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GENERIC FORMAL FIBERS OF POLYNOMIAL RING EXTENSIONS

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ABSTRACT. In this paper we explore the relationship between the dimension of the generic formal fiber of a Noetherian local domain R and the dimension of the generic formal fiber of the domain $R[X]$ localized at (M_R, X) where M_R is the maximal ideal of R and X is an indeterminate. Specifically, we show that if R is a universally catenary local Noetherian domain such that the dimension of the generic formal fiber of $R[X]_{(M_R,X)}$ is $\dim R$, then the dimension of the generic formal fiber of R is $\dim R - 1$. We also provide counter-examples showing that the converse does not hold.

1. Introduction. Let (R, M_R) be a local Noetherian domain with maximal ideal M_R, quotient field K and M_R-adic completion $\hat{R}$. The generic formal fiber ring of R is defined to be $\hat{R} \otimes_R K$. The dimension of the generic formal fiber of R is the Krull dimension of the generic formal fiber ring of R. In this setting we will denote the dimension of the generic formal fiber of R by $\alpha(R)$. Suppose that $\hat{P}$ is a prime ideal of $\hat{R}$ satisfying $\hat{P} \cap R = (0)$. Then we say that $\hat{P}$ is in the generic formal fiber of R.

If (R, M_R) is a complete local domain of dimension $n \geq 1$ which contains a field, Matsumura shows in [**4**, Example 1] that the dimension of the generic formal fiber of the localized polynomial ring $R[X]_{(M_R,X)}$ is $n - 1$. This implies that for every local Noetherian domain (A, M_A) of dimension n the dimension of the generic formal fiber ring of $A[X]_{(M_A,X)}$ is at least $n - 1$. This fact seems to indicate that there is little or no relationship between $\alpha(A)$ and $\alpha(A[X]_{(M_A,X)})$ except possibly in the case where $\alpha(A) = n - 1 = \dim(A) - 1$.

Heinzer, Rotthaus and Sally informally posed the following conjecture.

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Conjecture 1. *Let R be a universally catenary local Noetherian domain, M_R the maximal ideal of R and X an indeterminate. Then*

$$\alpha(R[X]_{(M_R,X)}) = \dim R \quad \textit{if and only if} \quad \alpha(R) = \dim R - 1.$$

We show that, while the forward direction of the conjecture is true, the backward direction is not. The failure of the backward direction is caused by an unusual phenomenon, namely, the existence of rings R such that there exists a prime ideal $\hat{P}$ in the generic formal fiber of R with the property that the quotient ring $\hat{R}/\hat{P}$ is algebraic over R. We produce examples of excellent local Noetherian domains of all dimensions which have a nonzero prime ideal with this property in the generic formal fiber.

2. The forward direction.

Theorem 2. *Let (R, M_R) be a universally catenary local Noetherian domain. If $\alpha(R[X]_{(M_R,X)}) = \dim R$, then $\alpha(R) = \dim R - 1$.*

Proof. Let Q be a prime ideal of $\hat{R}[[X]]$ with $htQ = \dim R$ and $Q \cap R[X] = 0$. Let $(\hat{R}[[X]]/Q)^{\sim}$ be the normalization of $(\hat{R}[[X]]/Q)$ and let

$$W = Q(R)(X) \cap \left(\frac{\hat{R}[[X]]}{Q}\right)^{\sim}.$$

Since $(\hat{R}[[X]]/Q)^{\sim}$ is a DVR, W is a DVR that birationally dominates $R[X]_{(M_R,X)}$. Moreover, by Theorem 1.6 in [**2**], $W/(M_R, X)W$ is a finite-dimensional vector space over $R[X]/(M_R, X) = R/M_R$. Let $V = W \cap Q(R)$. Then V is a DVR and we have maps

$$\frac{R}{M_R} \hookrightarrow \frac{V}{M_R V} \longrightarrow \frac{W}{(M_R, X)W}.$$

