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LINEAR COMBINATIONS OF ISOMETRIES

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ABSTRACT. span $(A_1, \dots, A_n)$ denotes the linear space spanned by Hilbert space operators $A_1, \dots, A_n$. It is known that if span (A, B) consists of normal operators, then A, B commute. Let MI denote the set of all scalar multiples of all isometries in a Hilbert space H. In this paper finite-dimensional linear spaces contained in MI will be investigated. Commutativity of such spaces will be described. An example will be given of two unilateral shifts A, B of infinite multiplicity such that span $(A, B) \subset MI$ and A, B do not commute.

1. Introduction. $B(H)$ is the algebra of all bounded linear operators in an infinite dimensional Hilbert space H. I denotes the identity operator.

In 1966 Sarason in [**6**] proved that each Hilbert space operator algebra which consists of commuting normal operators is reflexive. In 1969, Radjavi and Rosenthal in [**4**] noticed that the assumption of commutativity is not needed, i.e., if a linear space of Hilbert space operators consists of normal operators, then the operators commute, cf. also [**5**, Lemma 9.20].

In 1988 Conway and Szymanski [**2**] proved that the latter statement fails for hyponormal operators. These results suggest the general problem:

For which sets $\mathfrak{C}$ of operators, if a linear space S of operators is contained in $\mathfrak{C}$, then the operators from S commute.

The above-mentioned result of [**4**] gives one answer; for $\mathfrak{C}$ = the set of all normal operators.

The result of [**2**] shows that $\mathfrak{C}$ = the set of all hyponormal operators is "too big."

In this paper it is shown that the most important set of operators for the above problem is the set of all scalar multiples of all isometries in a Hilbert space which will be denote by MI; namely, two unilateral shifts

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A, B of infinite multiplicity will be found so that span $(A, B) \subset MI$ and A, B do not commute (Example 3.2); such an example is impossible if one of the shifts is of finite multiplicity, Proposition 2.10. This example shows that if a set $\mathfrak{C}$ of operators contains MI, then operators A and B can be found such that span $(A, B) \subset \mathfrak{C}$ and A and B do not commute.

This problem led us to investigate (finite-dimensional) linear spaces contained in MI. This investigation is carried out with the help of the mapping $\langle\ ,\ \rangle : B(H) \times B(H) \to B(H)$ defined by $\langle A, B\rangle = B^*A$ for $A, B \in B(H)$. It turns out that (Theorem 2.4) for a linear space $S \subset MI$ the restriction of $\langle\ ,\ \rangle$ to $S \times S$ induces naturally an inner product on S. An application to Cuntz algebras is given.

Lastly, some notation. If $M, N \subset B(H)$ are linear spaces, then

$$M + N = \{x + y : x \in M,\ y \in N\}.$$

If $A_1, \dots, A_n \in B(H)$, then span $(A_1, \dots, A_n)$ ($C^*(A_1, \dots, A_n)$, respectively) denote the linear space (C^*-algebra, respectively) generated by $A_1, \dots, A_n$.

2. Linear spaces contained in MI**.** The set MI is not a linear space – the sum of the unilateral shift of multiplicity one and its square is not in MI. In this section we examine linear spaces contained in MI.

Proposition 2.1. *$B \in MI$ if and only if there exists $\gamma \in C$ such that $B^*B = |\gamma|^2 I$.*

Proof. Suppose B^*B is a scalar multiple of I. If $B^*B = 0$, then $B = 0$, ($B = \gamma I$ with $\gamma = 0$). If $B^*B = |\gamma|^2 I$ with $\gamma \neq 0$, then B/γ is an isometry. The converse implication is obviously true. □

Proposition 2.2. *Suppose $A, B \in B(H)$ are such that* span $(A, B) \subset$ *MI. Then there is $\lambda \in C$ such that $B^*A = \lambda I$.*

Proof. Take an arbitrary complex number ν and compute

$$(*) \qquad (\nu A + B)^*(\nu A + B) = |\nu|^2 A^*A + \bar{\nu} A^*B + \nu B^*A + B^*B.$$

Suppose span $(A, B) \subset MI$. Then, for each $\nu \in C$, $(\nu A+B)^*(\nu A+B)$ is a scalar multiple of I, thus by $(*)$,

$$\bar{\nu}A^*B + \nu B^*A = \gamma(\nu)I$$

with some complex $\gamma(\nu)$, because $A, B \in MI$.

Let $\nu = 1$. Then $A^*B + B^*A = \gamma(1)I$.

Let $\nu = i$. Then $-iA^*B + iB^*A = \gamma(i)I$. Multiply both sides by $-i$ to get $-A^*B + B^*A = -i\gamma(i)I$.

Add the last equality to the equality for $\nu = 1$. As a result, B^*A is a scalar multiple of I. $\square$

Theorem 2.3. *Suppose $S \subset B(H)$ is a linear space. $S \subset MI$ if and only if, for each $A, B \in S$, there is a $\lambda \in C$ such that $B^*A = \lambda I$.*

Proof. Suppose $S \subset MI$. If $A, B \in S$, then span $(A, B) \subset S$; thus, span $(A, B) \subset MI$. Use Proposition 2.2.

