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A NEW CLASS OF NORMED SPACES WITH NONTRIVIAL GROUPS OF ISOMETRIES AND SOME ESTIMATES FOR OPERATORS WITH GIVEN ACTION

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ABSTRACT. We introduce and study a new class of finite-dimensional normed spaces with non-trivial groups of isometries. We call it the class of spaces with *distinguished bases*. The introduction of this new class is motivated by the following: (a) this class is a natural class that contains both the class of spaces with unconditional bases and the class of spaces spanned by characters in translation invariant function spaces on compact abelian groups; (b) there is a very simple formula for 2-summing norms of the operators that are diagonal with respect to distinguished bases.

An additional motivation for introduction of the notion of a distinguished basis is its relations with the problem of the estimate of norms of operators with given action. More precisely, let A be a $k \times k$ matrix and let V be a k-dimensional normed space. The set of all operators on V whose matrix with respect to some basis in V is A is denoted by $L_A(V)$. The problem is to estimate $\inf\{||T|| : T \in L_A(V)\}$, where $||\cdot||$ is one of the natural norms on the space of linear operators on V. The second part of the paper is devoted to some aspects of this general problem.

1. Introduction. Different classes of spaces with nontrivial groups of isometries play an important role in Banach Space Theory and in Asymptotic Theory of Finite-Dimensional Normed Spaces (see, for example, [**41**]). Three of such classes have been extensively studied. These are the classes of (1) spaces with unconditional bases, (2) spaces with symmetric bases, and (3) spaces with enough symmetries. The purpose of this paper is to introduce one more class, we call it the class of spaces with *distinguished bases*. The introduction of this new class is motivated by the following: (a) this class is a natural class that contains both the class of spaces with unconditional bases and the class of spaces spanned by characters in translation invariant function spaces

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on compact abelian groups; (b) there is a very simple formula for π_2-norms of the operators that are diagonal with respect to distinguished bases. This formula proves a special case of Conjecture 1 from [**8**].

At the end of the paper we discuss some more estimates for operators with given action; in particular, we give a counterexample to Conjecture 2 from [**8**].

We shall use the standard terminology and notation of Banach space theory, see [**41**]. For the theory of absolutely summing operators and π_2 norms we refer to [**35**]. By $B(X)$ we denote the unit ball of a normed space X. By I_V we denote the identity operator on a normed space V. The subscript V is omitted if it is clear from context.

2. Spaces with distinguished bases.

Definition 1. Let V be a k-dimensional normed space. A basis $\{e_i\}_{i=1}^k$ in V is called a *distinguished basis* if there exists a subgroup G in the group of isometries on V satisfying the condition:

An operator $S : V \to V$ satisfies $gS = Sg$ for every $g \in G$ if and only if S is diagonal with respect to $\{e_i\}_{i=1}^k$.

Proposition 1. (1) *The condition from Definition* 1 *is equivalent to: all elements in G are diagonal with respect to $\{e_i\}_{i=1}^k$, and for each $i \neq j$, $(i, j \in \{1, \dots, k\})$, there exists an element $g \in G$ such that the ith and the jth diagonal elements of g are different.*

(2) *The diagonal elements of $g \in G$ have absolute values* 1*. In particular, in the real case they are equal to ± 1. Hence in the real case $g^2 = I_V$ for every $g \in G$.*

(3) *The cardinality of G is at least k.*

Proof. The proof of (1) is straightforward. The statement (2) is a special case of the well-known fact on spectral properties of isometries.

(3) The fact that elements of G are simultaneously diagonalizable (see part (1)) implies that the group G is abelian. It will be convenient to identify G with the set of functions on $\{1, \dots, k\}$ (so that $g(i)$ is the ith diagonal element of g). It will be convenient to restate the condition

from (1) as: the functions from G separate points of $\{1, \dots, k\}$.

There is nothing to prove if G is infinite. (Observe that by (1) and (2) G can be infinite only in the complex case.) So we assume that G is finite. By the fundamental theorem of abelian groups (see, e.g., [**42**, Section 12.3]) G is the direct sum of cyclic groups. Let $g_1, \cdots, g_m$ be the generators of the cyclic groups. Then the cardinality of G is equal to $\#G = r_1 \cdots r_m$, where $r_1, \dots, r_m$ are the orders of $g_1, \dots, g_m$. The values of g_i are the r_ith roots of unity. Therefore the functions g_i split the set $\{1, \dots, k\}$ onto at most r_i subsets. Therefore the functions $g_1, \dots, g_m$, and, hence, G can separate at most $r_1 \cdots r_m$ points. We get $k \leq r_1 \cdots r_m = \#G$. $\square$

Basic examples of spaces with distinguished bases.

