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HEREDITARILY UNICOHERENT CONTINUA AND THEIR ABSOLUTE RETRACTS

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ABSTRACT. We investigate absolute retracts for classes of hereditarily unicoherent continua, tree-like continua, λ-dendroids, dendroids and some other related ones. The main results are: (1) the inverse limits of trees with confluent bonding mappings are absolute retracts of hereditarily unicoherent continua; (2) each tree-like continuum is embeddable in a special way in a tree-like absolute retract for the class of hereditarily unicoherent continua; (3) a dendroid is an absolute retract for hereditarily unicoherent continua if and only if it can be embedded as a retract into the Mohler-Nikiel universal smooth dendroid.

1. Introduction. According to a classical result of Borsuk [**3**, p. 138] each dendrite is an absolute retract for the class of all compacta. Consequently, any dendrite D is an absolute retract for each class $\mathcal{C}$ of compacta (abbreviated $\mathrm{AR}(\mathcal{C})$) such that $D \in \mathcal{C}$. More generally, if $\mathcal{C}_1 \subset \mathcal{C}_2$ for some classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of spaces, then

$$\mathcal{C}_1 \cap \mathrm{AR}(\mathcal{C}_2) \subset \mathrm{AR}(\mathcal{C}_1). \tag{1.1}$$

However, there are significant classes $\mathcal{C}$ of compact with some $\mathrm{AR}(\mathcal{C})$-spaces which need not be AR-spaces for all compacta. For example, continua of dimension at most n, connected and locally connected in dimension n are absolute retracts for the class of all compacta of dimension at most n, see [**29**, § 53, Theorems 1 and 1′, p. 347]. Thus, the opposite inclusion to (1.1) does not hold.

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More recently AR-spaces for some classes of continua had been studied, e.g., by Maćkowiak in [**34, 35**] and [**36**]. Among other results, he has shown the simplest Knaster indecomposable continuum, see, e.g., [**29**, § 48, Example 1, p. 204 and Figure 1, p. 205], and cones over zero-dimensional compact spaces are AR-spaces for the class of hereditarily unicoherent continua, see [**34**, Corollary 4, p. 181; Corollary 5 and the paragraph following it, p. 183]. He has also shown that the pseudo-arc is an AR for the class of hereditarily indecomposable metric continua, see [**35**, Corollary 18, p. 78] and compare [**36**] for a generalization of these results to the nonmetric case. Further, in [**14**], a large family of classes $\mathcal{K}$ of continua was proved to have only locally connected members of $\mathrm{AR}(\mathcal{K})$ and in [**15**] absolute retracts for tree-like continua were studied.

In [**32**, p. 811], Maćkowiak has shown that each tree-like continuum X can be embedded in a tree-like continuum $Y = \varprojlim\{Y_n, f_n\}$, where Y_n are dendroids and f_n are open. Our results presented in Theorem 4.5 are much stronger: we embed X in the inverse limit Y of trees with open bonding mappings in such a way that X is the kernel of Y, and we show that then Y is an absolute retract for the class of hereditarily unicoherent continua.

The classes of hereditarily unicoherent continua, tree-like continua, λ-dendroids and dendroids appear in a natural way in various regions of mathematical interest: the fixed point property, homogeneous spaces, continuous and upper semi-continuous decompositions, (hereditarily) indecomposable continua and many other areas of topology, and also out of topology. These classes are hereditary and they have many invariant properties with respect to numerous classes of mappings. They proved to be important and are among the most extensively studied classes of continua. By these reasons investigation of absolute retracts for the mentioned classes of continua is both interesting and important. As it has been shown in [**11, 12**] and [**13**], absolute retracts for these classes have many interesting, strong and useful properties as, for example, the generalized ε-push property, the arc approximation property, the property of Kelley, homeomorphic translation of arcs and many others. They also have similar properties as absolute retracts in the classical theory of retracts, namely, for any such class $\mathcal{C}$ a retract of a member of $\mathrm{AR}(\mathcal{C})$ is in $\mathrm{AR}(\mathcal{C})$ and each member of $\mathrm{AR}(\mathcal{C})$ is an absolute extensor for $\mathcal{C}$. Nevertheless, it follows from some results that are shown in the present paper, as the theorems on the inverse limits of trees with confluent bonding mappings, Theorem 3.6, and on embeddings of tree-like continua as kernels in absolute retracts for the class of hereditarily unicoherent continua, Theorem 4.5, that they form relatively large classes of continua.

The paper consists of six sections. After the introduction, some auxiliary concepts and results are collected in the second section. The third and the fourth sections form the main part of the paper. Section 3 is devoted mainly to studying the inverse limits of inverse sequences of trees with confluent bonding mappings. It is shown that any such continuum is an absolute retract for the class of hereditarily unicoherent continua. In Section 4 it is proved that each tree-like continuum X can be embedded in a tree-like absolute retract for the class of hereditarily unicoherent continua Y so that X is a kernel of Y and Y/X is a dendroid having the property of Kelley (thus being smooth). Section 5 summarizes the obtained results and provides some related examples. Section 6 contains general open problems and some particular questions that indicate directions of a further study in the area.

A long-term goal of our study is to find characterizations of absolute retracts for the mentioned classes. The following problem emerges from our investigations. Solving it seems to be the next step in this direction, compare Questions 5.5 and 6.3 and see also other related questions in Section 6 and in [**11, 12**] and [**13**].

1.2. Problem. Let continuum X be an absolute retract for the class of hereditarily unicoherent continua (tree-like continua, λ-dendroids, dendroids). Can X be represented as the inverse limit of an inverse sequence of trees with confluent bonding mappings? In particular, is each absolute retract for hereditarily unicoherent continua a tree-like continuum?

By a *space* we mean a topological space and a *mapping* means a continuous function. Given a space X and its subspace $Y \subset X$, a mapping $r : X \to Y$ is called a *retraction* if the restriction $r|Y$ is the identity. Then Y is called a *retract* of X. The reader is referred to [**3**] and [**22**] for needed information on these concepts.

Let $\mathcal{C}$ be a class of *compacta*, i.e., of compact metric spaces. Following

[**22**, p. 80], we say that a space $Y \in \mathcal{C}$ is an *absolute retract for the class* $\mathcal{C}$, abbreviated AR ($\mathcal{C}$), if for any space $Z \in \mathcal{C}$ such that Y is a subspace of Z, Y is a retract of Z. The concept of an AR space originally had been studied by Borsuk, see [**3**].

Let X be a metric space with a metric d. For a mapping $f : A \to B$, where A and B are subspaces of X, we define $d(f) = \sup\{d(x, f(x)) : x \in A\}$. The symbol $\mathbf{N}$ stands for the set of all positive integers.

By a *continuum* we mean a connected compactum. A continuum X is said to be *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected. X is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid*. A locally connected dendroid is called a *dendrite*. A *tree* means a graph containing no simple closed curve or, in other words, a dendrite being the union of finitely many arcs.

A continuum is said to be *decomposable* provided that it can be represented as the union of two of its proper subcontinua. Otherwise it is said to be *indecomposable*. A continuum is said to be *hereditarily decomposable* provided that each of its subcontinua is decomposable. A hereditarily unicoherent and hereditarily decomposable continuum is called a λ-*dendroid*. A continuum is said to be *tree-like* (*arc-like, circle-like*) provided that it is the inverse limit of an inverse sequence of trees (arcs, circles, respectively).