Now we claim $V/M_R V$ is a finite-dimensional vector space over $R/M_R = K$. To see this, let K_V and K_W be the residue fields of V and W, respectively. Since $K \subseteq K_V \subseteq K_W$ and K_W is a finite K-vector space,

K_V is also a finite K-vector space. Now V/M_RV is an Artinian ring and so $l(V/M_RV) < \infty$. Let

$$(0) = I_0 \subset I_1 \subset \cdots \subset I_s = \frac{V}{M_RV}$$

be a normal series of distinct ideals. So $I_{i+1}/I_i \cong K_V$ for every $i = 0, 1, \ldots, s-1$. This implies that the sequence

$$0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \frac{I_2}{I_1} \longrightarrow 0$$

is exact and $\dim_K I_1 + \dim_K I_2/I_1 = \dim_K I_2 < \infty$. By induction, $\dim_K(V/M_RV) < \infty$. By Theorem 2.5 in [**2**], there is a prime ideal P of $\hat{R}$ in the generic formal fiber of R corresponding to V. Since $\dim V = 1$, we have $\operatorname{ht} P = \dim R - 1$. It follows that $\alpha(R) = \dim R - 1$. $\square$

3. A counterexample to the converse. The converse of Theorem 2, however, is not true. In this section we provide a counterexample. Specifically, for each integer $n \geq 2$, we construct an excellent regular local ring (R, M_R) such that $\dim R = n$, $\alpha(R) = n-1$ and $\alpha(R[X]_{(M_R,X)}) = n-1$.

Example. Let K be a field of characteristic 0 and t a variable. Fix an integer $n \geq 2$. Let $\{w_i\}_{i \in I} \subseteq K[[t]]$ be a transcendence basis of $K[[t]]$ over $K[t]$ and let $w_1, w_2, \ldots, w_{n-1} \in \{w_i\}_{i \in I}$ be $n-1$ distinct elements in the transcendence basis. We define $\Omega = \{w_i\}_{i \in I} - \{w_1, w_2, \ldots, w_{n-1}\}$ and

$$D = K(t)(w_i \mid w_i \in \Omega) \cap K[[t]].$$

Note that D is a DVR with $\hat{D} = K[[t]]$. Since every DVR containing a field of characteristic 0 is excellent, we have that D is an excellent ring. Moreover, the transcendence degree of $K[[t]]$ over D is $n-1$. We define $R = D[y_1, \ldots, y_{n-1}]_{(t, y_1, \ldots, y_{n-1})}$ to be the localized polynomial ring in $n-1$ variables over D. Then R is an excellent RLR with $\hat{R} = K[[t, y_1, \ldots, y_{n-1}]]$.

Lemma 3. *Let R be defined as in the example above, and let $\hat{P} = (y_1 - w_1, y_2 - w_2, \dots, y_{n-1} - w_{n-1})\hat{R}$. Then $\hat{P} \cap R = (0)$. In other words $\hat{P}$ is the generic formal fiber of R.*

Proof. Suppose $\hat{P} \cap R \neq (0)$. Then $\hat{P} \cap D[y_1, \dots, y_{n-1}] \neq (0)$, and let $f(y_1, \dots, y_{n-1}) \in \hat{P} \cap D[y_1, \dots, y_{n-1}]$. Then we have

$$\begin{aligned} f(y_1, \dots, y_{n-1}) &= \hat{g}_1(y_1 - w_1) + \hat{g}_2(y_2 - w_2) + \cdots \\ &\quad + \hat{g}_{n-1}(y_{n-1} - w_{n-1}) \\ &\quad \text{for some } \hat{g}_i \in \hat{R}. \end{aligned}$$

It follows that $f(w_1, \dots, w_{n-1}) = 0$. But, since $w_1, \dots, w_{n-1}$ are algebraically independent over D, we must have that $f(y_1, \dots, y_{n-1}) = 0$. Hence, $\hat{P} \cap D[y_1, \dots, y_{n-1}] = (0)$, a contradiction. Therefore, the lemma holds. □

The dimension of R is n and $\hat{P}$ is a prime ideal of height $n-1$ in the generic formal fiber of R. Thus $\alpha(R) = n - 1 = \dim R - 1$.