Example 1. Any 1-unconditional basis $\{e_i\}_{i=1}^k$ is distinguished.

In fact, let G be the set of all matrices diagonal with respect to $\{e_i\}_{i=1}^k$ with ± 1 on the diagonal. They are isometries by the definition of an 1-unconditional basis. It is clear that G satisfies the condition from Proposition 1 (1).

Example 2. Let A be a compact abelian group. For each function f on A and for each $b \in A$ we define the b-translation of $f = f(a), a \in A$ by $f_b(a) = f(ab)$. Let X be a translation invariant function space on A. Translation invariance means that $f_b \in X$ for all $f \in X$ for all $b \in A$, and $||f_b||_X = ||f||_X$. Let $\{x_i\}_{i=1}^k$ be a finite set of characters of A contained in X. Consider the linear span V of $\{x_i\}_{i=1}^k$ as a subspace of X. Then $\{x_i\}_{i=1}^k$ is a distinguished basis of V.

In fact, by the definition of a translation invariant space we get: $f \mapsto f_b$ is an isometry on X. By the definition of a character we get

$$(x_i)_b = x_i(b)x_i. \tag{1}$$

Hence the space V is invariant under translation. Let G be the group of isometries of V that we obtain in such a way. By (1) each element of G is diagonal with respect to the basis $\{x_i\}_{i=1}^k$. Also, since $x_i \neq x_j$, for

$i \neq j$, then there exists $b \in A$ such that $x_i(b) \neq x_j(b)$. The assertion follows by Proposition 1 (1). $\square$

Example 3. Let X and Y be two finite-dimensional normed spaces with distinguished bases. By $L(X, Y)$ we denote the space of all linear operators from X to Y. Let α be a norm on $L(X, Y)$ satisfying the conditions:

(A) $\alpha(AB) \leq \alpha(A)||B||$ for every $A \in L(X, Y)$ and $B \in L(X, X)$;

(B) $\alpha(CA) \leq ||C||\alpha(A)$ for every $A \in L(X, Y)$ and $C \in L(Y, Y)$.

Then the space $(L(X, Y), \alpha)$ has a distinguished basis.

In fact, let $\{x_i\}_{i=1}^n$ be a distinguished basis in X, and let $\{y_i\}_{i=1}^m$ be a distinguished basis in Y. We identify $L(X, Y)$ with the space of all $m \times n$-matrices in a natural way. Let G_X and G_Y be groups corresponding to the distinguished bases $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^m$. Condition (A) implies that the right multiplication by an element of G_X is an isometry on $(L(X, Y), \alpha)$. Condition (B) implies that the left multiplication by an element of G_Y is an isometry on $(L(X, Y), \alpha)$. We consider the group G_L of isometries generated by these isometries on $(L(X, Y), \alpha)$. Let $\{l_{i,j}\}_{i=1,}^{m}{}_{j=1}^{n}$ be a basis in $L(X, Y)$ consisting of all matrices with only one nonzero entry. It is easy to see that each element in G_X and G_Y is diagonal with respect to this basis. Hence each element in G_L is diagonal with respect to it. Observe, also, that for each pair of different elements in this basis we can find an element in G_X or an element in G_Y such that the corresponding diagonal entries are different. Hence G_L satisfies the condition from Proposition 1 (1), and $\{l_{i,j}\}_{i=1,}^{m}{}_{j=1}^{n}$ is a distinguished basis in $(L(X, Y), \alpha)$. $\square$

Remark 1. It is known (see [**11**]) that a normed space of dimension n, $n \neq 2, 4$ in the real case, different from a Euclidean space, cannot have two different 1-symmetric bases (that is, two 1-symmetric bases that cannot be obtained from each other by permutations and multiplications by scalars). See also [**18**] and [**37**, Section 8 in Chapter IX], for closely related results. The uniqueness of 1-unconditional bases in complex Banach spaces was studied by Kalton and Wood [**22**]. They proved a very strong uniqueness result (see Theorem 6.1 and Lemma

5.2 in [**22**]). In the real case the situation is different; it was studied by Lacey and Wojtaszczyk [**27**] and by Randrianantoanina (see Theorem 4 in [**36**]). Another type of uniqueness result was discovered by Lindenstrauss and Pełczyński [**28**]. They proved that a λ-unconditional basis in l_1^n, or l_∞^n, is $c(\lambda)$-equivalent to the unit vector basis in the space (see the proof of Theorem 6.1 in [**28**]).

Distinguished bases do not have any of the uniqueness properties introduced in the above-mentioned papers. In fact, let A_N, where N is a positive integer, be the Cantor group, that is $A_N = \{-1, 1\}^N$ with the coordinatewise product. Consider $l_1(A_N)$. This space has two quite different distinguished bases: the unit vector basis and the basis consisting of all characters of the group A_N.