Let $\mathcal{D}_0$ denote the class of dendrites, $\mathcal{D}$–the class of dendroids, $\lambda\mathcal{D}$–of λ-dendroids, $\mathcal{TL}$–of tree-like continua and $\mathcal{HU}$–the class of hereditarily unicoherent ones. Then

$$\mathcal{D}_0 \subset \mathcal{D} \subset \lambda\mathcal{D} \subset \mathcal{TL} \subset \mathcal{HU}. \tag{1.3}$$

As was mentioned previously, according to the result of Borsuk, we have

$$\mathrm{AR}\,(\mathcal{D}_0) = \mathcal{D}_0 \subset \mathrm{AR}\,(\mathcal{D}) \cap \mathrm{AR}\,(\lambda\mathcal{D}) \cap \mathrm{AR}\,(\mathcal{TL}) \cap \mathrm{AR}\,(\mathcal{HU}).$$

Note that the class of absolute retracts of all unicoherent continua coincides with the class of retracts of the Hilbert cube, thus it also coincides with the class of absolute retracts of all compacta. This class is relatively well studied, and we do not investigate it here.

2. Auxiliary concepts and results. In this section we collect concepts and results used in the body of the paper, mostly introduced and studied in our very recent papers, and therefore perhaps not acknowledged to the reader. The aim of the section is to support the reader in understanding our arguments applied in proofs of results in the next section. We start with recalling the following concepts and results taken from [**11**, Section 2].

A class $\mathcal{S}$ of nonempty spaces is called *unionable* provided that for all members X, Y of $\mathcal{S}$ if $X \cap Y \in \mathcal{S}$, then $X \cup Y \in \mathcal{S}$.

2.1. Observation [**11**, Observation 2.2]. The following classes of spaces are unionable: compact spaces of dimension less than or equal to n, continua, hereditarily unicoherent continua, tree-like continua, λ-dendroids, dendroids, dendrites.

Let X and Y be two disjoint spaces, $U \subset X$ a closed subset of X, and let $f : U \to Y$ be a mapping. In the disjoint union $X \oplus Y$ define an equivalence relation $\sim$ by $u \sim f(u)$ for each $u \in U$. Then the quotient space $(X \oplus Y)/\sim$ is denoted by $X \cup_f Y$, see [**19**, Definition 6.1, p. 127] and compare [**41**, p. 42].

A class $\mathcal{S}$ of nonempty spaces is called *functionally unionable* provided that, for all members U, X, Y of $\mathcal{S}$ such that $U = \operatorname{cl} U \subset X$ and for each mapping $f : U \to Y$ if $f(U) \in \mathcal{S}$, then $X \cup_f Y \in \mathcal{S}$. Each functionally unionable class of spaces is unionable, but not conversely, [**11**, Observation 2.7 and Remark 2.8].

2.2. Theorem [**11**, Theorem 2.5]. *Let $\mathcal{S}$ be a unionable class of spaces. If $X \in \mathrm{AR}(\mathcal{S})$ and $Y \in \mathcal{S}$ is a retract of X, then $Y \in \mathrm{AR}(\mathcal{S})$.*

2.3. Proposition [**11**, Proposition 2.10]. *All the classes of continua listed in Observation* 2.1 *are functionally unionable.*

2.4. Theorem [**11**, Theorem 2.11]. *Let a class $\mathcal{S}$ of spaces be functionally unionable. Then the following two conditions are equivalent*:

(2.4.1) $Y \in \mathrm{AR}(\mathcal{S})$;

(2.4.2) *for each space $X \in \mathcal{S}$, for each closed subspace $U \subset X$ such* *that* $U \in \mathcal{S}$ *and for each mapping* $f : U \to Y$ *with* $f(U) \in \mathcal{S}$ *there exists a mapping* $f^* : X \to Y$ *such that* $f^*|U = f$.

A continuum X is said to have the *property of Kelley* provided that, for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points p_n converging to p, there exists a sequence of subcontinua K_n of X containing p_n and converging to the continuum K, see, e.g., [**23**, p. 167] or [**40**, Definition 16.10, p. 538].

A continuum X is said to have the *arc approximation property* provided that for each point $x \in X$, for each subcontinuum K of X containing x, there exists a sequence of arcwise connected subcontinua K_n of X containing x and converging to the continuum K, see [**17**, Section 3, p. 113].

2.5. Proposition [**13**, Proposition 3.2]. *If a continuum* X *has the arc approximation property and contains no simple triod, then each proper subcontinuum of* X *is an arc.*

Investigating absolute retracts for some classes of continua we have found that the following concept of the arc property of Kelley that joins the arc approximation property and the property of Kelley turns out to be both natural and useful.

A continuum X is said to have the *arc property of Kelley*, see [**11**, Definition 3.3], provided that, for each point $p \in X$, for each subcontinuum K of X containing p and for each sequence of points p_n converging to p, there exists a sequence of arcwise connected subcontinua K_n of X containing p_n and converging to the continuum K.

2.6. Proposition [**11**, Proposition 3.4]. *A continuum has the arc property of Kelley if and only if it has the arc approximation property and the property of Kelley.*

To formulate the next result some definitions are in order first. A dendroid X is said to be *smooth* provided that there is a point $v \in X$, called an *initial point* of X, such that for each point $x \in X$ and for each sequence $\{x_n\}$ of points of X which tends to x, the sequence of arcs

vx_n is convergent and it has the arc vx as its limit. It is known that the class of all smooth dendroids has a universal element, i.e., there is a smooth dendroid that contains all other smooth dendroids, see [**16**, Corollary 2, p. 165], [**20**, Theorem 3.1, p. 992] and [**39**]. Since each dendroid having the property of Kelley is smooth, see [**18**, Corollary 5, p. 730], we have the following results.

2.7. Corollary [**11**, Corollary 3.6]. *Each member of* $\mathrm{AR}(\mathcal{D})$ *is a smooth dendroid* (*with the property of Kelley*).

2.8. Corollary [**11**, Corollary 3.7]. *Let* $\mathcal{K}$ *be any class of continua listed in* (1.3). *Then any member of* $\mathrm{AR}(\mathcal{K})$ *has the arc property of Kelley.*

Let X be a hereditarily unicoherent continuum, and let $\mathcal{F}(X)$ be the family of all subcontinua of X intersecting all arc components of X. The intersection of all members of the family $\mathcal{F}(X)$ is named the *kernel* of X and is denoted by $\mathrm{Ker}(X)$. Since a continuum X is hereditarily unicoherent if and only if the intersection of all members of any family of subcontinua of X is a continuum, the kernel $\mathrm{Ker}(X)$ is a subcontinuum of X.

If a closed subset C of a continuum X is given, then X/C is the quotient space obtained by shrinking C to a point. Thus, if C is a continuum, the quotient mapping $q : X \to X/C$ is monotone. See [**44**, Chapter 7, p. 122] for the details.

2.9. Theorem [**12**, Theorem 3.5]. *If* X *is a hereditarily unicoherent, not arcwise connected continuum, then* $\mathrm{Ker}(X)$ *is the smallest continuum* Y *such that* X/Y *is arcwise connected, i.e.,* X/Y *is a dendroid.*

2.10. Theorem [**12**, Theorem 3.7]. *For each hereditarily unicoherent, not arcwise connected continuum* X, *the kernel* $\mathrm{Ker}(X)$ *contains all nondegenerate indecomposable subcontinua of* X.

A subcontinuum T of a continuum X is said to be *terminal* in X provided that, for each subcontinuum K of X the condition $K \cap T \neq \varnothing$

implies $K \subset T$ or $T \subset K$. Note that, according to the definition, the whole continuum X is a terminal subcontinuum of itself, and that each singleton is terminal.

