is denoted by $\delta(A)$. For the remainder of this article, we assume that $\delta(A) > 0$; otherwise, our results are applicable to $-A$. It follows that $S(A) = ww^t$ for some $w \in \mathbf{R}^n$. Note that if $Cw \neq 0$ for some $C \in \mathcal{M}_n(\mathbf{R})$, then the congruence CAC^t is also an almost skew-symmetric matrix. Also, every principal submatrix of an almost skew-symmetric matrix is either skew-symmetric or almost skew-symmetric.

Given an almost skew-symmetric matrix $A \in \mathcal{M}_n(\mathbf{R})$ with symmetric part $S(A) = ww^t$, the *variance* of A is defined by

$$\mathrm{v}(A) = \frac{\|K(A)w\|_2^2}{\|w\|_2^2}.$$

Notice that $w/\|w\|_2$ is the unit eigenvector of $S(A)$ corresponding to the simple eigenvalue $w^t w$. It readily follows that if $\mathrm{v}(A) = 0$, then $(\delta(A), w)$ is an eigenpair for both A and A^t. That is, $\mathrm{v}(A)$ is a measure of how close $\delta(A)$ is to being a normal eigenvalue of A.

3. Eigenvalues and singular values. Under our simplifying assumption that $\delta(A) > 0$, an almost skew-symmetric matrix A is weakly positive stable, namely, for all $\lambda \in \sigma(A)$, $\operatorname{Re}\lambda \geq 0$. In the following theorem, we present another important inequality that is satisfied by every eigenvalue of A. In turn, this inequality will lead to a localization of $\sigma(A)$ via a curve in the complex plane.

Theorem 3.1. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix. Let λ be an eigenvalue of A. Then*

$$(\operatorname{Im}\lambda)^2 \operatorname{Re}\lambda \leq (\delta(A) - \operatorname{Re}\lambda)\ [\mathrm{v}(A) + \operatorname{Re}\lambda\,(\operatorname{Re}\lambda - \delta(A))]\,. \tag{3.1}$$

Proof. If $\lambda = \delta(A)$ or $\operatorname{Re}\lambda = 0$, then clearly (3.1) holds. So assume that $\lambda \neq \delta(A)$ and $\operatorname{Re}\lambda \neq 0$. Let $y = w/\|w\|_2 \in \mathbf{R}^n$ be the unit eigenvector of $S(A) = ww^t$ corresponding to the simple eigenvalue $\delta(A)$. Then, there exists a unitary $U \in \mathcal{M}_n(\mathbf{R})$, whose first column is y, such that

$$U^t S(A) U = \operatorname{diag}\{\delta(A), 0, \dots, 0\}.$$

Moreover, as $U^t K(A) U$ is real skew-symmetric, we have

$$U^t K(A) U = \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix}, \tag{3.2}$$

where $K_1 \in \mathcal{M}_{n-1}(\mathbf{R})$ is skew-symmetric and $u \in \mathbf{R}^{n-1}$. Consequently,

$$U^t A U = \begin{bmatrix} \delta(A) & -u^t \\ u & K_1 \end{bmatrix}$$

and the matrix

$$U^t (A - \lambda I) U = \begin{bmatrix} \delta(A) - \lambda & -u^t \\ u & K_1 - \lambda I \end{bmatrix}$$

is singular. Thus, its Schur complement of the leading entry is singular [**7**, p. 21], i.e., $0 \in \sigma(E)$, where

$$E = K_1 - \lambda I + \frac{1}{\delta(A) - \lambda}\, uu^t.$$

The symmetric and skew-symmetric parts of E are

$$M = \frac{(\delta(A) - \operatorname{Re}\lambda)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}\ uu^t - \operatorname{Re}\lambda\, I$$

and

$$N = K_1 - i \operatorname{Im}\lambda\, I + \frac{i \operatorname{Im}\lambda}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} uu^t, \tag{3.3}$$

respectively. Since $\sigma(E) \subseteq F(E)$ and $F(M) = \operatorname{Re} F(E)$ (see [**8**, Properties 1.2.5, 1.2.6]) it follows that $0 \in F(M)$, which, in turn, implies

$$\operatorname{Re}\lambda \in \frac{(\delta(A) - \operatorname{Re}\lambda)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} F(uu^t).$$

Since $F(uu^t)$ coincides with the interval $[0, u^t u]$ and since $\delta(A) > 0$, we have that

$$\operatorname{Re}\lambda \leq \frac{(\delta(A) - \operatorname{Re}\lambda)(u^t u)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}$$

or, equivalently,

$$(\operatorname{Im}\lambda)^2 \leq (\delta(A) - \operatorname{Re}\lambda)\left(\frac{u^t u}{\operatorname{Re}\lambda} - (\delta(A) - \operatorname{Re}\lambda)\right).$$