Remark 2. The norms satisfying the conditions (A) and (B) from Example 3 have been studied in the theory of operator ideals, see [**33**] (and also [**19**, p. 17] and [**41**, p. 19]).

Relations with some other classes of spaces with nontrivial groups of isometries.

(1) There exist spaces with distinguished bases that do not have enough symmetries. An example of such space can be constructed in the following way. We consider the group G of all operators on $\mathbf{R}^k$ that are diagonal with respect to the unit vector basis of $\mathbf{R}^k$, and the diagonal elements are ± 1. By Theorem 3.1 of [**18**] there exists a norm on $\mathbf{R}^k$, whose group of isometries coincide with G. It is clear that the corresponding normed space does not have enough symmetries.

(2) There exist spaces with distinguished bases that do not have $1-$unconditional bases. This assertion follows, for example, from the results of Gordon and Lewis [**17**] (see also [**19**, Section 12]). They proved that many of the spaces described in Example 3 are far from having $1-$unconditional bases.

Theorem 1. *There exist finite dimensional normed spaces with enough symmetries that do not have distinguished bases.*

Proof. Our proof is based on the fact that there exist irreducible representations of symmetric groups that do not have subgroups satisfying the conditions of Proposition 1. We are not going to describe the conditions under which this phenomenon happens in any generality; we shall rather give a completely concrete example.

Consider the symmetric group S_5 and consider its irreducible representation corresponding to the frame with three rows: the first row is of length 3, the second row and the third row are of length 1 (we use the terminology from [**2**, Chapter IV]). According to the well-known formula, see formula (4.4) from [**2**, p. 119], the dimension of this representation is equal to 6. This representation is faithful (see the discussion in the introduction to Chapter IV of [**2**]) and real.

For readers who are not familiar with the representation theory we summarize this discussion in the following way: there exists a subgroup F of the orthogonal group $O(6)$ on $\mathbf{R}^6$ satisfying the conditions:

(A) If $T : \mathbf{R}^6 \to \mathbf{R}^6$ is a linear operator satisfying $fT = Tf$ for all $f \in F$, then $T = \lambda I_{\mathbf{R}^n}$ for some $\lambda \in \mathbf{R}$.

(B) The group F is isomorphic to S_5.

Let $F^* = F \cup (-F) \subset O(6)$, where by $-F$ we denote the set of all elements of F multiplied by -1. By a result of Gordon and Loewy, [**18**, Theorem 3.1], there exists a norm on $\mathbf{R}^6$ such that F^* is the group of (all) isometries of the corresponding normed space, which we will denote by X_F. The property (A) of the group F implies that X_F has enough symmetries.

To show that X_F does not have a distinguished basis we need to show that $F \cup (-F)$ does not have a subgroup satisfying the condition of the part (1) of Proposition 1. It is easy to see that it is enough to show that F does not have such subgroup. By Proposition 1 it is enough to show that F does not contain an abelian subgroup G of cardinality ≥ 6 such that $g^2 = I$ for every $g \in G$. By the condition (B) it is enough to verify that any abelian subgroup H in S_5 satisfying $h^2 = e$ for all $h \in H$ (where e is the unit permutation) has cardinality ≤ 4.

We prefer to prove the following more general result.

Lemma 1. *Let $H \subset S_n$ be an abelian subgroup satisfying $h^2 = e$ for all $h \in H$. Then $\#H \leq 2^{[n/2]}$.*

Proof. By induction. The result is obvious for $n = 1, 2$. Suppose that we have proved it for all positive integers $\leq n - 1$.

Consider a subgroup $H \subset S_n$ satisfying the conditions. If H is not transitive on $\{1, \dots, n\}$, we can split $\{1, \dots, n\}$ onto two invariant subsets A and B. By the induction hypothesis $\#(H|_A) \leq 2^{[\#A/2]}$ and $\#(H|_B) \leq 2^{[\#B/2]}$. Also $\#H \leq \#(H|_A)\#(H|_B)$. The inequality follows.

In the case when H is transitive we prove that n is even and that $\#H \leq n$. This estimate is sufficient for our purposes because $n \leq 2^{[n/2]}$ for every even $n \in \mathbf{N}$.

Claim. *Let H be transitive. Then n is even. If $h \in H$ is such that $h \neq e$, then h does not fix any element of $\{1, \dots, n\}$.*

Proof. We start with the second statement. Assume the contrary. Let $h \in H$ be such that $h(i) = i$ and $h(j) \neq j$ for some $i, j \in \{1, \dots, n\}$. Observe that the condition $h^2 = e$ implies that each element in H is a product of disjoint transpositions. By transitivity there exist $g \in H$ such that (i, j) is one of the transpositions in the product representing g. Since H is abelian, then $hg = gh$. We get $j = g(i) = gh(i) = hg(i) = h(j) \neq j$. This contradiction proves the second statement.