2.11. Theorem [**12**, Theorem 3.8]. *For each hereditarily unicoherent, not arcwise connected continuum X, the kernel* $\operatorname{Ker}(X)$ *contains all proper nondegenerate terminal subcontinua of any continuum $Y \subset X$.*

3. Absolute retracts as inverse limits. In the previous papers [**11, 12**] and [**13**], the authors studied necessary conditions under which a continuum belongs to $\operatorname{AR}(\mathcal{HU})$, $\operatorname{AR}(\mathcal{TL})$, $\operatorname{AR}(\lambda\mathcal{D})$ or to $\operatorname{AR}(\mathcal{D})$. Until now only a few examples of such continua are known which are not dendrites (compare again Maćkowiak results in [**34, 35**] and [**36**], cited in the beginning of the paper). In the present section we will show that the inverse limits of inverse sequences of trees with confluent bonding mappings are members of $\operatorname{AR}(\mathcal{HU})$, see Theorem 3.6. This is one of the main results of the paper. Using this theorem it is possible to get large classes of continua in $\operatorname{AR}(\mathcal{HU})$, $\operatorname{AR}(\mathcal{TL})$, etc., which are not locally connected. But the authors do not know whether each member of $\operatorname{AR}(\mathcal{HU})$ or of $\operatorname{AR}(\mathcal{TL})$, $\operatorname{AR}(\lambda\mathcal{D})$ or of $\operatorname{AR}(\mathcal{D})$ can be represented as the inverse limit of an inverse sequence of trees with confluent bonding mappings, compare Problem 1.2 and Questions 6.2.

A mapping $f : X \to Y$ between continua is said to be *confluent* provided that, for each subcontinuum Q of Y and for each component K of $f^{-1}(Q)$ the equality $f(K) = Q$ holds. Obviously, each monotone mapping is confluent, and also *open* mappings, i.e, such surjections that map open subsets of the domain onto open subsets of the range, are known to be confluent, see [**44**, Theorem 7.5, p. 148]. For properties of this class of mappings, see, e.g., [**33**]. Now we intend to show that a continuum which is the inverse limit of an inverse sequence of trees with confluent bonding mappings is in the class $\operatorname{AR}(\mathcal{HU})$. Some introductory and auxiliary material is necessary first.

By a *graph* we mean a 1-dimensional, finite simplicial complex. In particular, a tree can be seen as an (acyclic) graph if a finite set of its vertices (that contains the set of all ramification points and of all end points of the tree) is fixed.

Let $f : X \to Y$ be a mapping between graphs X and Y. We say that f is *piecewise homeomorphic* provided that there is a finite set $V_X \subset X$ containing all ramification points and all end points of X such that for each component C of $X \setminus V_X$ the partial mapping $f|C : C \to f(C)$ is a homeomorphism. Elements of V_X and of $V_Y = f(V_X)$ are called vertices of the graphs X and Y, respectively, for the mapping f.

Observe that any open mapping between trees can be considered as a piecewise homeomorphic one. More precisely, we have the following assertion which is a particular case of Whyburn's theorem (1.1) in [**44**, p. 182]. A short outline of its proof is given below for the reader's convenience.

3.1 Assertion. *Let $f : X \to Y$ be an open mapping between trees X and Y. Then one can extend the sets of vertices of X and of Y so that f considered as a mapping between the trees X and Y with the new sets of vertices is piecewise homeomorphic.*

Outline of proof. Let W_X and W_Y be the sets of vertices of X and of Y, respectively. Put

$$V_Y = f(W_X) \cup W_Y \quad \text{and} \quad V_X = f^{-1}(V_Y).$$

One can verify that f maps (X, V_X) onto (Y, V_Y) in a piecewise homeomorphic way. □

A similar assertion holds for monotone mappings under an additional assumption.

3.2 Assertion. *Let a monotone mapping $f : X \to Y$ between trees X and Y be such that*

(3.2.1) *the set $F = \{y \in Y : f^{-1}(y)$ is nondegenerate$\}$ is finite.*

Then one can extend the sets of vertices of X and of Y so that f considered as a mapping between the graphs X and Y with the new sets of vertices is piecewise homeomorphic.

Outline of proof. Let W_X and W_Y be the sets of vertices of X and of

Y, respectively. Putting

$$V_Y = W_Y \cup F \quad \text{and} \quad V_X = W_X \cup \operatorname{bd} f^{-1}(V_Y)$$

and, proceeding as previously, we are done. □

As a consequence of the above two assertions we have a lemma.

3.3. Lemma. *Given trees X and Y, let a mapping $f : X \to Y$ be either open or monotone satisfying conditions* (3.2.1)*. Then*

(3.3.1) *there is a positive integer p and points $y_1, \dots, y_p \in Y$ such that for each $\delta > 0$ there are connected and open subsets $U_1, \dots, U_p$ of Y with $y_i \in U_i$ and $\operatorname{diam} U_i < \delta$ for each $i \in \{1, \dots, p\}$ such that for each component C of $Y \setminus (U_1 \cup \dots \cup U_p)$ and for each component K of $f^{-1}(C)$ the partial mapping $f|K : K \to C$ is a homeomorphism of K onto C.*

Recall that a space is said to be *connected between two of its subsets A and B* provided that there is no closed and open subset C of the space such that $A \subset C$ and $B \cap C = \varnothing$, see [**29**, § 46, IV, p. 142]. Maćkowiak has shown in [**34**, Proposition 1, p. 177], the following result that we will use in the sequel. We copy it here only for the reader's convenience.

3.4 Proposition (Maćkowiak). *If A and B are closed subsets of a subcontinuum X of a hereditarily unicoherent continuum Y and if a closed subset L of Y is connected between A and B, then there is a subcontinuum P of $X \cap L$ which intersects both A and B.*

Proposition 3.4 will be used to show the next one.

3.5 Proposition. *Let X be a subcontinuum of a hereditarily unicoherent continuum Y, and let U be an open subset of Y such that $X \setminus U$ has finitely many, say q, components $K_1, \dots, K_q$. Then there are pairwise disjoint closed subsets $L_1, \dots, L_q$ of Y such that*

(3.5.1) *$K_i \subset L_i$ for each $i \in \{1, \dots, q\}$;*

(3.5.2) *$L_i \cap U = \varnothing$ for each $i \in \{1, \dots, q\}$;*

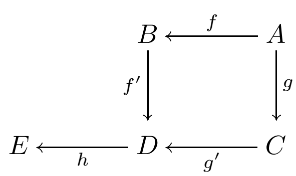
(3.5.3) $Y = (L_1 \cup \cdots \cup L_q) \cup U$.