Denoting by e_1 the first standard basis vector in $\mathbf{R}^n$, observe that

$$\begin{aligned} u^t u &= \left\| \begin{bmatrix} 0 \\ u \end{bmatrix} \right\|_2^2 = \left\| U \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix} e_1 \right\|_2^2 \\ &= \left\| U \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix} U^t U e_1 \right\|_2^2 \\ &= \|K(A)y\|_2^2 = \mathrm{v}(A), \end{aligned} \tag{3.4}$$

completing the proof of (3.1). □

Corollary 3.2. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix with $\mathrm{v}(A) < \delta^2(A)/4$. Then for every eigenvalue λ of A,*

$$\operatorname{Re}\lambda \notin \left(\frac{\delta(A) - \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2}, \frac{\delta(A) + \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2}\right).$$

Proof. Suppose that λ is an eigenvalue of A. Since we readily have that

$$0 \leq \frac{\delta(A) - \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2} \quad \text{and} \quad \delta(A) \geq \frac{\delta(A) + \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2},$$

let us consider the case $\mathrm{Re}\,\lambda \neq 0$ and $\lambda \neq \delta(A)$. Then, by (3.1),

$$(\delta(A) - \mathrm{Re}\,\lambda)\left(\frac{\mathrm{v}(A)}{\mathrm{Re}\,\lambda} + \mathrm{Re}\,\lambda - \delta(A)\right) \geq 0,$$

that is, as $0 \leq \mathrm{Re}\,\lambda \leq \delta(A)$,

$$(\mathrm{Re}\,\lambda)^2 - \delta(A)\mathrm{Re}\,\lambda + \mathrm{v}(A) \geq 0.$$

Since $\mathrm{v}(A) < \delta^2(A)/4$, the proof is complete. $\square$

Definition 3.3. Prompted by (3.1), we define the *shell* of an almost skew-symmetric matrix $A \in \mathcal{M}_n(\mathbf{R})$ to be the curve in the complex plane given by

$$\Gamma(A) = \left\{ x + iy \in \mathbf{C} : x, y \in \mathbf{R} \text{ and } y^2 = (\delta(A) - x)\left(\frac{\mathrm{v}(A)}{x} + x - \delta(A)\right) \right\}.$$

The curve $\Gamma(A)$ depends only on the variance $\mathrm{v}(A)$ and the nonzero eigenvalue of $S(A)$. It is symmetric with respect to the real axis which it intercepts at $\delta(A)$. If $\mathrm{v}(A) \geq \delta^2(A)/4$, then $\Gamma(A)$ consists of an unbounded branch. If $\mathrm{v}(A) < \delta^2(A)/4$, then $\Gamma(A)$ consists of an unbounded branch and a bounded branch, and it also intercepts the real axis at the points

$$\frac{\delta(A) \pm \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2}.$$

By Theorem 3.1, the shell $\Gamma(A)$ yields a localization of the spectrum of A, as specified by (3.1). The various possible configurations of the shell of an almost skew-symmetric matrix are illustrated in the next example.

Example 3.4. We have taken 5×5 almost skew-symmetric matrices A, B and C with variances $\mathrm{v}(A) = 1.75$, $\mathrm{v}(B) = 4.5$ and $\mathrm{v}(C) = 1$. Also $\delta(A) = 4$, $\delta(B) = 4$ and $\delta(C) = 2$. The eigenvalues of each matrix are marked with $+$'s. The shell $\Gamma(A)$ consists of one bounded and one unbounded branch. The bounded branch surrounds a real eigenvalue of A and the unbounded branch isolates the rest of the spectrum. The shell $\Gamma(B)$ is connected and all the eigenvalues of B are located in the region between $\Gamma(B)$ and the imaginary axis. The shell $\Gamma(C)$ can be loosely described as an inverted α-curve.