The first statement follows from the second because the product of disjoint transpositions moves only even amount of numbers.

To derive the lemma from the claim observe that H cannot contain more than one permutation whose representation as a product of disjoint transpositions contains $(1, i)$, because the product of such permutations would have (at least two) fixed points. Hence H cannot have more than n elements. This proves the lemma and the theorem. □

Remark. The statement about symmetric groups mentioned at the beginning of the proof of the theorem can be derived from known results on permutation groups (see Theorem 5.8A and Exercise 5.8.2 in [**10**]). We have preferred to give the elementary proof for the convenience of the reader.

The spaces with distinguished bases have the following important property:

Theorem 2. *Let V be a normed space with a distinguished basis $\{e_i\}_{i=1}^k$. Let T be an operator diagonal with respect to $\{e_i\}_{i=1}^k$, with the diagonal elements $\{d_i\}_{i=1}^k$. Then*

$$\pi_2(T) = \left(\sum_{i=1}^k |d_i|^2 \right)^{1/2}. \tag{2}$$

Proof. The fact that

$$\pi_2(T) \geq \left(\sum_{i=1}^k |d_i|^2 \right)^{1/2} \tag{3}$$

is a special case of a result of Pietsch [**32**] (see also Proposition 2.a.1 from [**23**]).

So it remains to prove the estimate from above only.

It is well known (see, e.g., [**19**, Proposition 4.3 and Theorem 5.11]) that π_2-norm is self-dual with respect to the trace duality.

Therefore

$$\pi_2(T) = \sup_{S:V\to V} \frac{\mathrm{tr}(TS)}{\pi_2(S)}. \tag{4}$$

Let S be such that we have equality in (4). Let G be a group of isometries of V corresponding to the distinguished basis $\{e_i\}_{i=1}^k$. Let $g \in G$. Observe that $\pi_2(gSg^{-1}) = \pi_2(S)$ and $Tg = gT$. Hence

$$\mathrm{tr}(TgSg^{-1}) = \mathrm{tr}(gTSg^{-1}) = \mathrm{tr}(TS).$$

Let

$$\tilde{S} = \int_G gSg^{-1} d\mu(g),$$

where μ is the normalized Haar measure on G. Then $\pi_2(\tilde{S}) \leq \pi_2(S)$ and $\mathrm{tr}(T\tilde{S}) = \mathrm{tr}(TS)$. Hence

$$\pi_2(T) \leq \frac{\mathrm{tr}(T\tilde{S})}{\pi_2(\tilde{S})}.$$

Standard verification shows that $g\tilde{S} = \tilde{S}g$ for every $g \in G$. Hence $\tilde{S}$ is diagonal with respect to $\{e_i\}_{i=1}^k$. Let $s_1, \dots, s_n$ be the diagonal elements of $\tilde{S}$. Since they are the eigenvalues of $\tilde{S}$, by (3) we get

$$\pi_2(\tilde{S}) \geq \left(\sum_{i=1}^{k} |s_i|^2\right)^{1/2}.$$

Using this inequality and the fact that $\mathrm{tr}\,(T\tilde{S}) = \sum_{i=1}^k d_i s_i$, we get the desired inequality. $\square$

Let V be an n-dimensional normed space.

Definition 2. Let $\{e_i\}_{i=1}^n$ be a basis in V. The *distinguished basic constant* of $\{e_i\}_{i=1}^n$ is defined by

$$\mathrm{dbc}\,(\{e_i\}_{i=1}^n) = \inf_G \sup_{g \in G} ||g||,$$

where the infimum is taken over all groups G of operators on V satisfying the condition

(5) $\quad gT = Tg \quad \text{for all } g \in G \Longleftrightarrow T$ is diagonal w.r.t. $\{e_i\}_{i=1}^n$.

The *distinguished basic constant* of V is defined by

$$\mathrm{dbc}\,(V) = \inf_{\{e_i\}} \mathrm{dbc}\,(\{e_i\}_{i=1}^n).$$

This definition is the natural analogue of the well-known definitions of basic constants and asymmetry constants, see [**13**, p. 349], [**31**, p. 104] and [**41**, pp. 2 and 133].