Let X be an indeterminate. The following lemma shows that $\alpha(R[X]_{(X_R,X)}) = n-1$, providing a counterexample to the converse of Theorem 2.

Lemma 4. *Let R be as in the example above. For every prime ideal $\hat{Q}$ of $\hat{R}[[X]]$ with ht$\hat{Q} = n$, we have that $\hat{Q} \cap R[X] \neq (0)$.*

Proof. Let $\hat{Q}$ be a prime ideal of height n in $\hat{R}[[X]]$. Note that if $t \in \hat{Q}$, then the lemma holds. So, assume $t \notin \hat{Q}$. Then $\hat{Q} + (t)$ is an $(X, y_1, \dots, y_{n-1}, t)$-primary ideal of $\hat{R}[[X]]$. So $\hat{R}[[X]]/\hat{Q} + (t)$ is a finite K-vector space. By Theorem 8.4 in [**3**], with $A = K[[t]]$, $I = (t)$ and $M = \hat{R}[[X]]/\hat{Q}$, $\hat{R}[[X]]/\hat{Q}$ is a finite $K[[t]]$-module. Now suppose $\hat{Q} \cap R[X] = (0)$. Then we have the following commutative diagram of embeddings:

$$\begin{array}{ccc} K[[t]] & \xrightarrow{\text{finite}} & \hat{R}[[X]]/\hat{Q} \\ {\scriptstyle \text{trdeg}\, n-1} \uparrow & & \uparrow \\ D & \xrightarrow[\text{trdeg}\, n]{} & R[X]_{(M_R,X)} \end{array}$$

But this is a contradiction. Hence, $\hat{Q} \cap R[X] \neq (0)$ as desired. □

Note that the ring R of the example has $\alpha(R) = n-1 = \dim R-1$, but $\alpha(R[X]_{(M_R,X)}) = n-1$. So the backward direction of the conjecture fails.

4. A condition that ensures the converse holds. In this section we explore a related question. Suppose (R, M_R) is a local Noetherian domain and $\hat{P}$ is in the generic formal fiber of R. Under what conditions can we be assured that there is a prime ideal $\hat{Q}$ of $\hat{R}[[X]]$ with $\hat{P} \subset \hat{Q}$, $\operatorname{ht} \hat{Q} > \operatorname{ht} \hat{P}$ and such that $\hat{Q}$ is in the generic formal fiber of $R[X]_{(M_R,X)}$? The following proposition gives such a condition and therefore shows that there are many cases where the converse of Theorem 2 does hold.

Proposition 5. *Let R be a local Noetherian domain with maximal ideal M_R. Let $\hat{P}$ be a prime ideal of $\hat{R}$ with $\hat{P} \cap R = (0)$. Suppose that $\hat{R}/\hat{P}$ is not algebraic over R, and let $w \in \hat{R} - \hat{P}$ be such that the image $\bar{w}$ of w in $\hat{R}/\hat{P}$ is transcendental over R. Let $\hat{Q}$ denote the prime ideal $\hat{Q} = (\hat{P}, X - w)$ of $\hat{R}[[X]]$. Then*

(a) *$\hat{Q}$ is in the generic formal fiber of $R[X]_{(M_R,X)}$.*

(b) *If $\hat{P}$ is maximal in the generic formal fiber of R, the prime ideal $\hat{Q}$ is maximal in the generic formal fiber of $R[X]_{(M_R,X)}$.*

Proof. (a) $\hat{Q} = (\hat{P}, X - w)$ is a prime ideal of $\hat{R}[[X]]$. If $f(X) \in \hat{Q} \cap R[X]$ is nonzero, then $f(X) = \hat{q}(X) + \hat{h}(X)(X - w)$ for some $\hat{q}(X) \in \hat{P}\hat{R}[[X]]$ and $\hat{h}(X) \in \hat{R}[[X]]$. Thus, $f(w) = \hat{q}(w) \in \hat{P}$ and $\bar{w}$ is algebraic over R, a contradiction. Hence, $f(X) = 0$ and it follows that $\hat{Q} \cap R[X] = (0)$.