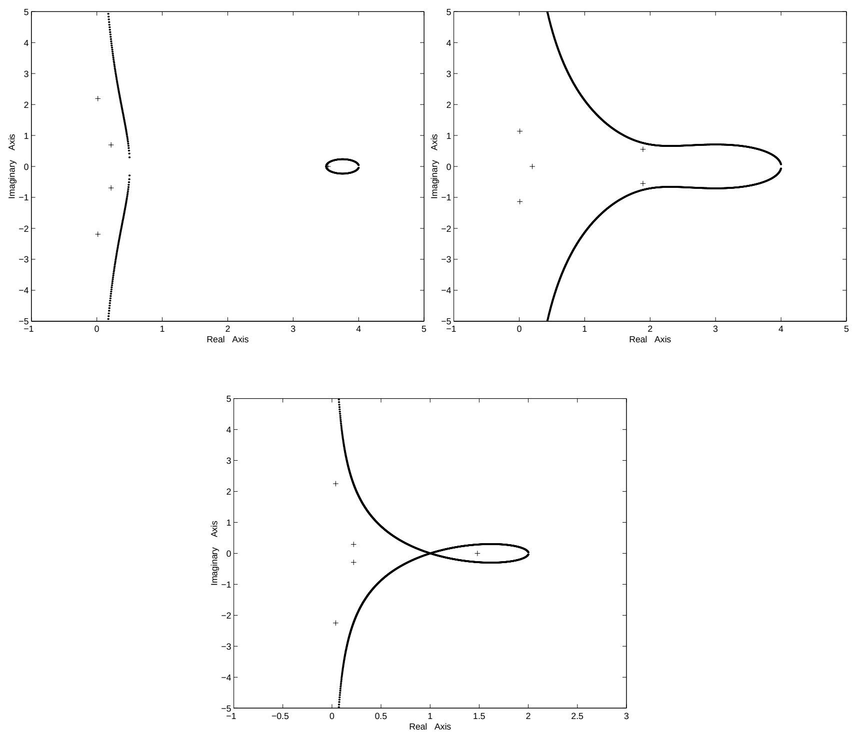

FIGURE 1. The shells $\Gamma(A)$, $\Gamma(B)$ and $\Gamma(C)$.

In the following theorem, we examine the possibility of eigenvalues of A lying on $\Gamma(A)$.

Theorem 3.5. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix, and let $\lambda \in \sigma(A) \cap \Gamma(A)$. Then either $\lambda = \delta(A)$, or*

$$\lambda = \frac{\delta(A) \pm \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2} \quad \text{and} \quad \mathrm{v}(A) < \frac{\delta^2(A)}{4}.$$

Proof. Let $\lambda \neq \delta(A)$ be an eigenvalue of A with $\operatorname{Re}\lambda \neq 0$. Let also E, M and N be as defined in the proof of Theorem 3.1. Equality in (3.1), namely,

$$(\operatorname{Im}\lambda)^2 = (\delta(A) - \operatorname{Re}\lambda)\left(\frac{\mathrm{v}(A)}{\operatorname{Re}\lambda} + \operatorname{Re}\lambda - \delta(A))\right)$$

holds if and only if

$$\operatorname{Re}\lambda = \frac{\delta(A) - \operatorname{Re}\lambda}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}\, u^t u$$

or, equivalently, if and only if the matrix M is singular negative semi-definite. In this case, the eigenvalue $0 \in \sigma(M)$ is simple and corresponds to the eigenvector u that appears in (3.2). Moreover, the matrix E is singular and $0 \in \partial F(E)$ (the boundary of the numerical range) because $\operatorname{Re} F(E) = F(M)$. Thus 0 must be a normal eigenvalue of E (see [**8**, Theorem 1.6.6]) and every corresponding eigenvector belongs to $\operatorname{null}(M) \cap \operatorname{null}(N) = \operatorname{span}\{u\}$. Hence, u is an eigenvector of N in (3.3) corresponding to the eigenvalue 0. Furthermore, the vector u is an eigenvector of the rank one matrix

$$\frac{\operatorname{Im}\lambda}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}\, uu^t$$

corresponding to the simple eigenvalue

$$\frac{\operatorname{Im}\lambda\,(u^t u)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}.$$

As a consequence, the quantity

$$i\operatorname{Im}\lambda - \frac{i\operatorname{Im}\lambda(u^t u)}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} = i\operatorname{Im}\lambda\left(1 - \frac{u^t u}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}\right)$$

is an eigenvalue of the matrix K_1 appearing in (3.2) with corresponding eigenvector u. Thus,

$$\frac{u^t K_1 u}{u^t u} = i\operatorname{Im}\lambda\left(1 - \frac{u^t u}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}\right).$$

The same arguments applied to $\bar{\lambda} \in \sigma(A)$ yield

$$\frac{u^t K_1 u}{u^t u} = i \operatorname{Im} \bar{\lambda}\left(1 - \frac{u^t u}{(\delta(A) - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\bar{\lambda})^2}\right),$$

and hence $\operatorname{Im}\lambda = 0$. Thus the existence of eigenvalues on $\Gamma(A)$ hinges on the equation in λ

$$(\delta(A) - \lambda)\left(\frac{\mathrm{v}(A)}{\lambda} + \lambda - \delta(A)\right)$$

having real solutions. If $\mathrm{v}(A) > \delta^2(A)/4$, this equation has no real solutions. Otherwise, the real solutions lead to eigenvalues as stated in the theorem. $\square$

Remark 3.6. We note that many of the results herein can be generalized to *almost skew-Hermitian matrices*, that is, complex matrices whose Hermitian part has rank one. In particular, the eigenvalue inequalities can be stated for almost skew-Hermitian matrices with the introduction of a constant $\gamma \in \mathbf{R}$ representing a shift of the shell along the imaginary axis. This constant would not be known a priori; it would be introduced in (3.2), where the zero (1,1) entry of $\frac{1}{2}U^*(A - A^*)U$ would be replaced by $\gamma\, i$.