Our next purpose is to estimate

$$\sup\{\operatorname{dbc}(V):\ \dim V = n\}.$$

John proved the following (by now well-known) estimate for the Banach-Mazur distance [**20**] (see [**31**, p. 144]):

$$d(V, l_2^n) \le \sqrt{n}.$$

This estimate and the fact that $\operatorname{dbc}(l_2^n) = 1$ imply that $\operatorname{dbc}(V) \le \sqrt{n}$ for every n-dimensional space V. It turns out that this estimate from above cannot be significantly improved, at least in the real case. More precisely, we prove

Theorem 3. *There exists an absolute constant $c > 0$ such that*

$$\sup\{\operatorname{dbc}(V) : V \text{ is real and } \dim V = n\} \ge c\sqrt{n}.$$

Remark. Similar results are already known for unconditional basic constant [**12**] and for asymmetry constant [**29**].

Our proof of Theorem 3, as well as the proof in [**29**], is based on the approach invented by Gluskin [**15**]. More precisely, we shall use the following result of Szarek [**39**].

Consider $\mathbf{R}^n$ with its standard inner product, and denote by $||\cdot||_2$ the corresponding norm. Following [**39**] we will say that a linear operator $T : \mathbf{R}^n \to \mathbf{R}^n$ satisfies the condition $M_{k,\alpha}$ (M for "mixing") if

there exists a subspace $H \subset \mathbf{R}^n, \dim H \ge k$, such that $||P_{H^\perp}Tx||_2 \ge \alpha||x||_2\ \forall x \in H$.

Here $P_{H^\perp}$ denotes the orthogonal projection onto $H^\perp$.

Theorem 4 [**39**, Theorem 1.4]. *Given $\delta > 0$ there exists a norm $||\cdot||_B$ on $\mathbf{R}^n$ such that whenever T satisfies $(M_{k,\alpha})$ with some $k \ge \delta n$. Then $||T : B \to B|| \ge c\alpha\sqrt{n}$, where c depends only on δ.*

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be diagonal with respect to some basis in $\mathbf{R}^n$ with diagonal entries equal to ± 1.

Claim. *If the number of* 1*'s on the diagonal of* T *is between* $n/4$ *and* $3n/4$*, then* T *satisfies* $(M_{k,\alpha})$ *with* $k = [n/4]$ *and* $\alpha = 1$*.*

Proof. In fact, let $M_1 \subset \mathbf{R}^n$ be the eigenspace corresponding to 1 and let M_{-1} be the eigenspace corresponding to -1. It is easy to see that we can find subspaces $L_1 \subset M_1$ and $L_{-1} \subset M_{-1}$ such that L_1 and L_{-1} are orthogonal, and $\dim L_1 = \dim L_{-1} \geq [n/4]$.

Let $S : L_1 \to L_{-1}$ be an isometry. We introduce a subspace $H \subset \mathbf{R}^n$ by

$$H = \{l + Sl : \ l \in L_1\}.$$

Observe that for $x = l + Sl \in H$ we have $Tx = l - Sl$, hence $Tx \perp H$ and $||P_{H^\perp} Tx||_2 = ||Tx||_2 = ||x||_2$. $\square$

Proof of Theorem 3. We are going to show that the space B constructed in Theorem 4 satisfies $\operatorname{dbc}(B) \geq c\sqrt{n}$ where c is the constant that corresponds to some $\delta < 1/4$ in Theorem 4.

By the claim above it is enough to prove that for any group G of operators on $\mathbf{R}^n$ satisfying the condition of Proposition 1 (1), there exists an operator $T \in G$ such that the amounts of 1's and -1's in the diagonal representation of T are $\geq n/4$.

As in the proof of Proposition 1 we identify G with the corresponding subgroup of the multiplicative group of functions on $\{1, \dots, n\}$ with the values in $\{-1, 1\}$. It is enough to prove that for at least one of the functions the sum of its values is $\leq n/2$.

By the fundamental theorem of abelian groups (see [**42**, Section 12.3]) there exist elements $g_1, \dots, g_l \in G$ such that G is the direct product of the groups generated by $g_1, \dots, g_l$.

The functions $g_1, \dots, g_l$ separate points of $\{1, \dots, n\}$, see Proposition 1 (1). Therefore, to each collection f from $\{-1, 1\}^l$ there corresponds one or none point x from $\{1, \dots, n\}$ satisfying $f = \{g_i(x)\}_{i=1}^l$. In this way the set $\{1, \dots, n\}$ is mapped onto a subset Z in $\{-1, 1\}^l$.

Let $g \in G$. We need to estimate $\sum_{i=1}^n g(i)$. Let $g = g_{i_1} \cdots g_{i_p}$ be the (uniquely determined) representation of g as a product of some of

$\{g_1, \dots, g_l\}$. Then

$$\sum_{i=1}^{n} g(i) = \sum_{z \in Z} w_A(z),$$

where w_A is the Walsh function on $\{-1,1\}^l$ corresponding to $\{i_1, \dots, i_p\} \subset \{1, \dots, l\}$.