(b) Obviously, $\hat{Q}$ is extended from $\hat{R}[X]_{(M_{\hat{R}},X)}$ and $\dfrac{\hat{R}[X]_{(M_{\hat{R}},X)}}{\hat{Q}} \cong \hat{R}/\hat{P}$. Thus every prime ideal $\hat{W} \subseteq \hat{R}[[X]]$ which contains $\hat{Q}$ is extended from a prime ideal of $\hat{R}$ that contains $\hat{P}$. Since $\hat{P}$ is maximal in the generic formal fiber, so is $\hat{Q}$. □

Corollary 6. *Let R be a local Noetherian domain with maximal ideal M_R. Let $\dim R = d$ and $\alpha(R) = d-1$. Suppose that there is a prime ideal $\hat{P}$ of $\hat{R}$ in the generic formal fiber of R with $ht\hat{P} = d-1$ and such that $\hat{R}/\hat{P}$ is not algebraic over R. Then $\alpha(R[X]_{(M_R,X)}) = d$.*

Proof. This follows directly from Proposition 5. □

We mentioned at the beginning that if (R, M_R) is a local Noetherian domain of dimension n, and if X is a variable over R, then $\alpha(R[X]_{(M_R,X)}) \geq n-1$. Although $\alpha(R)$ and $\alpha(R[X]_{(M_R,X)})$ seem not to be connected, at least if $\alpha(R) \leq n-2$, we do have the following result.

Corollary 7. *Let R be a local Noetherian domain of dimension n with maximal ideal M_R. If $\hat{P} \subseteq \hat{R}$ is a maximal ideal in the generic formal fiber of R of height t and if $\hat{R}/\hat{P}$ is not algebraic over R, then there is a maximal prime ideal of height $t+1$ in the generic formal fiber of $R[X]_{(M_R,X)}$. In particular, if $t < n-2$, then the generic formal fiber of $R[X]_{(M_R,X)}$ contains maximal ideals of distinct heights.*

Proof. This follows directly from Proposition 5. □

In view of Proposition 5, it must be that for R in our example of the previous section, if $\hat{P}$ is a nonzero prime ideal of $\hat{R}$ that is in the generic formal fiber of R and satisfies $\operatorname{ht}\hat{P} = n-1$, then the ring $\hat{R}/\hat{P}$ is algebraic over R. The following lemma shows exactly this fact.

Lemma 8. *Let R be defined as in the example of the previous section with $\dim R = n$. Let $\hat{P} \subseteq \hat{R}$ be a prime ideal in the generic formal fiber of R with $\operatorname{ht}\hat{P} = n-1$. Then the ring $\hat{R}/\hat{P}$ is algebraic over R.*

Proof. Let $\hat{P}$ be a nonzero prime ideal of $\hat{R}$ such that $\hat{P} \cap R = (0)$ and $\operatorname{ht}\hat{P} = n-1$. Then $t \notin \hat{P}$ and the ideal $\hat{P} + (t)$ is $M_{\hat{R}} = (t, y_1, \ldots, y_{n-1})\hat{R}$-primary. Therefore, $\hat{R}/(\hat{P} + (t))$ is a finite-dimensional vector space over K and, by [**3**, Theorem 8.4], the ring

$\hat{R}/\hat{P}$ is a finite $K[[t]]$-module. Consider the commutative diagram:

$$\begin{array}{ccccc} & R & \hookrightarrow & \dfrac{\hat{R}}{\hat{P}} & \\ \text{trdeg } n-1 & \uparrow & & \uparrow & \text{finite, algebraic} \\ & D & \hookrightarrow & K[[t]] & \\ & & \text{trdeg } n-1 & & \end{array}$$