Proof. In Proposition 3.4 substitute $A_1 = K_1$ and $B_1 = K_2 \cup \cdots \cup K_q$ and put $L_1' = Y \setminus U$. Then, according to Proposition 3.4, the set L_1' is not connected between A_1 and B_1, so there are two disjoint closed subsets L_1 and L_2' of L_1' such that $A_1 = K_1 \subset L_1$, $B_1 \subset L_2'$ and $L_1' = L_1 \cup L_2'$. In this way L_1 is defined. Next use Proposition 3.4 again, with $L_1 \cup U$ in place of U, and with $A_2 = K_2$ and $B_2 = K_3 \cup \cdots \cup K_q$. Then L_2' is not connected between A_2 and B_2, so there are two disjoint closed subsets L_2 and L_3' of L_2' such that $A_2 = K_2 \subset L_2$, $B_2 \subset L_3'$ and $L_2' = L_2 \cup L_3'$. Continuing in this manner, we construct the pairwise disjoint closed subsets $L_1, \ldots, L_q$ of Y that satisfy conditions (3.5.1)–(3.5.3). □

To prove the next result, we need the following auxiliary concept. Let A, B, C, D and E be metric spaces, and let

$$f : A \to B, \quad f' : B \to D, \quad g : A \to C, \quad g' : C \to D, \quad h : D \to E$$

be mappings. Given $\varepsilon > 0$ we say that the diagram



commutes up to ε (or is ε-*commutative*) provided that

$$\rho(h(f'(f(a))), h(g'(g(a)))) < \varepsilon \quad \text{for each } a \in A,$$

where ρ stands for the metric in E.

Given $\varepsilon > 0$ and a mapping f between compacta, we denote by $\delta(f, \varepsilon)$ a positive number that satisfies the conclusion of the definition of uniform continuity of f for the number ε.

The following notation will be used. Given an inverse sequence $\mathbf{S} = \{X_n, f_n\}$ of compact spaces X_n with bonding mappings $f_n : X_{n+1} \to X_n$, where the set of positive integers $\mathbf{N}$ is taken as the directed set of indices, we denote by $X = \varprojlim \mathbf{S}$ its inverse limit and by $\pi_n : X \to X_n$ the projections. Further, let $f_m^n : X_n \to X_m$ be the bonding mapping $f_m^n = f_m \circ f_{m+1} \circ \cdots \circ f_{n-1}$ of $\mathbf{S}$ for $m < n$ and $f_n^n = \mathrm{id}|X_n$. In particular, $f_n^{n+1} = f_n$.

3.6. Theorem. *Let* $\mathbf{S} = \{X_n, f_n\}$ *be an inverse sequence of trees* X_n *with confluent bonding mappings* f_n. *Then* $X = \varprojlim \mathbf{S} \in \mathrm{AR}\,(\mathcal{HU})$.

Proof. Since every confluent mapping between trees is the composition of a monotone and an open mapping, see [**30**, Corollary 5.2, p. 109], we may assume without loss of generality that all the bonding mappings $f_n : X_{n+1} \to X_n$ are either monotone or open. Moreover, since every monotone mapping between trees is the limit of monotone mappings satisfying condition (3.3.1), we may assume by [**37**, Lemma 1, p. 73] that the monotone bonding mappings f_n are such that the set $\{x \in X_n : f_n^{-1}(x)$ is nondegenerate$\}$ is finite.

Let Y be a hereditarily unicoherent continuum with $X \subset Y$. For each $n \in \mathbf{N}$ define an equivalence relation $\sim_n$ in Y by

$$x \sim_n y \Longleftrightarrow x = y \text{ or } x, y \in X \text{ and } \pi_n(x) = \pi_n(y) \in X_n.$$

Since the class $\mathcal{HU}$ is functionally unionable (Proposition 2.3), we conclude that $Y_n = Y/\sim_n$ is a hereditarily unicoherent continuum. Let $\sigma_n : Y \to Y_n$ be the quotient mapping. Define $g_n : Y_{n+1} \to Y_n$ by $g_n(y) = \sigma_n(\sigma_{n+1}^{-1}(y))$ for any $y \in Y_{n+1}$, note that this is a well-defined surjective mapping, and that $Y = \varprojlim\{Y_n, g_n\}$. To define the needed retraction $r : Y \to X$ we will construct, for each $n \in \mathbf{N}$, retractions $r_n : Y_n \to X_n$ such that the diagram

$$\text{(3.6;}k,m,n) \qquad \begin{array}{ccccc} & & Y_m & \xleftarrow{\;g_m^n\;} & Y_n \\ & & \big\downarrow{\scriptstyle r_m} & & \big\downarrow{\scriptstyle r_n} \\ X_k & \xleftarrow[\;f_k^m\;]{} & X_m & \xleftarrow[\;f_m^n\;]{} & X_n \end{array}$$

commutes up to ε_m, for every $k, m, n \in \mathbf{N}$ with $k \leq m \leq n$ where $\lim \varepsilon_m = 0$.

Let $r_1 : Y_1 \to X_1$ be an arbitrary retraction, and assume that for some $n \in \mathbf{N}$ and for each $k < n$ we have defined retractions $r_k : Y_k \to X_k$ such that the diagram (3.6;k', m', n') commutes up to $\varepsilon_{m'}$ for every k', m', n' satisfying $k' \le m' \le n' < n$. For each $m' < n$ let $\eta_{m'}$ be a number satisfying $0 < \eta_{m'} < \varepsilon_{m'}$ and such that the diagram (3.6;k', m', n') commutes up to $\eta_{m'}$ for every k', m', n' satisfying $k' \le m' \le n' < n$. Choose $\delta < \min\{\varepsilon_n, \delta(f_1^n, \varepsilon_1 - \eta_1), \dots, \delta(f_{n-1}^n, \varepsilon_{n-1} - \eta_{n-1})\}$.

We apply Lemma 3.3 to the mapping $f_{n-1} = f_{n-1}^n : X_n \to X_{n-1}$. Let points $x_1, \dots, x_p \in X_{n-1}$ and connected and open subsets $U_1, \dots, U_p$ of X_{n-1} be as in Lemma 3.3, i.e., such that $x_i \in U_i$ and $\operatorname{diam} U_i < \delta$ for each $i \in \{1, \dots, p\}$ and that (putting $U = U_1 \cup \dots \cup U_p$ for shortness), for each component K of $f_{n-1}^{-1}(X_{n-1} \setminus U)$ the partial mapping $f_{n-1}|K : K \to f_{n-1}(K)$ is a homeomorphism onto a component of $X_{n-1} \setminus U$. Let $K_1, \dots, K_q$ for some $q \in \mathbf{N}$ be all the components of $f_{n-1}^{-1}(X_{n-1} \setminus U)$. By Proposition 3.5 there are closed mutually disjoint subsets $L_1, \dots, L_q$ of Y_n satisfying the following conditions:

(3.6.1) $K_i \subset L_i$ for each $i \in \{1, \dots, q\}$;

(3.6.2) $L_i \cap g_{n-1}^{-1}(r_{n-1}^{-1}(U)) = \varnothing$ for each $i \in \{1, \dots, q\}$;

(3.6.3) $Y_n = L_1 \cup \dots \cup L_q \cup g_{n-1}^{-1}(r_{n-1}^{-1}(U))$.

If $y \in L_i$ for some $i \in \{1, \dots, q\}$, then $r_{n-1}(g_{n-1}(y)) \notin U$, and we define $r_n(y)$ as the only point of K_i satisfying $f_{n-1}(r_n(y)) = r_{n-1}(g_{n-1}(y))$. Thus the diagram (3.6;$k', n-1, n$) commutes for $y \in L_i$ and for arbitrary $k' \le n-1$.

Observe that $(g_{n-1})^{-1}(r_{n-1}^{-1}(U)) \cap X_n = f_{n-1}^{-1}(U)$ is an open subset of the tree X_n that has finitely many components $V_1, \dots, V_s$, for some $s \in \mathbf{N}$, and each of the components is mapped onto U_i for some $i \in \{1, \dots, p\}$. Therefore $g_{n-1}^{-1}(r_{n-1}^{-1}(U))$ can be written as the union $W_1 \cup \dots \cup W_s$ of open mutually disjoint subsets W_i of Y_n satisfying $V_i \subset W_i$ for each $i \in \{1, \dots, s\}$. Consider now the open set W_i for some $i \in \{1, \dots, s\}$. The mapping $r_n|\operatorname{bd} W_i$ has already been defined. Since $\operatorname{cl} V_i$ is an absolute retract, we can extend the retraction $r_n|\operatorname{bd} W_i$ to a retraction $r_n|\operatorname{cl} W_i : \operatorname{cl} W_i \to \operatorname{cl} V_i$. In this way the definition of r_n is finished.