For any matrix $X \in \mathcal{M}_n(\mathbf{C})$, we order and denote its singular values by

$$\sigma_1(X) \geq \ldots \geq \sigma_n(X) \geq 0.$$

Recall that $\sigma_1(X) = \|X\|_2$ (the spectral norm) and

$$\sum_{j=1}^{n} \sigma_j^2(X) = \operatorname{trace}(X^t X) = \|X\|_F^2 \quad \text{(the Frobenius norm)}.$$

The following result is a generalization of [**6**, Proposition 2.4].

Theorem 3.7. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix. Consider the matrix K_1 in* (3.2)*, and let $\|K_1\|_F$ denote its Frobenius norm. Then the following hold:*

(i) $\sigma_1(A) \geq \delta(A)$.

(ii) $\|A\|_F^2 = \sum_{j=1}^n \sigma_j^2(A) = \delta^2(A) + 2\mathrm{v}(A) + \|K_1\|_F^2$.

(iii) $\sigma_n^2(A) \leq \frac{\|K_1\|_F^2 + 2\mathrm{v}(A)}{n-1}$.

Proof. By [**1**, Proposition III.5.1, p. 73], we have

$$\delta(A) = w^t w = \sigma_1(S(A)) \leq \sigma_1(A),$$

proving (i). Referring to (3.2) in the proof of Theorem 3.1 and the notation thereof, and since the singular values of A and U^tAU coincide, we have

$$\begin{aligned}\sum_{j=1}^n \sigma_j^2(A) &= \mathrm{trace}\,\big((U^tAU)^t(U^tAU)\big)\\ &= \mathrm{trace}\begin{pmatrix} \delta^2(A) + u^tu & \delta(A)u^t + u^tK_1 \\ -\delta(A)u - K_1u & uu^t - K_1^2 \end{pmatrix}\\ &= \delta^2(A) + u^tu + \mathrm{trace}\,(uu^t) - \mathrm{trace}\,(K_1^2).\end{aligned}$$

Then (ii) follows by recalling that $\mathrm{trace}(uu^t) = u^tu = \mathrm{v}(A)$ (see (3.4)) and observing that $\|K_1\|_F^2 = -\mathrm{trace}\,(K_1^2)$. To prove (iii), we use parts (i) and (ii) to obtain

$$\begin{aligned}(n-1)\,\sigma_n^2(A) + \delta^2(A) &\leq (n-1)\,\sigma_n^2(A) + \sigma_1^2(A)\\ &\leq \sum_{j=1}^n \sigma_j^2(A) \leq \delta^2(A) + 2\mathrm{v}(A) + \|K_1\|_F^2. \qquad \square\end{aligned}$$

4. Inverses, principal pivot transforms and Schur complements. Almost skew-symmetric matrices are not necessarily invertible, e.g., the all ones matrix. Interestingly, however, the class of invertible almost skew-symmetric matrices is inverse closed. In what follows, we abbreviate $(A^{-1})^t$ by A^{-t}.

Theorem 4.1. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an invertible almost skew-symmetric matrix. Then A^{-1} is also almost skew-symmetric and $K(A^{-1}) = K(A^{-t}AA^{-t})$.*

Proof. We have that $S(A) = ww^t$ for some $w \in \mathbf{R}^n$. Thus $A = -A^t + 2ww^t$ and as $-A^t$ is by assumption invertible, by the Sherman-Morrison formula for the inverse of rank-one perturbations of invertible matrices (see, e.g., [**7**, p. 19]) we have

$$A^{-1} = -A^{-t} - \frac{2}{1 + 2w^t(-A^{-t})w}(-A^{-t})ww^t(-A^{-t})$$

or, equivalently,

$$S(A^{-1}) = \frac{1}{2w^tA^{-t}w - 1}A^{-t}ww^tA^{-t}. \tag{4.1}$$

It is now clear that $S(A^{-1})$ has rank one. Also the quantity $2w^tA^{-t}w-1$ in (4.1) must be positive; otherwise, $S(A^{-1})$ would be negative semi-definite, contradicting the fact that A and thus A^{-1} are positive stable. Thus A^{-1} is almost skew-symmetric. Since, of course, $S(A^{-1})$ is symmetric, we also have

$$A^{-1}ww^tA^{-1} = A^{-t}ww^tA^{-t}$$

or, equivalently,

$$AA^{-t}S(A) = S(A)A^{-1}A^t.$$

It follows that

$$\frac{AA^{-t}A + A}{2} = \frac{A^t + A^tA^{-1}A^t}{2},$$

which implies that $K(A) = K(A^tA^{-1}A^t)$. Considering A^{-1} instead of A in this argument completes the proof. □

When A and $K(A)$ are both invertible, more can be said about the symmetric, the skew-symmetric part and the variance of A^{-1} in relation to the corresponding quantities of A.