This forces the extension $R \hookrightarrow \frac{\hat{R}}{\hat{P}}$ to be algebraic. $\square$

The situation where a prime ideal $\hat{P}$ in the generic formal fiber of a local Noetherian domain R exists with the quotient ring $\hat{R}/\hat{P}$ algebraic over R seems fairly rare. For example, suppose R is a countable local Noetherian domain with maximal ideal M_R and that $\hat{R}$ is equidimensional with $\dim R = d$. Then, by Proposition 4.10 in [**2**], we know $\alpha(R) = d - 1$. So we know there are prime ideals of height $d - 1$ in the generic formal fiber of R. Let $\hat{P}$ be such a prime in the generic formal fiber of R. The algebraic closure of a countable field is again countable and the one-dimensional complete local ring $\hat{R}/\hat{P}$ is uncountable. Thus $\hat{R}/\hat{P}$ is not algebraic over R.

For the construction of our example, we have used an (almost trivial) property of DVRs, namely: suppose that V is a DVR with completion $\hat{V}$, and let L be a field between the quotient field of V and the quotient field of the completion $\hat{V} : Q(V) \subseteq L \subseteq Q(\hat{V})$, then the intersection ring $W = L \cap \hat{V}$ is a DVR with completion $\hat{W} = \hat{V}$. Following the notation of [**1**], we say that a local Noetherian analytically irreducible domain R has the *Noetherian intersection property* (NIR) if, for any field L such that $Q(R) \subseteq L \subseteq Q(\hat{R})$, the ring $\hat{R} \cap L$ is Noetherian and has completion $\hat{R}$. In [**5**], excellent RLRs of all dimensions have been constructed which have the property (NIR). We can use these examples to produce more examples of local Noetherian rings which have a prime ideal $\hat{P}$ in the generic formal fiber so that $\hat{R}/\hat{P}$ is an algebraic extension of R. More specifically,

Example. We will construct for all $n, t \in \mathbf{N}$ with $n \geq 2$ an excellent regular local ring S of dimension $n + t$ such that $\alpha(S) = n + t - 2$ and there is a maximal prime ideal $\hat{P} \subseteq \hat{S}$ of height t in the generic formal fiber of S with $\hat{S}/\hat{P}$ algebraic over S.

Consider R_n as defined in [**5**]. Then the following facts hold, see [**5**] for proofs.

(i) R_n is Noetherian.

(ii) The completion of R_n is $K[[Y_1, \dots, Y_{n-1}, T]]$, so R_n is an RLR and $\dim R_n = n$.

(iii) R_n is excellent.

(iv) The transcendence degree of $\hat{R}_n$ over R_n is infinite.

(v) $\alpha(R_n) = 0$.

(vi) $K[[Y_1]] \subseteq R_n$.

(vii) [**1**, Corollary 2.3 and Example 2.4]. R_n has the property (NIR), that is, for every intermediate field $Q(R_n) \subseteq L \subseteq \hat{R}_n$, the intersection ring $D = \hat{R}_n \cap L$ is local Noetherian with completion $\hat{D} = \hat{R}_n$. Moreover, the intersection ring D is excellent.

Let $\{w_i\}_{i \in I} \subseteq \hat{R}_n$ be a transcendence basis of $\hat{R}_n$ over R_n, and let $\omega_1, \dots, \omega_t \in \{\omega_i\}_{i \in I}$ be t distinct elements. Define $\Omega = \{\omega_i\}_{i \in I} - \{\omega_1, \dots, \omega_t\}$ and let

$$D = Q(R_n)(\omega_i \mid \omega_i \in \Omega) \cap \hat{R}_n.$$

Then D is an excellent RLR with $\hat{D} = \hat{R}_n$ and $\alpha(D) = 0$.