Note that if $y \in W_i$ for some $i \in \{1, \dots, s\}$, then both points $f_{n-1}(r_n(y))$ and $r_{n-1}(g_{n-1}(y))$ are elements of $\operatorname{cl} f_{n-1}(V_i) = \operatorname{cl} U_j$ for some $j \in \{1, \dots, p\}$ and therefore the diagram (3.6;$n-1, n-1, n$) is δ-commutative. By the choice of δ the diagrams (3.6;k, m', n) are $\varepsilon_{m'}$-commutative for each $k \le m' \le n$.

For each $n \in \mathbf{N}$, let $\rho_n : Y \to Y_n$ be the projection for the inverse sequence $\{Y_n, g_n\}$. By [**38**, Theorem 2, p. 40], the sequence of retractions $\{r_n : n \in \mathbf{N}\}$ induces the existence of a unique surjective mapping $r : Y \to X$ such that the diagrams

$$(3.6.4) \qquad \begin{array}{ccccc} & & Y_n & \xleftarrow{\rho_n} & Y \\ & & {\scriptstyle r_n}\downarrow & & \downarrow{\scriptstyle r} \\ X_k & \xleftarrow[f_k^n]{} & X_n & \xleftarrow[\pi_n]{} & X \end{array}$$

commute up to ε_n for every $k, n \in \mathbf{N}$ with $k \le n$. Since r_ns are retractions, the diagram (3.6;k, m, n) commutes (exactly) for each $x \in X_n \subset Y_n$. Therefore, the diagram (3.6.4) commutes (exactly) for $x \in X$ with $r(x) = \langle r_1(x_1), r_2(x_2), \dots \rangle = \langle x_1, x_2, \dots \rangle = x$ so r is the needed retraction. □

3.7. Corollary. *Knaster type continua, i.e., inverse limits of arcs with open bonding mappings, are in the class* AR ($\mathcal{HU}$).

Corollary 3.7 together with Theorem 2.4 generalize (in the realm of metric spaces) Theorem 2 of [**34**, p. 179].

Since each *fan*, i.e., a dendroid having exactly one ramification point, with the property of Kelley is the inverse limit of an inverse sequence of finite fans with confluent bonding mappings, see [**9**, Theorem 3, p. 75], we have the following corollary.

3.8. Corollary. *Each fan with the property of Kelley is in the class* AR ($\mathcal{HU}$).

The next corollary is due to Maćkowiak, see [**34**, p. 183].

3.9. Corollary. *Each cone over a zero-dimensional compactum is in the class* AR ($\mathcal{HU}$).

To show the sequential corollary we recall some auxiliary facts. As was said previously, the class of all smooth dendroids (see the definition just before Corollary 2.7 above) has a universal element. The one constructed in [**39**] is called the *Mohler-Nikiel universal smooth dendroid*. Its construction as the inverse limit of an inverse sequence of trees with open bonding mappings is recalled in [**10**, p. 14]. It is known, see [**10**, Theorem 3.21, p. 14], that it has the property of Kelley. Further, the property of Kelley implies smoothness of dendroids, [**18**, Corollary 5, p. 730].

3.10. Corollary. *The Mohler-Nikiel universal smooth dendroid is in the class* $\mathrm{AR}\,(\mathcal{HU})$.

Corollary 3.10 and Theorem 2.2 imply the next result.

3.11. Corollary. *Each retract of the Mohler-Nikiel universal smooth dendroid is in the class* $\mathrm{AR}\,(\mathcal{HU})$.

3.12. Theorem. *A dendroid is a member of* $\mathrm{AR}\,(\mathcal{D})$ *if and only if it is a retract of the Mohler-Nikiel universal smooth dendroid.*

Proof. One implication follows from Corollary 3.11 and from (1.1). To see the other one, note that if a dendroid is in $\mathrm{AR}\,(\mathcal{D})$, then it has the property of Kelley by Corollary 2.8, so it is smooth by [**18**, Corollary 5, p. 730], and therefore embeddable in the Mohler-Nikiel universal smooth dendroid. □

As a consequence of Theorem 3.12 and Corollary 3.11, we get the following.

3.13. Corollary. $\mathrm{AR}\,(\mathcal{D}) \subset \mathrm{AR}\,(\mathcal{HU})$.

It follows from the above corollary and from (1.1) that all four considered classes of absolute retracts coincide in the realm of dendroids. So we have the next corollary.

3.14. Corollary. $\mathrm{AR}\,(\mathcal{D}) = \mathcal{D} \cap \mathrm{AR}\,(\lambda\mathcal{D}) = \mathcal{D} \cap \mathrm{AR}\,(\mathcal{TL}) = \mathcal{D} \cap \mathrm{AR}\,(\mathcal{HU})$.

The following question is related to Theorem 3.12, compare also Question 6.5.

3.15. Question. Is every dendroid having the property of Kelley a retract of the Mohler-Nikiel universal smooth dendroid?

4. Tree-like continua as kernels of absolute retracts. Now we will prove that any tree-like continuum is a kernel of some tree-like continuum in AR$(\mathcal{HU})$. First we need a construction, a definition and a lemma.

For given trees X and Y and for a piecewise homeomorphic mapping $f : X \to Y$ we will construct a tree X^* containing X and an open extension $f^* : X^* \to Y$ of f. The tree X^* is obtained from X by adding needed parts of the tree Y at points where f is not interior, i.e., $f(p) \notin \operatorname{int} f(U)$ for some $U \subset X$ with $p \in \operatorname{int} U$; obviously a mapping is open if and only if it is interior at each point of its domain. Precisely we have the following construction.

4.1. Construction. Let $f : X \to Y$ be a piecewise homeomorphic mapping between trees. For a vertex $x \in V_X \subset X$, let $T(x)$ be the union of all components C of $Y \setminus \{f(x)\}$ such that there is an open neighborhood U of x in X satisfying $f(U) \cap C = \varnothing$. Denote by X^* the union $X \cup \cup\{T(x) : x \in V_X\}$ with the natural topology, i.e., $x \in \operatorname{cl} T(x) = T(x) \cup \{x\}$ and $T(x) \cap T(y) = \varnothing$ for $x \neq y$. Define $f^* : X^* \to Y$ in a natural way, i.e.,

$$f^*(p) = \begin{cases} f(p) & \text{if } p \in X, \\ p & \text{if } p \in T(x) \text{ for some } x \in V_X. \end{cases}$$

One can check that f^* is an open extension of f. It will be called the *minimal open extension* of f.

A mapping $f : X \to Y$ between continua X and Y is said to be *monotone relative to a point* $p \in X$ provided that for each continuum $Q \subset Y$ with $f(p) \in Q$ the preimage $f^{-1}(Q)$ is connected. For dendroids X, in particular for trees, monotoneity relative to $p \in X$ is equivalent, see [**31**, Corollary 2.10, p. 732], to the condition

(4.2) for each point $x \in X$ the partial mapping $f|px$ is monotone.

Further, the following result is known, see [**8**, Corollary 3, p. 145].