Theorem 4.2. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an invertible almost skew-symmetric matrix with $S(A) = ww^t$ and assume that $K(A)$ is also invertible. Then the following hold*:

(i) $K(A^{-1}) = K(A)^{-1}$.

(ii) $S(A^{-1}) = (K(A)^{-1}w)\,(K(A)^{-1}w)^t$.

(iii) $\mathrm{v}(A^{-1}) = \|K(A)^{-2}w\|_2^2 \,/\, \|K(A)^{-1}w\|_2^2$.

Proof. Since $K(A)^{-1}$ exists, applying the Sherman-Morrison formula to $A = K(A) + ww^t$, we obtain

$$\begin{aligned} A^{-1} &= K(A)^{-1} - (1 + w^t K(A)^{-1} w)^{-1} \; K(A)^{-1} S(A) K(A)^{-1} \\ &= K(A)^{-1} - K(A)^{-1} ww^t K(A)^{-t}. \end{aligned}$$

Since $K(A)^{-1}$ is skew-symmetric and since $K(A)^{-1}ww^tK(A)^{-1}$ has rank one and is symmetric, the expressions in (i) and (ii) follow by the uniqueness of the decomposition of a matrix into symmetric and skew-symmetric summons. Part (iii) follows from (i), (ii) and the definition of variance. □

As almost skew-symmetric matrices bequeath their low rank symmetric part to their principal submatrices and inverses, and because of the well-known connection of submatrices of inverses and Schur complements, one expects that Schur complements of almost skew-symmetric matrices are (almost) skew-symmetric. This is indeed true; however, there is a more general matrix transformation that preserves almost skew-symmetry, namely, the principal pivot transform (also known as 'exchange' or 'sweep operator' in statistics; see [**12**]). It can be defined relative to any principal submatrix of A but for reasons of brevity, we define it here relative to the leading block in the partition of $A \in \mathcal{M}_n(\mathbf{C})$ given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.2}$$

where A_{11} is an invertible submatrix. Then, the *principal pivot transform* of A relative to A_{11} is defined by

$$\operatorname{ppt}(A, A_{11}) = \begin{pmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12} \\ A_{21}(A_{11})^{-1} & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{pmatrix}. \tag{4.3}$$

The matrices A and $\operatorname{ppt}(A, A_{11})$ are related as follows: If $x = (x_1^t, x_2^t)^t$ and $y = (y_1^t, y_2^t)^t$ in $\mathbf{C}^n$ are partitioned conformally to A in (4.2), then

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{if and only if} \quad \operatorname{ppt}(A, A_{11}) \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}.$$

Notice that the trailing principal submatrix of $\mathrm{ppt}\,(A, A_{11})$ coincides with the Schur complement of A_{11} in A, denoted by A/A_{11}.

Theorem 4.3. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be partitioned as in (4.2). Suppose A_{11} is invertible and $\operatorname{rank} S(A) = 1$. Then $\operatorname{rank} S(\mathit{ppt}\,(A, A_{11})) = 1$. In particular, if A is an almost skew-symmetric matrix, then so is $\mathit{ppt}\,(A, A_{11})$.*

Proof. As shown in [**12**, Lemma 3.4], $\mathrm{ppt}\,(A, A_{11})$ admits the factorization $\mathrm{ppt}\,(A, A_{11}) = C_1 C_2^{-1}$, where

$$C_1 = \begin{pmatrix} I & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & I \end{pmatrix}.$$

Consider now the congruence of $\mathrm{ppt}\,(A, A_{11})$ given by

$$C_2^t\, \mathrm{ppt}\,(A, A_{11})\, C_2 = C_2^t\, C_1.$$

Observe that

$$C_2^t\, C_1 + C_1^t C_2 = \begin{pmatrix} A_{11} + A_{11}^t & A_{12} + A_{21}^t \\ A_{12}^t + A_{21} & A_{22} + A_{22}^t \end{pmatrix} = 2S(A). \tag{4.4}$$

Thus $\operatorname{rank} S(\mathrm{ppt}(A, A_{11}) = \operatorname{rank} S(C_2^t\, C_1) = \operatorname{rank} S(A)$. Suppose now that A is almost skew-symmetric and $S(A) = ww^t$. As C_2 is invertible, it follows that $C_2^t\, w \neq 0$. Thus $\mathrm{ppt}(A, A_{11})$ is almost skew-symmetric with symmetric part $S(\mathrm{ppt}(A, A_{11})) = C_2^t\, w\, w^t\, C_2$. □

The following corollary is due to the fact that the Schur complement appears as a principal submatrix in the principal pivot transform.