Hence we have reduced our problem to the following. Consider the Cantor group $\{-1,1\}^l$ endowed with the counting measure, and a subset $Z \subset \{-1,1\}^l$ of cardinality n. Show that there exist a Walsh function w_A on $\{-1,1\}^l$ such that

$$|\langle w_A, \chi_Z \rangle| \leq \frac{n}{2}, \tag{6}$$

where χ_Z is the characteristic function of Z and the scalar product is computed with respect to the counting measure.

Observe that

$$||\chi_Z||^2 = \sum_A \frac{|\langle w_A, \chi_Z \rangle|^2}{||w_A||^2}, \tag{7}$$

where the norms correspond to the scalar product described above.

Observe that $||\chi_Z||^2 = n$, $||w_A||^2 = 2^l$. There are 2^l summands in the righthand side of (7). Hence (7) implies that

$$2^l \frac{|\langle w_A, \chi_Z \rangle|^2}{||w_A||^2} \leq ||\chi_Z||^2$$

for some A. This inequality implies $|\langle w_A, \chi_Z \rangle| \leq \sqrt{n}$. Since $\sqrt{n} \leq n/2$ for $n \geq 4$, we have proved (6).

3. Estimates of π_2-norms and extension norms for operators with given action. Now we turn to the following problem. Let A be a $k \times k$ matrix and let V be a k-dimensional space. By $L_A(V)$ we denote the set of all operators on V whose matrix with respect to some basis is A. The problem is to estimate from below, and, if possible, to minimize the values of some natural norms over the set $L_A(V)$. Some aspects of this problem were studied in the well-known papers on basic constants (Gluskin [**16**] and Szarek [**38**] and [**39**]). In these papers the problem was studied in the case when A is diagonal with 0s and 1s on the diagonal, and the norm is the usual operator norm. A systematic study of this problem for arbitrary A and extension norms, defined below, was initiated in [**4**] and [**7**] (and continued in [**8**], [**9**] and [**30**]). This study led to the study of this problem for the π_2-norm, see [**8**]. The remaining part of the present paper is devoted to some problems that arise naturally in connection with the results of [**4**], [**7**] and [**8**]. We refer to $L_A(V)$ as to the set of operators *with given action*.

For a Banach space X containing V as a subspace and for a linear operator $T : V \to V$ by $e(T, X)$ we denote the infimum of norms of operators $\tilde{T} : X \to V$ satisfying $\tilde{T}|_V = T$. By $e(T)$ we denote the supremum of $e(T, X)$ taken over all Banach spaces X containing isometric copies of V and over all isometric embeddings $V \subset X$. It can be shown that $e(\cdot)$ is a norm on $L(V)$. We call this norm the *extension norm*.

Let V be a k-dimensional normed space and let A be a $k \times k$-matrix. The problem mentioned above is to estimate

$$E(A, V) := \inf_{T \in L_A(V)} e(T).$$

The well-known Kadets-Snobar theorem, see [**21**], can be restated as:

$$E(I, V) \leq \sqrt{k},$$

where I is the identity matrix of order k.

Let D be a diagonal matrix with the numbers $\{d_i\}_{i=1}^k$ on the diagonal. One of the natural directions of generalization of the Kadets-Snobar theorem is to ask:

whether

$$E(D, V) \leq \left(\sum_{i=1}^{k} |d_i|^2 \right)^{1/2} \tag{8}$$

for any V? (See [**4**], where a stronger assertion was conjectured. See also the discussion below.)

Observe that the Pietsch factorization theorem [**34**] (see [**35**, p. 11]) implies that $e(T) \leq \pi_2(T)$ for every T. Hence

$$E(A, V) \leq \inf_{T \in L_A(V)} \pi_2(T).$$

Let D be a diagonal $k \times k$ matrix with the entries $d_1, \ldots, d_k$ on the diagonal. Let V be a k-dimensional normed linear space.

In [**8**, Conjecture 1, p. 73], the following stronger form of (8) was conjectured:

$$\inf_{T \in L_D(V)} \pi_2(T) = \left(\sum_{i=1}^{k} |d_i|^2 \right)^{1/2} \tag{9}$$

Theorem 2 implies that this conjecture is true for spaces with distinguished bases. In fact, observe that

$$\inf_{T \in L_D(V)} \pi_2(T) \geq \left(\sum_{i=1}^{k} |d_i|^2 \right)^{1/2}$$

is an immediate consequence of (3).

The inequality

$$\inf_{T \in L_D(V)} \pi_2(T) \leq \left(\sum_{i=1}^{k} |d_i|^2 \right)^{1/2}$$

immediately follows from Theorem 2.