4.3. Proposition. *For each* $n \in \mathbf{N}$*, let* X_n *be a dendroid,* $p_n \in X_n$ *and* $f_n : X_{n+1} \to X_n$ *be a mapping which is monotone relative to* p_{n+1} *and such that* $f_n(p_{n+1}) = p_n$*. Then the inverse limit* $\varprojlim\{X_n, f_n\}$ *is a dendroid.*

4.4. Lemma. *Given two trees* X *and* Y *with* $X \cap Y = \{p\}$*, define* $Z = X \cup Y$*, and let* $f : Z \to Y$ *be a piecewise homeomorphic retraction. Let* $f^* : Z^* \to Y$ *be the minimal open extension of* f *as in the Construction* 4.1*. Denote by* $q_1 : Z^* \to Z^*/(X \cup f(X))$ *and* $q_2 : Y \to Y/f(X)$ *the quotient mappings. Then there is the unique mapping* $g : Z^*/(X \cup f(X)) \to Y/f(X)$ *such that the diagram*

$$\begin{array}{ccc} Z^* & \xrightarrow{\quad f^* \quad} & Y \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ Z^*/(X \cup f(X)) & \xrightarrow[\quad g \quad]{} & Y/f(X) \end{array}$$

commutes. Moreover, the mapping g *is open, monotone relative to the point* $q_1(X \cup f(X))$*, and each component of* $(Z^*/(X \cup f(X)) \setminus q_1(X \cup f(X))$ *is mapped homeomorphically under* g *onto a component of* $Y/f(X) \setminus q_2(f(X))$*.*

Proof. Just define $g(x) = (q_2 \circ f^* \circ q_1^{-1})(x)$ for each point $x \in Z^*/(X \cup f(X))$. Openness of g follows from that of f^*. For any component C of $Z^*/(X \cup f(X)) \setminus q_1(X \cup f(X))$ the partial mapping $g|C$ is a homeomorphism by its construction, and g is monotone relative to the point $q_1(X \cup f(X))$ by (4.2). □

4.5. Theorem. *For each tree-like continuum* X *there is a tree-like continuum* Y *containing* X *such that* $X = \mathrm{Ker}\,(Y)$ *and* $Y \in \mathrm{AR}\,(\mathcal{HU})$*. Moreover,* Y *is the inverse limit of trees with open, thus confluent, bonding mappings.*

Proof. Represent X as the inverse limit $X = \varprojlim\{X_n, f_n\}$ of trees X_n with piecewise homeomorphic bonding mappings f_n. Recall that

$\pi_n : X \to X_n$ denotes the nth projection mapping. For each $k \in \mathbf{N}$, choose a point $v_k \in X$ in such a way that

(4.5.1) X is the only subcontinuum of X containing almost all points of the set $\{v_k : k \in \mathbf{N}\}$.

In the disjoint union $X_1 \oplus \cdots \oplus X_n$ identify pairs of points $\pi_k(v_k)$ and $f_{k-1}(\pi_k(v_k))$ for $k \in \{2, \dots, n\}$, and put $L_n = X_1 \cup \cdots \cup X_n$ with the above identifications, i.e., $X_i \cap X_j = \varnothing$ if $|i-j| > 1$ and $X_i \cap X_{i+1} = \{\pi_{i+1}(v_{i+1})\} = \{f_i(\pi_{i+1}(v_{i+1}))\}$ for $i \in \{1, \dots, n-1\}$. For each $n \in \mathbf{N}$, denote by $r_n : L_{n+1} \to L_n$ the natural retraction such that $r_n | X_{n+1} = f_n$ and let $G = \varprojlim\{L_n, r_n\}$. Since the bonding mappings r_n are retractions, we may assume that the sets L_n are naturally embedded in G. Observe that, under this assumption, $X = \varprojlim\{X_n, f_n\} \subset G$. The threads $\{x_1, x_2, \dots\}$ corresponding to the points of any L_n have constant coordinates x_k for $k > n$, while the threads corresponding to the points of X have infinitely many coordinates mutually different. Thus $X \cap L_n = \varnothing$ for each n. We have also $G = \operatorname{cl}(\cup\{L_n : n \in \mathbf{N}\}) = X \cup \cup\{L_n : n \in \mathbf{N}\}$.

Any subcontinuum of G intersecting X and its complement $G \setminus X$ must contain almost all v_ks. Thus X is terminal in G by (4.5.1).

Using Construction 4.1, we will define inductively retractions r_n^* which extend retractions r_n to some larger domain trees L_{n+1}^*. Let $L_1^* = L_1$ and $r_1^* : L_2^* \to L_1^*$ be the minimal open extension of r_1 as in 4.1. Next let $r_2^* : L_3^* \to L_2^*$ be the minimal open extension of $r_2 : L_3 \to L_2^*$ understood as a mapping into the already defined space L_2^*. Continuing in the same way, we define open retractions $r_n^* : L_{n+1}^* \to L_n^*$ for each $n \in \mathbf{N}$. Define $Y = \varprojlim\{L_n^*, r_n^*\}$. Then Y is in $\mathrm{AR}(\mathcal{HU})$ by Theorem 3.6, and $X \subset G \subset Y$.

We will show that $X = \operatorname{Ker}(Y)$. Since X is terminal in G, we get $X \subset \operatorname{Ker}(Y)$ by Theorem 2.11. To prove the other inclusion we will show that Y/X is arcwise connected. This will end the proof by Theorem 2.9.

Let $\sigma_n : Y \to L_n^*$ be the projections of the inverse limit $\varprojlim\{L_n^*, r_n^*\}$. Then $\sigma_n(X) = X_n$ for each $n \in \mathbf{N}$ by the construction. For each $n \in \mathbf{N}$ there is the unique mapping $g_n : L_{n+1}^*/X_{n+1} \to L_n^*/X_n$ such that the diagram

$$\begin{array}{ccc} L_{n+1}^* & \xrightarrow{r_n^*} & L_n^* \\ \downarrow{\scriptstyle q_{n+1}} & & \downarrow{\scriptstyle q_n} \\ L_{n+1}^*/X_{n+1} & \xrightarrow[g_n]{} & L_n^*/X_n \end{array}$$

commutes, where q_n and q_{n+1} are the natural quotient mappings. The space Y/X is homeomorphic to $\varprojlim\{L_n^*/X_n, g_n\}$ and $\varprojlim q_n$ is the natural quotient mapping between X and Y/X.

By Lemma 4.4 and by the definition of the mapping r_n^* we infer that for each $n \in \mathbf{N}$ the mapping g_n is monotone relative to the point $q_{n+1}(X_{n+1})$. Applying Proposition 4.3 the space Y/X is arcwise connected. □

4.6. Corollary. *A tree-like continuum X is a member of* AR$(\mathcal{HU})$ *if and only if X is a retract of the inverse limit of trees with open* (*equivalently: with confluent*) *bonding mappings.*

Proof. Since the class of hereditarily unicoherent continua is (functionally) unionable, see Proposition 2.3, one implication follows from Theorems 2.2 and 3.6. The other one is a consequence of Theorem 4.5. □

4.7. Corollary. *There exist non-arcwise connected λ-dendroids in* AR$(\mathcal{HU})$. *In particular, they have the arc property of Kelley and thus each of their arc components is dense.*

4.8. Remarks. 1) A non-arcwise connected λ-dendroid X with uncountably many arc components, each of which is dense in X, has been constructed by Krasinkiewicz and Minc in [**26**, Example 4, p. 285].