Corollary 4.4. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix partitioned as in* (4.2) *and such that A_{11} is invertible. Then $S(A/A_{11})$ has rank at most one.*

Example 4.5. In this example we illustrate that a Schur complement of an almost skew-symmetric matrix can indeed be skew-symmetric. Let

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 5 \\ 2 & -1 & 2 \end{pmatrix}.$$

The Schur complement of the leading submatrix $A_{11} = (2)$ in A is the skew symmetric matrix

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}.$$

This section concludes with the following new result on pseudo-tournaments.

Corollary 4.6. *Let $T \in \mathcal{M}_n(\mathbf{R})$ and suppose C is a principal pivot transform of $T + \frac{1}{2}I$. Then, T is a pseudo-tournament if and only if $C - \frac{1}{2}I$ is a pseudo-tournament. In particular, suppose that $\frac{1}{2} \notin \sigma(T)$. Then, T is a pseudo-tournament if and only if $(T + \frac{1}{2}I)^{-1} - \frac{1}{2}I$ is a pseudo-tournament.*

Proof. If T is a pseudo-tournament then $\operatorname{rank} S(T + \frac{1}{2}I) = 1$ and thus, by Theorem 4.3, $\operatorname{rank} S(C) = 1$. It follows that $C - \frac{1}{2}I$ is a pseudo-tournament. Conversely, if $C - \frac{1}{2}I$ is a pseudo-tournament, then $\operatorname{rank} S(C) = 1$. The principal pivot transform is an involution [**12**, Theorem 3.1]; that is, $T + \frac{1}{2}I$ is a principal pivot transform of C and thus, by Theorem 4.3, it must also have rank one symmetric part. It follows that T is a pseudo-tournament. The second part of the theorem follows similarly. □

5. Levinger's transformation and the numerical range. Given an almost skew-symmetric matrix $A \in \mathcal{M}_n(\mathbf{R})$ with $S(A) = ww^t$, recall that Levinger's transformation of A is given by $\mathcal{L}(A, a) = (1-a)A + aA^t$, $a \in \mathbf{R}$. Observe that

$$\mathcal{L}(A, a) = S(A) + (1 - 2a)K(A), \tag{5.1}$$

which means that, for every $a \in \mathbf{R}$, $\mathcal{L}(A, a)$ is also an almost skew-symmetric matrix with $\mathcal{L}(A, 0) = A$, $\mathcal{L}(A, 1) = A^t$ and $\mathcal{L}(A, \frac{1}{2}) = S(A)$. Moreover,

$$\mathcal{L}(A, \tfrac{1}{2} + a) = \mathcal{L}(A^t, \tfrac{1}{2} - a). \tag{5.2}$$

From (5.1) we have that $S(\mathcal{L}(A, a)) = ww^t$ and

$$\mathrm{v}\,(\mathcal{L}(A, a)) = \frac{\|K(\mathcal{L}(A, a))\,w\|_2^2}{\|w\|_2^2} = (1 - 2a)^2 \mathrm{v}(A).$$

Because of symmetry in (5.2) with respect to a, in the remainder of this discussion we need only consider $a \in (-\infty, \frac{1}{2}]$. As a consequence of the above, the results of the previous section can be adapted for $\mathcal{L}(A, a)$ ($a \in (-\infty, \frac{1}{2}]$). We summarize them in the following theorem.

Theorem 5.1. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix. If λ_a is an eigenvalue of $\mathcal{L}(a, A)$, $a \in (-\infty, \frac{1}{2}]$, then*

$$(\operatorname{Im} \lambda_a)^2 \, Re\, \lambda_a \leq (\delta(A) - Re\, \lambda_a) \left[(1-2a)^2 \, v(A) + Re\, \lambda_a \,(Re\, \lambda_a - \delta(A))\right].$$

Moreover, for every $a \in (-\infty, \frac{1}{2}]$ such that $(1-2a)^2 \; v(A) < \delta^2(A)/4$, the real part of every eigenvalue of $\mathcal{L}(a, A)$ lies outside the interval

$$\left(\frac{\delta(A) - \sqrt{\delta^2(A) - 4(1-2a)^2 \, \mathrm{v}(A)}}{2}, \; \frac{\delta(A) + \sqrt{\delta^2(A) - 4(1-2a)^2 \, \mathrm{v}(A)}}{2} \right).$$

Note that the condition $(1-2a)^2 \, \mathrm{v}(A) < \delta^2(A)/4$ in the above theorem is always true when a is "close enough" to $\frac{1}{2}$. In fact, as $a \to \frac{1}{2}$, the open interval in the second part of the above theorem converges to $(0, \delta(A))$. Generally speaking, Theorem 5.1 provides information (bounds) regarding the behaviour of $\sigma(A)$ under Levinger's transformation. In that respect, we also have the following result.