Conversely, suppose $A \in S$. Let $B = A$. Then there is a $\lambda \in C$ such that $A^*A = \lambda I$. By Proposition 2.1, $A \in MI$. $\square$

Consider now the mapping $\langle\ ,\ \rangle : B(H) \times B(H) \to B(H)$ defined by $\langle A, B\rangle = B^*A$ for $A, B \in B(H)$. This mapping is linear in the first variable, antilinear in the second variable, $\langle A, A\rangle = A^*A \geq 0$ and $\langle A, A\rangle = A^*A = 0$ if and only if $A = 0$ for each $A \in B(H)$. Let CI denote all scalar multiples of I. Theorem 2.3 now reads:

Theorem 2.4. *Suppose $S \subset B(H)$ is a linear space. $S \subset MI$ if and only if $\langle\ ,\ \rangle : S \times S \to B(H)$, the restriction of $\langle\ ,\ \rangle$ to $S \times S$, takes only values in CI.*

Using this result, on each linear space $S \subset MI$, we introduce an inner product denoted also by $\langle\ ,\ \rangle$, slightly abusing notation: $\langle\ ,\ \rangle : S \times S \to C$. If $A, B \in S$, then there is a $\lambda \in C$ such that $B^*A = \lambda I$. By definition, we let $\langle A, B\rangle = \lambda$. By the comments before the statement of Theorem 2.4, this is, indeed, an inner product.

Theorem 2.4 shows also that the only linear subspaces of $B(H)$, the restriction to which of the mapping $\langle\ ,\ \rangle : B(H) \times B(H) \to B(H)$,

$\langle A, B\rangle = B^*A$, $A, B \in B(H)$ is a scalar-valued inner product, are the linear spaces contained in MI.

For a moment, let us fix a linear space $S \subset MI$ with the inner product $\langle\ ,\ \rangle$. The norm induced on S by $\langle\ ,\ \rangle$ is $\langle A, A\rangle^{1/2} = |\gamma|$, where $\gamma \in C$ is such that $A^*A = |\gamma|^2 I$ (cf. Proposition 2.1), $A \in S$. Thus vectors of norm one in S are precisely isometries. If $A, B \in S$, then A, B are orthogonal if $\langle A, B\rangle = 0$, which means $B^*A = 0$.

Since $A, B \in MI$, this condition is equivalent to mutual orthogonality of the ranges of A and B. Therefore, an orthonormal system in S consists of isometries with mutually orthogonal ranges.

The following theorem characterizes finite-dimensional linear spaces contained in MI.

Theorem 2.5. *Suppose $S \subset B(H)$ is a finite-dimensional linear space. The following conditions are equivalent*

(a) $S \subset MI$.

(b) *If $T_1, \dots, T_n \in B(H)$*span S, *then for each $i, j = 1, \dots, n$, there is a $\lambda_{ij} \in C$ such that $T_i^* T_j = \lambda_{ij} I$.*

(c) *There exist isometries $A_1, \dots, A_k \in B(H)$ with mutually orthogonal ranges ($A_i^* A_j = \delta_{ij} I$, $i, j = 1, \dots k$, δ_{ij} is the Kronecker symbol) such that $S =$* span $(A_1, \dots, A_k)$.

Proof. (a) $\Rightarrow$ (b) is clear by Theorem 2.3.

(b) $\Rightarrow$ (a). If $A, B \in S$, then $A = a_1 T_1 + \dots + a_n T_n$, $B = \beta_1 T_1 + \dots + \beta_n T_n$ for some $a_1, \dots, a_n, \beta_1, \dots, \beta_n \in C$ and

$$B^*A = \sum_{i,j=1}^{n} \bar{\beta}_i \alpha_j T_i^* T_j = \sum_{i,j=1}^{n} \bar{\beta}_i \alpha_j \lambda_{ij} I.$$

Use Theorem 2.3.

(a) $\Rightarrow$ (c). S with the inner product $\langle\ ,\ \rangle$ is a finite-dimensional Hilbert space (see Theorem 2.4). Therefore S has an orthonormal basis $A_1, \dots, A_k$. By the comments preceding the statement of this theorem, $A_1, \dots, A_k$ are isometries with mutually orthogonal ranges.

(c) $\Rightarrow$ (a) is proved similarly as (b) $\Rightarrow$ (a). $\square$

In [**1**], Cuntz introduced a class of C^*-algebras generated by isometries, cf. also [**3**]. A Cuntz algebra O_n is the universal C^*-algebra generated by isometries $S_1, S_2, \dots, S_n$ satisfying the condition $S_1S_1^* + \dots + S_nS_n^* = I$. These isometries have orthogonal ranges: $S_i^*S_j = 0$ for $i, j = 1, \dots, n$, $i \neq j$. The following corollary follows immediately from Theorem 2.5.

Corollary 2.6. *If $S_1, \dots, S_n$ are generators of the Cuntz algebra O_n, then* span $(S_1, \dots, S_n) \subset MI$.