Our next purpose is to prove the inequality (8) for a class of spaces. (It can be shown that this class of spaces contains spaces without distinguished bases.)

Let us denote by I the canonical mapping from l_2^k into l_∞^k. We say that V has the *Dvoretzky-Rogers constant* 1 if there exist operators $\alpha : l_2^k \to V$ and $\beta : V \to l_\infty^k$ such that $I = \beta\alpha$ and $||\alpha|| = ||\beta|| = 1$.

Proposition 2. *The inequality* (8) *is valid for spaces with the Dvoretzky-Rogers constant* 1.

Proof. Let V be a space with the Dvoretzky-Rogers constant 1, α and β be as in the corresponding definition, and let X be a Banach space containing V as a subspace.

We need to find an operator $T : V \to V$ and its extension $\tilde{T} : X \to V$ such that $T \in L_D(V)$ and $||\tilde{T}|| \le (\sum_{i=1}^k |d_i|^2)^{1/2}$.

Let $C = \beta^{-1}B(l_\infty^k)$. Then $C \supset B(V)$. The Hahn-Banach theorem implies that there exists a projection $P : X \to V$ satisfying $P(B(X)) \subset C$.

Let $e_1, \dots, e_k \in V$ be the pre-images of the unit vector basis of l_∞^k under β.

We define T by the equations $Te_i = d_i e_i$, $i = 1, \dots, k$. Let $\tilde{T} = TP$. It is clear that $\tilde{T}$ is an extension of T to X.

To show that $||\tilde{T}|| \le (\sum_{i=1}^k |d_i|^2)^{1/2}$ we observe that since $||\alpha|| = 1$, then

$$||\tilde{T}(x)|| \le ||\alpha^{-1}\tilde{T}(x)||_{l_2^k}.$$

Let $x \in B(X)$. Since $P(B(X)) \subset C$, the formula for $\tilde{T}$ implies that

$$\alpha^{-1}\tilde{T}(x) = \sum_{i=1}^k d_i \omega_i \alpha^{-1} e_i, \text{ where } |\omega_i| \le 1. \tag{10}$$

Since $\{\alpha^{-1}e_i\}_{i=1}^k$ is the unit vector basis in l_2^k, then (10) implies

$$||\alpha^{-1}\tilde{T}(x)||_{l_2^k} \le \left(\sum_{i=1}^k |d_i|^2\right)^{1/2}. \qquad \square$$

Remark 1. It is well known (see [**19**, Theorem 0.10], [**31**, Corollary 15.2] or [**41**, Proposition 14.6]) that for every k-dimensional space X there exist operators $\gamma : l_1^k \to V$ and $\delta : V \to l_\infty^k$ such that $||\gamma|| = ||\delta|| = 1$ and $I = \delta\gamma$, where I is the canonical mapping from l_1^k into l_∞^k. Using the same argument as in Proposition 2 we can prove

$$E(D, V) \le \sum_{i=1}^k |d_i|. \tag{11}$$

Hence

$$E(D,V) \le \sqrt{k}\left(\sum_{i=1}^{k} |d_i|^2\right)^{1/2}. \tag{12}$$

Estimates (11) and (12) can be improved using the result of Giannopoulos [**14**]. It should be mentioned that straightforward applications of the result from [**14**] improve (12) in a nontrivial way only for sequences $\{d_i\}_{i=1}^k$ that are "small" perturbations of sequences with "many" zeros. For example, combining Theorem 1 from [**14**] and the argument of Proposition 2 we get: if at least half of the elements of $\{d_i\}_{i=1}^k$ are zeros, then

$$E(D,V) \le 2c\left(\sum_{i=1}^{k} |d_i|^2\right)^{1/2},$$

where c is the absolute constant from [**14**, Theorem 1].

Remark 2. Szarek [**40**] proved that there exist spaces with large Dvoretzky-Rogers constants.

Let A be an operator in a k-dimensional normed space V. By $\{\alpha_i\}_{i=1}^k$ we denote the sequence of eigenvalues of A (each eigenvalue is listed according to its algebraic multiplicity). Let $\rho = |\sum_{i=1}^k \alpha_i|$, the absolute value of the trace of A, and let $\beta = \sum_{i=1}^k |\alpha_i|^2$.

In [**26**] (see also [**25**] and [**24**]) the following inequality was proved for the case $A = I$ (the identity operator on a k-dimensional normed space V).

$$E(A,V) \le f_A(k, n(k)) \quad \text{with } n(k) := \left\{ \begin{array}{cc} k(k+1)/2 & \mathbf{R} \\ k^2 & \mathbf{C} \end{array} \right\}, \tag{13}$$

where $f_A(k,n) := \rho/n + \sqrt{(n-1)(n\beta-\rho^2)}/n$.