Proposition 5.2. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix with score variance $v(A)$. Then for every $a \in (-\infty, \frac{1}{2}]$ such that $(1-2a)^2 v(A) < \delta^2(A)/4$, the matrix $\mathcal{L}(a, A)$ has a real eigenvalue*

$$\lambda(a, A) \geq \frac{\delta(A) + \sqrt{\delta^2(A) - 4(1-2a)^2 \, v(A)}}{2}$$

and $n-1$ complex eigenvalues whose real parts are not greater than

$$\frac{\delta(A) - \sqrt{\delta^2(A) - 4(1-2a)^2 \, \mathrm{v}(A)}}{2}.$$

Proof. The conclusions hold for $a = \frac{1}{2}$ as $\mathcal{L}(A, \frac{1}{2}) = S(A)$ and $\mathrm{v}\,(\mathcal{L}(A, \frac{1}{2})) = 0$. Moreover, the set

$$\{a \in (-\infty, \tfrac{1}{2}] : (1-2a)^2 \mathrm{v}(A) < \delta^2(A)/4\}$$

coincides with the interval $(a_0, \frac{1}{2}]$, where

$$a_0 = \frac{2 - \sqrt{\delta^2(A)/\mathrm{v}(A)}}{4}.$$

Since the eigenvalue $\delta(A) \in \sigma(\mathcal{L}(A, \frac{1}{2}))$ is simple, by Theorem 5.1 and the continuity of the eigenvalues of $\mathcal{L}(A, a)$ with respect to a, the proof is complete. □

Corollary 5.3. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix with variance $v(A) < \delta^2(A)/4$. Then A has exactly one real eigenvalue in the interval*

$$\left[\frac{\delta(A) + \sqrt{\delta^2(A) - 4v(A)}}{2},\, \delta(A)\right]$$

and $n-1$ complex eigenvalues with real parts in the interval

$$\left[0,\, \frac{\delta(A) - \sqrt{\delta^2(A) - 4\mathrm{v}(A)}}{2}\right].$$

Levinger's transformation allows us to address a question raised in [**9**]. Consider a *generalized tournament* $T \in \mathcal{M}_n(\mathbf{R})$, namely, a matrix with nonnegative entries satisfying $T + T^t = ee^t - I$, where e denotes the all ones vector. The *score vector* of T is defined as $s = Te$. The *score variance* of T is defined as

$$\mathrm{sv}\,(T) = \frac{s^t s}{n} - \frac{(n-1)^2}{4}.$$

The problem posed in [**9**] can be stated in terms of $\mathrm{sv}\,(T)$ as follows: *Does $\rho(\mathcal{L}(T, a)) > (n-2)/2$ imply that $sv\,(\mathcal{L}(T, a)) < (n-1)/4$?* The converse is known to be true.

As noted for the variance of Levinger's transformation of almost skew-symmetric matrices, it can also be seen that the score variance of $\mathcal{L}(T, a))$ is given by

$$\mathrm{sv}\,(\mathcal{L}(A, a)) = (1 - 2a)^2\, \mathrm{sv}\,(T).$$

Also note that for every $a \in [0, \frac{1}{2}]$, the matrix $\mathcal{L}(T, a)$ is a generalized tournament. Exploiting these facts, we can construct the following example that yields a negative answer to the above question.

Example 5.4. Consider the tournament matrix

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The score variance of T is $\mathrm{sv}\,(T) = 1.2$. For $a = 0.04$, $\mathrm{sv}\,(\mathcal{L}(T, 0.04)) = 1.0157$ and $\rho(\mathcal{L}(T, 0.04)) = 1.5138$. That is, although $\rho(\mathcal{L}(T, 0.04)) > 1.5$, we have $\mathrm{sv}(\mathcal{L}(T, 0.04)) > 1$.

Let us now look at the behavior of normal eigenvalues under Levinger's transformation, first for real matrices in general, and then for almost skew-symmetric ones.

Theorem 5.5. *Let $A \in \mathcal{M}_n(\mathbf{R})$ have normal eigenvalues $\lambda_j = \mu_j + i\nu_j$, where $\mu_j, \nu_j \in \mathbf{R}$ have algebraic multiplicities m_j, $j = 1, \ldots, k$. Then for every $a \in (-\infty, \frac{1}{2}]$, the quantities $\lambda_j(a) = \mu_j + i(1 - 2a)\nu_j$ are normal eigenvalues of $\mathcal{L}(A, a)$ with algebraic multiplicities at least m_j, $j = 1, \ldots, k$.*