2) Theorem 4.5 leads to the existence of a large family of such λ-dendroids as mentioned above. Indeed, each non-arcwise connected λ-dendroid X is the kernel of a non-arcwise connected λ-dendroid Y, all arc components of which are dense in Y (since $Y \in$ AR$(\mathcal{HU})$). Further, each λ-dendroid Y constructed according to Theorem 4.5 has

the arc property of Kelley, again since $Y \in \mathrm{AR}\,(\mathcal{HU})$.

Theorem 4.5 allows us to prove the following nontrivial inclusions.

4.9. Corollary. $\mathrm{AR}\,(\mathcal{D}) \subset \mathrm{AR}\,(\lambda\mathcal{D}) \subset \mathrm{AR}\,(\mathcal{TL}) \subset \mathrm{AR}\,(\mathcal{HU})$.

Proof. The first inclusion is proved in Corollary 3.14.

Let $X \in \mathrm{AR}\,(\lambda\mathcal{D})$ be a subset of a tree-like continuum T. By Theorem 4.5 we may assume that a tree-like continuum $Y \in \mathrm{AR}\,(\mathcal{HU})$ contains a homeomorphic copy X' of X such that $X' = \mathrm{Ker}\,(Y)$. Since $X \in \lambda\mathcal{D}$, the continuum Y is in $\lambda\mathcal{D}$ by Theorem 2.10. We identify X and X' by a homeomorphism, obtaining a tree-like continuum $Z = T \cup Y$ by Proposition 2.3. Since Y is in $\mathrm{AR}\,(\mathcal{HU})$, there exists a retraction $r_1 : Z \to Y$; and because $X \in \mathrm{AR}\,(\lambda\mathcal{D})$, there exists a retraction $r_2 : Y \to X$. The restriction $r_2 \circ r_1|T : T \to X$ is the required retraction. Hence $X \in \mathrm{AR}\,(\mathcal{TL})$.

The proof of the last inclusion is similar. □

5. Further results on tree-like continua. Consider the following conditions that a continuum X can satisfy:

(5.1.1) X is the inverse limit of an inverse sequence of trees with confluent bonding mappings;

(5.1.2) X is in $\mathrm{AR}\,(\mathcal{TL})$;

(5.1.3) X is in $\mathrm{AR}\,(\mathcal{HU})$;

(5.1.4) X is a tree-like continuum having the arc property of Kelley;

(5.1.5) X is a hereditarily unicoherent continuum having the arc property of Kelley.

The following diagram shows known implications between conditions (5.1.1)–(5.1.5).

$$\begin{array}{ccccc}
(5.1.1) & \xrightarrow{Thm.3.6} & (5.12) & \xrightarrow{Cor.4.7} & (5.1.3) \\
 & & \Big\downarrow{\scriptstyle Cor.2.8} & & \Big\downarrow{\scriptstyle Cor.2.8} \\
 & & (5.1.4) & \xrightarrow[trivial]{} & (5.1.5)
\end{array}$$

Since solenoids satisfy condition (5.1.5) but they are not tree-like and they do not belong to AR$(\mathcal{HU})$, see [**13**, Corollary 3.18], then the class in (5.1.5) is different from the other ones. Thus we have

- (5.1.5) $\not\Rightarrow$ (5.1.3) and (5.1.5) $\not\Rightarrow$ (5.1.4).

Further, as is shown in [**13**, Example 3.14 and Remark 3.16], the Ingram continuum defined in [**24**] is a tree-like (but not arc-like) continuum with the arc property of Kelley that is not in AR$(\mathcal{HU})$. Therefore,

- (5.1.4) $\not\Rightarrow$ (5.1.3), and consequently (5.1.4) $\not\Rightarrow$ (5.1.2).

Additionally we will show that there exists an arc-like continuum, Example 5.4, that satisfies (5.1.4) and does not satisfy (5.1.1). To show the example, we need some auxiliary results.

5.2. Proposition. *Let* $\mathbf{S} = \{X_n, f_n\}$ *be an inverse sequence of trees* X_n *with confluent bonding mappings* f_n. *If* $X = \varprojlim \mathbf{S}$ *does not contain simple triods, then all trees* X_n *are arcs.*

Proof. By Theorem 3.6 the continuum X is in AR$(\mathcal{HU})$. Thus X has the arc property of Kelley by Corollary 2.8, whence it follows by Proposition 2.5 that each proper subcontinuum of X is an arc. Suppose that some X_n is not an arc, and let $T \neq X_n$ be a triod in X_n. Thus each component of $\pi_n^{-1}(T)$ is a proper subcontinuum of X, so it is an arc. Since the projection $\pi_n : X \to X_n$ is confluent, [**7**, Corollary 7, p. 5] and since for each component C of $\pi_n^{-1}(T)$ the restriction $\pi_n|C : C \to \pi_n(C) = T$ is confluent, [**4**, p. 213]. So we have a confluent mapping of an arc C onto a triod T, a contradiction with [**33**, p. 74]. □

The following concept is introduced in [**13**, Definition 3.10]. A continuum X is said to have the *local property of Kelley at a point* $p \in X$ provided that there exists a neighborhood $U(p)$ of p such that, for each continuum $K \subset U(p)$ with $p \in K$ and for each sequence of points $\{p_n\}$ converging to p, there is a sequence of continua $\{K_n\}$ with $p_n \in K_n$ converging to K. Note that the property of Kelley at a point as defined in [**43**, p. 292] implies the local property of Kelley at the point in the sense defined above.

The next result is shown as [**13**, Theorem 3.10].

5.3. Theorem. *If each nondegenerate proper subcontinuum of a continuum X is an arc, and if X has the local property of Kelley at each of its points, then X has the propety of Kelley.*

5.4. Example. There exists an arc-like continuum having the arc property of Kelley, which is not the inverse limit of an inverse sequence of trees with confluent bonding mappings.

Proof. Let X be the simplest arc-like indecomposable continuum with exactly three end points, a, b and c (here the term "end point" is understood in the sense of [**2**, Section 5, p. 660]. The continuum X is obtained as the intersection of unions of $1/2^k$-chains $\mathcal{C}_k$ of (closed) disks in the plane such that

(5.4.1) for each $n \in \{0, 1, 2, \dots\}$ the chain $\mathcal{C}_{3n+1}$ goes from a to c through b; $\mathcal{C}_{3n+2}$ goes from b to c through a; and $\mathcal{C}_{3n+3}$ goes from a to b through c;

(5.4.2) for each $k \in \mathbf{N}$ the chain $\mathcal{C}_{k+1}$ refines $\mathcal{C}_k$

(see [**21**, p. 142] and [**41**, pp. 7–8]). The continuum X can also be seen as the inverse limit of an inverse sequence of closed unit intervals $X_n = [0, 1]$ with the same piecewise linear bonding mappings $f_n = f : [0, 1] \to [0, 1]$ determined by

$$f(0) = \tfrac{1}{2}, \quad f\big(\tfrac{1}{2}\big) = 1, \quad f(1) = 0$$

and being linear on $[0, 1/2]$ and $[1/2, 1]$.

Suppose that X is the inverse limit of an inverse sequence of trees T_n with confluent bonding mappings g_n. Since X is arc-like, it is atriodic, [**23**, p. 259]. Applying Proposition 5.2 we see that each factor space T_n is an arc. Since inverse limits of arcs with confluent bonding mappings are the same as ones of arcs with open bonding mappings, called *Knaster type continua*, see [**5**, pp. 224, 231]; compare [**42**, p. 455], X has at most two end points, see [**28**, p. 48].