Now define $S = D[X_1, \dots, X_t]_{(M_D, X_1, \dots, X_t)}$ where M_D is the maximal ideal of D and $X_1, \dots, X_t$ are indeterminates. Then S is an excellent RLR with $\hat{S} = K[[Y_1, \dots, Y_{n-1}, T, X_1, \dots, X_t]]$. Let $\hat{P} = (X_1 - \omega_1, \dots, X_t - \omega_t)\hat{S}$. We claim that $\hat{P}$ is a maximal prime ideal of height t in the generic formal fiber of S and that $\hat{S}/\hat{P}$ is algebraic over S.

To see this, let $f(X_1, \dots, X_t) \in \hat{P} \cap D[X_1, \dots, X_t]$. Then

$$f(X_1, \dots, X_t) = \hat{g}_1(X_1 - \omega_1) + \cdots + \hat{g}_t(X_t - \omega_t)$$

for some $\hat{g}_i \in \hat{S}$. Now $f(\omega_1, \dots, \omega_t) = 0$. So, since $\omega_1, \dots, \omega_t$ are algebraically independent over D and $f(X_1, \dots, X_t) \in D[X_1, \dots, X_t]$, it must be that $f(X_1, \dots, X_t) = 0$. Hence, $\hat{P} \cap D[X_1, \dots, X_t] = (0)$

and it follows that $\hat{P} \cap S = (0)$. So $\hat{P}$ is in the generic formal fiber of S. Obviously,

$$\frac{\hat{S}}{\hat{P}} = \frac{K[[Y_1, \dots, Y_{n-1}, T, X_1, \dots, X_t]]}{(X_1 - \omega_1, \dots, X_t - \omega_t)} \cong K[[Y_1, \dots, Y_{n-1}, T]] = \hat{R}_n$$

and the map

$$S = D[X_1, \dots, X_t]_{(M_D, X_1, \dots, X_t)} \longrightarrow \frac{\hat{S}}{\hat{P}} = K[[Y_1, \dots, Y_{n-1}, T]] = \hat{R}_n$$

which sends X_i to ω_i for $i = 1, \dots, t$ is an embedding. But this implies that S contains a transcendence basis of $K[[Y_1, \dots, Y_{n-1}, T]]$ over R_n. Hence, $\hat{S}/\hat{P}$ is algebraic over S. Note that $\hat{P}$ is a maximal ideal in the generic formal fiber of S and that $\operatorname{ht}(\hat{P}) = t$.

We have that $S \subset \hat{D}[X_1, \dots, X_t] \subset \hat{D}[[X_1, \dots, X_t]]$. Example 2 in [**4**] shows that $\alpha(\hat{D}[X_1, \dots, X_t]_{(M_{\hat{D}}, X_1, \dots, X_t)}) = n + t - 2$. It follows that $\alpha(S) \geq n + t - 2$. Since S contains a power series ring in one variable $K[[Y_1]]$ and has residue class field K, by [**4**, Theorem 3] we obtain that $\alpha(S) \leq n + t - 2$, thus $\alpha(S) = n + t - 2$. The formal fiber ring of S contains maximal ideals of height t and of height $n + t - 2$ and (some) of the maximal prime ideals $\hat{P}$ of height t have the property that $\hat{S}/\hat{P}$ is algebraic over S.

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REFERENCES

1. W. Heinzer and C. Rotthaus, *Formal fibers and complete homomorphic images*, Proc. Amer. Math. Soc. **120** (1994), 359–369.

2. W. Heinzer, C. Rotthaus and J. Sally, *Formal fibers and birational extensions*, Nagoya Math. J. **131** (1993), 1–38.

3. H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, 1986.

4. ———, *On the dimension of formal fibres of a local ring*, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata (Kinokuniya, Tokyo, 1987), pp. 261–266.

5. C. Rotthaus, *On rings with low dimensional formal fibres*, J. Pure Appl. Algebra **71** (1991), 287–296.

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