Proof. By the definition of normal eigenvalues, there exists unitary $V \in \mathcal{M}_n(\mathbf{C})$ such that

$$V^*AV = V^*S(A)V + V^*K(A)V = \lambda_1 I_{m_1} \oplus \ldots \oplus \lambda_k I_{m_k} \oplus B,$$

for some B of order $n - (m_1 + \ldots + m_k)$. Thus

$$\begin{aligned} V^*\mathcal{L}(A, a)V &= V^*S(A)V + (1 - 2a)V^*K(A)V \\ &= \lambda_1(a) I_{m_1} \oplus \ldots \oplus \lambda_k(a) I_{m_k} \oplus \mathcal{L}(B, a). \qquad \square \end{aligned}$$

It follows by the above proposition that the normal eigenvalues of any $A \in \mathcal{M}_n(\mathbf{R})$ move vertically under Levinger's transformation. At the same time, we know that for every $a \in (-\infty, \frac{1}{2}]$, $\sigma(\mathcal{L}(A,a) \subset F(\mathcal{L}(A,a))$ with $\mathrm{Re}\, F(\mathcal{L}(A,a)) = F(S(A)) = [0, \delta(A)]$. That is, the real parts of the eigenvalues of $\mathcal{L}(A,a)$ always belong to $[0, \delta(A)]$. Therefore we have shown the following result, which is in agreement with [**2**, Theorem 3.2 (ii)].

Corollary 5.6. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix and λ_0 be a normal eigenvalue of A. Then either $\lambda_0 = \delta(A)$, which remains a normal eigenvalue of $\mathcal{L}(A,a)$ for every $a \in (-\infty, \frac{1}{2}]$, or $Re\, \lambda = 0$.*

Moreover, as seen in the next result, the line $\mathrm{Re}\, z = 0$ intersects non-trivially the boundary of the numerical range of an almost skew-symmetric matrix.

Proposition 5.7. *Let $A \in \mathcal{M}_n(\mathbf{R})$ be an almost skew-symmetric matrix with numerical range $F(A)$. Then $\partial F(A) \cap \{z \in \mathbf{C} : Re\, z = 0\}$ is a line segment.*

Proof. As in the proof of Theorem 3.1, there exists unitary $U \in \mathcal{M}_n(\mathbf{R})$ such that

$$U^t A U = \begin{pmatrix} \delta(A) & 0 \\ 0 & 0_{n-1} \end{pmatrix} + \begin{pmatrix} 0 & -u^t \\ u & K_1 \end{pmatrix},$$

where K_1 is skew-symmetric. It is clear that for every unit $x \in \mathbf{C}^n$, $\mathrm{Re}\,(x^* U^t A U x) = 0$ if and only if $x = [0, x_1^t]^t$, for some $x_1 \in \mathbf{C}^{n-1}$. Thus, $\partial F(A) \cap \{z \in \mathbf{C} : \mathrm{Re}\, z = 0\} = F(K_1)$, proving the claim as the numerical range of any skew-symmetric matrix is a (vertical) line segment. $\square$

Remark 5.8. Referring to the above proposition and its proof, notice that

$$\partial F(A) \cap \{z \in \mathbf{C} : \mathrm{Re}\, z = 0\} = F(K_1) = \{0\}$$

if and only if $K_1 = 0$ or, equivalently,

$$U^t A U = \begin{pmatrix} \delta(A) & -u^t \\ u & 0 \end{pmatrix}.$$

In such a case, the characteristic polynomial of A is $\lambda^{n-2}(\lambda^2 - \delta(A)\lambda + u^t u)$.

Example 5.9. The distinctive feature of the numerical range described in the above proposition is illustrated by the following figures. The numerical ranges are shaded and they correspond to the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ -1 & 2 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

which have $\mathrm{v}\,(A_1) = 3.5$, $\delta(A_1) = 4$ and $\mathrm{v}\,(A_2) = 1$, $\delta(A_2) = 2$.

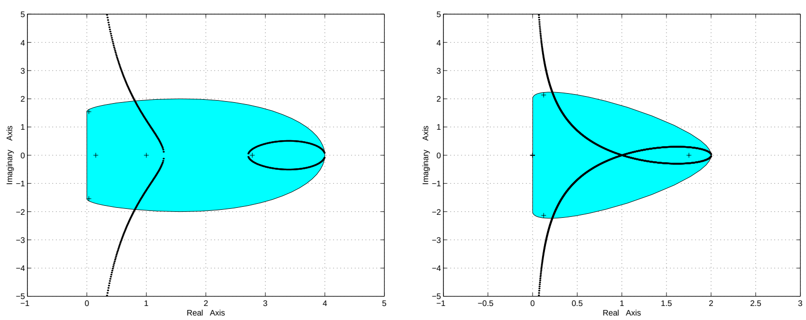

FIGURE 2. The numerical ranges (shaded) and shells of A_1 and A_2.

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