A certain form of "the converse" of this result, Corollary 2.9, is also true. It relaxes the assumptions of (thus generalizes) Corollary V.4.7 of [**3**]. To prove it we need some preparation.

It is a simple exercise in linear algebra to prove.

Remark 2.7. If $T_1, \dots, T_n$, $S_1, \dots, S_k$ are linear mappings from a linear space X into itself and span $(S_1, \dots, S_k) \subset$ span $(T_1, \dots, T_n)$, then $S_1X + \dots + S_kX \subset T_1X + \dots + T_nX$.

It is also easy to check.

Remark 2.8. If $T_1, \dots, T_n, S_1, \dots, S_k \in B(H)$ and span $(S_1, \dots, S_k) \subset$ span $(T_1, \dots, T_n)$, then $C^*(S_1, \dots, S_k) \subset C^*(T_1, \dots, T_n)$.

Corollary 2.9. $T_1, \dots, T_n \in B(H)$. *If, for each $i, j = 1, \dots, n$, there is a $\lambda_{ij} \in C$ such that $T_i^*T_j = \lambda_{ij}I$ and $T_1H + \dots + T_nH = H$, then $C^*(T_1, \dots, T_n)$ is isomorphic to a Cuntz algebra.*

Proof. By Theorem 2.5, there exist isometries $A_1, \dots, A_k \in B(H)$ with mutually orthogonal ranges such that span $(T_1, \dots, T_n)$ = span $(A_1, \dots, A_k)$. Since $T_1H + \dots + T_nH = H$, it follows from Remark 2.7 that $A_1H + \dots + A_kH = H$, that is to say $A_1A_1^* + \dots + A_kA_k^* = I$. By Remark 2.8, $C^*(T_1, \dots, T_n) = C^*(A_1, \dots, A_k)$. Finally, Corollary V.4.7 of [**3**] concludes the proof: $C^*(A_1, \dots, A_k)$ is isomorphic to the Cuntz algebra O_k. □

Examples of isometries with mutually orthogonal ranges are well known. For the sake of completeness, let us give one. If $e_1, e_2, \ldots$ is an orthonormal basis of H, define $A_0 e_n = e_{kn}$, $A_1 e_n = e_{kn+1}, \ldots, A_{k-1} e_n = e_{kn+(k-1)}$, $n \in N$.

Using the fundamental theorem of arithmetic one proves that $A_0, \ldots, A_{k-1}$ are unilateral shifts of infinite multiplicity. It is clear that their ranges are mutually orthogonal.

Proposition 2.10. *If $S \subset MI$ is a finite-dimensional linear space which contains a unilateral shift A of finite multiplicity, then $S = \operatorname{span}(A)$.*

Proof. Suppose $B \in S$ is in the orthogonal complement of $\operatorname{span}(A)$. Then $A^*B = 0$. This means that $BH \subset \ker A^* = H \Theta AH$ the wandering subspace of A which, by the assumption, is finite-dimensional. Since $B \in MI$, this is possible only if $B = 0$. □

3. Commutativity. Proposition 2.10 says in particular that, if $S \subset MI$ is a finite-dimensional linear space which contains a unilateral shift of finite multiplicity, then S is commutative, but in the most obvious, "one-dimensional," way. In this section we will show that, in general, if $S \subset MI$ is a finite-dimensional linear space, this is really the only way S can be commutative.

Theorem 3.1. *Suppose $S \subset MI$ is a finite-dimensional linear space, S is commutative if and only if $\dim S = 0$ or $\dim S = 1$.*

Proof. Suppose $\dim S \neq 0$. Select an isometry $A \in S$. Take $B \in S$ in the orthogonal complement of $\operatorname{span}(A)$. Since $B \in MI$, by Proposition 2.1 there is a $\gamma \in C$ such that $B^*B = |\gamma|^2 I$. Then $(AB-BA)^*(AB-BA) = B^*A^*AB - B^*A^*BA - A^*B^*AB + A^*B^*BA = 2|\gamma|^2 I$.

If S is commutative, then $\gamma = 0$, hence $B = 0$ and $S = \operatorname{span}(A)$.

The converse is obvious. □

Now it is clear how to get an example of two operators $A, B \in B(H)$

such that span $(A, B) \subset MI$ and A, B do not commute.

Example 3.2. Let A, B be isometries with mutually orthogonal ranges, i.e., $B^*A = 0$. By Theorem 2.5, span $(A, B) \subset MI$.

A, B are orthogonal in span (A, B), thus $\dim \text{span}\,(A, B) = 2$. By Theorem 3.1, A, B do not commute.

Corollary 3.3. *If $\mathcal{C} \subset B(H)$ is a class of operators that contain MI, then there are $A, B \in \mathcal{C}$ such that* span $(A, B) \subset \mathcal{C}$ *and A, B do not commute.*

A fairly complicated example illustrating this corollary for the class of hyponormal operators was given in [**2**, Example 2.4]. Even though one of the operators in that example is a unilateral shift of infinite multiplicity, it is not clear that the other one should be an isometry.

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