We will show that X has the local property of Kelley at each of its points. Let $p \in X$. If p is an end point of X, i.e., if $p \in \{a, b, c\}$, then X has the property of Kelley at p (and consequently it has the local property of Kelley at p) by [**27**, p. 380]. If p is not an end point of X, it follows from the construction of X that p has a neighborhood homeomoprhic to the product of the Cantor set and an arc (since for small subcontinua containing the coordinate of $p_n \in X_n = [0,1]$ the restrictions of the bonding mappings are homeomorphisms). Therefore, X has the local property of Kelley at p. It is proved in [**1**, Theorem 8, p. 168] (compare also [**1**, Remarks, p. 168]), that each proper subcontinuum of X is an arc. Applying Theorem 5.3 we conclude that X has the property of Kelley. Since each proper subcontinuum of X is an arc, X has the arc property of Kelley. □

The following question seems to be interesting in the light of Problem 1.2 in the introduction. Namely, Example 5.4 is our candidate for solving Problem 1.2 in the negative.

5.5 Question. Is the continuum X of Example 5.4 a member of AR$\,(\mathcal{HU})$?

Let us recall that, given a class $\mathcal{S}$ of spaces, a *universal element* of $\mathcal{S}$ is a member of $\mathcal{S}$ in which each member of $\mathcal{S}$ can be embedded. The reader is referred to the introductions of [**37**] and [**25**] and to [**6**, p. 741] for information about the existence of universal elements for various classes of continua.

Using Corollaries 2.7 and 3.10, we obtain the following.

5.6. Corollary. *The Mohler-Nikiel universal smooth dendroid is a universal element in the class* AR$\,(\mathcal{D})$.

It is shown in [**25**, Corollary 4.2, pp. 740] that there is no universal element in the class $\lambda\mathcal{D}$. This result and Theorem 4.5 imply the next one.

5.7. Theorem. *There is no universal element in the class* AR$\,(\lambda\mathcal{D})$.

Proof. If X is a λ-dendroid, then the continuum Y guaranteed by Theorem 4.5 is also a λ-dendroid according to Theorem 2.10. Therefore, a universal element X_0 in the class AR$\,(\lambda\mathcal{D})$ must contain homeomorphic copies of all continua Y of Theorem 4.5 for all λ-

dendroids X and, consequently, X_0 would be a universal element in the class $\lambda\mathcal{D}$, a contradiction. □

It is shown in [**37**, p. 72] that there exists a universal tree-like continuum. From this result combined again with Theorem 4.5 we can conclude the following.

5.8. Theorem. *There exists a universal element T_0 in the class* AR $(\mathcal{TL})$*. Furthermore,*

(5.8.1) Ker (T_0)*, and therefore also T_0, is a universal element in the class $\mathcal{TL}$;*

(5.8.2) *T_0 is the inverse limit of an inverse sequence of trees with confluent bonding mappings.*

Proof. Let T be a universal tree-like continuum, and let T_0 be the continuum Y guaranteed by Theorem 4.5 for the continuum $X = T$. Thus we can assume that T_0 is the inverse limit of trees with confluent bonding mappings. Since the kernel Ker $(T_0) = T$ is a universal tree-like, T_0 is also a universal tree-like continuum. □

The above result implies the next one.

5.9. Corollary. *If $X \in \mathcal{TL}$, then $X \in$* AR $(\mathcal{HU})$ *if and only if X is a retract of T_0 for each (for some) embedding of X in T_0.*

A continuum X is said to be an *absolute terminal retract* provided that if X is embedded in a continuum Y in such a way that the embedded copy X' is a terminal subcontinuum of Y, then X' is a retract of Y, see [**15**, Definition 4.1]. A compactum X is called

a) an *approximative absolute retract*, written AAR, provided that whenever X is embedded in a compactum, or equivalently in the Hilbert cube, Y, for each $\varepsilon > 0$ there is a mapping $f : Y \to X'$ such that $d(f|X') < \varepsilon$;

b) an *approximative absolute neighborhood retract*, written AANR, provided that whenever X is embedded in a compactum, or equivalently in the Hilbert cube, Y, for each $\varepsilon > 0$ there are a neighborhood U of the embedded copy X' of X in Y and a mapping $f : U \to X'$ such that $d(f|X') < \varepsilon$.

In [**15**] relations were shown between absolute retracts for the classes of tree-like continua ($\mathcal{TL}$), λ-dendroids ($\lambda\mathcal{D}$), dendroids ($\mathcal{D}$), arc-like continua ($\mathcal{AL}$) and arc-like λ-dendroids ($\lambda\mathcal{AL}$), as well as between AANR-continua and absolute terminal retracts. We quote principal results of [**15**] because they give an essential information on absolute retracts for the considered classes of continua. Namely, we have the following three theorems [**15**, Theorems 3.3, 4.5 and Corollary 5.5]. The second one is the main result of [**15**].

5.10. Theorem. *Let $\mathcal{K}$ be any of the following classes of continua: $\mathcal{TL}, \lambda\mathcal{D}, \mathcal{D}, \mathcal{AL}$ and $\lambda\mathcal{AL}$. Then each member of* AR$(\mathcal{K})$ *is an* AAR.

5.11. Theorem. *A continuum X is an* AANR *if and only if X is an absolute terminal retract.*

5.12. Theorem. *Let $\mathcal{K} \in \{\mathcal{TL}, \lambda\mathcal{D}, \mathcal{D}, \mathcal{AL}, \lambda\mathcal{AL}\}$. If a continuum X is in* AR$(\mathcal{K})$*, then for each $\varepsilon > 0$ there are a tree $T \subset X$ and a mapping $f : X \to T$ such that $d(f) < \varepsilon$.*

Besides, the following important result is shown in [**15**, Corollary 3.7].

5.13. Theorem. *Each member of* AR$(\mathcal{TL})$ *has the fixed point property.*

6. Problems. We close the paper stating some problems and questions concerning the subject. They are related to the conjecture stated in the introduction. The most general are the following.

6.1. Problems. Find intrinsic, i.e., structural, characterizations of the following classes of continua

a) AR$(\mathcal{D})$, b) AR$(\lambda\mathcal{D})$, c) AR$(\mathcal{TL})$, d) AR$(\mathcal{HU})$.

The authors wonder if the property described in Theorem 3.6 characterizes the class AR $(\mathcal{HU})$. Namely, the next question is of a special interest in view of the results of the paper.

6.2. Questions. Can every element of

a) AR $(\mathcal{D})$, b) AR $(\lambda\mathcal{D})$, c) AR $(\mathcal{TL})$, d) AR $(\mathcal{HU})$

be represented as the inverse limit of trees with confluent bonding mappings, see Question 5.5?

We know that the classes AR $(\mathcal{D})$, AR $(\lambda\mathcal{D})$ and AR $(\mathcal{TL})$ are mutually different. However, we still cannot answer the following question.

6.3. Question. Is it true that AR $(\mathcal{HU})$ = AR $(\mathcal{TL})$? (Equivalently, is each member of AR $(\mathcal{HU})$ a tree-like continuum, see Corollary 4.9?).

6.4. Question. Does there exist a universal element in the class AR $(\mathcal{HU})$?

In the following question the concept of an absolute retract is not used. If answered in the affirmative, it would give a characterization of the class AR $(\mathcal{D})$.

6.5. Question. Is every dendroid having the property of Kelley the inverse limit of an inverse sequence of trees with confluent bonding mappings?

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