The truth of (13) and the fact that it is best possible, for $k = 2$, $\mathbf{R}$, A diagonalizable, and V_2 an unconditional space, has been shown in [**6**].

Straightforward verification shows that $f_A(k,n) \le \sqrt{\beta}$ and $\lim\limits_{n\to\infty} f_A(k,n) = \sqrt{\beta}$. (Note that the aforementioned inequality is an equality if and only if $\rho = \sqrt{\beta}$, i.e., if and only if $\sum_{i\neq j} \alpha_i \overline{\alpha_j} = 0$.)

Our next purpose is to prove the inequality (13) in a special case where there is no apparent distinguished basis.

Let V be a two-dimensional subspace of $l_1^{(3)}$. Then it is easy to see, by simple algebraic considerations, that V is isometric to one of the subspaces $V(a,b)$ where

$$V(a,b) := \operatorname{span}\{(1,a,0),\,(0,b,1)\};\;\; 1 \geq a,b \geq 0.$$

Now consider the following matrix $\tilde{T}$ given by:

$$\begin{aligned}\rho\tilde{T} = & d_1 \begin{bmatrix} a+b+b^2(1+a) & a(1+b) & -ab(1+b) \\ a(a+b) & a^2(1+b)+b^2(1+a) & b(a+b) \\ -ab(1+a) & b(1+a) & a+b+a^2(1+b) \end{bmatrix} \\ & + \frac{d_1-d_2}{4} \begin{bmatrix} \varepsilon_1(a,b) & \varepsilon_2(a,b) & \varepsilon_3(a,b) \\ \varepsilon_4(a,b) & \varepsilon_5(a,b) & \varepsilon_4(b,a) \\ \varepsilon_3(b,a) & \varepsilon_2(b,a) & \varepsilon_1(b,a) \end{bmatrix},\end{aligned}$$

where $\rho := a+b+a^2(1+b)+b^2(1+a)$, $\varepsilon_1(a,b) := (a^2-b^2)(2+ab)-(2(a+b)+a^3+3ab^2))$, $\varepsilon_2(a,b) := (b-a)^2+b(b^2-a^2)-4a$, $\varepsilon_3(a,b) := (a^2-b^2)(b^2+b)+2(a+b)^2+4ab^2+2(a+b)$, $\varepsilon_4(a,b) := (a+b)(2(b-a)+a(b^2-a^2)+2(a^2+b^2)$, $\varepsilon_5(a,b) := (a-b)(a^2-b^2)-4(a^2+b^2)$.

Consider $\tilde{T}$ as an operator from $l_1^{(3)}$ onto V with respect to the basis $(v_1,\,v_2) = ((1,a,0),\,(0,b,1))$. Now it is a direct check that if $(w_1,\,w_2) = ((1,a+b,1),\,(-1,b-a,1))$, then $\tilde{T}w_1 = d_1w_1$ and $\tilde{T}w_2 = d_2w_2$. Furthermore if we replace (a,b) by $[(a+b/2),(a+b/2)]$ in the above formulas, then $\tilde{T}$ is transformed into the operator $\tilde{T}_u$, where $\tilde{T}_u$ is an operator from $l_1^{(3)}$ onto the two-dimensional subspace V_u with unconditional basis $(z_1,\,z_2) = (1,a+b,1),\,(-1,0,1)$ and $\tilde{T}_u(z_1,\,z_2) = (d_1z_1,\,d_2z_2)$. Furthermore if $|d_1-d_2|$ is sufficiently small, then the sign configuration of $\tilde{T}$ is clearly

$$(\operatorname{sign} d_1)\begin{bmatrix} + & + & - \\ + & + & + \\ - & + & + \end{bmatrix},$$

and the norms of $\tilde{T}$ and $\tilde{T}_u$ can be calculated from the absolute column sums (all equal):

$$\|\tilde{T}\| = \frac{(a+b)(a^2(d_1-d_2)+(1+b)(b(d_1-d_2)+2d_2)+a(d_1+d_2+2bd_2))}{2(a+a^2+b+a^2b+b^2+ab^2)}$$

and

$$\|\tilde{T}_u\| = \frac{(2+a+b)(ad_1+bd_1+2d_2)}{4+a^2+2b+b^2+2a(1+b)}.$$

Next, in the case $|d_1-d_2|$ is sufficiently small, we calculate

$$\|\tilde{T}_u\| - \|\tilde{T}\| = ((d_2-d_1)(a+b)^2+4d_2)q,$$

where $q = (a-b)^2(1+a+b)/(2(a+a^2+b+a^2b+b^2+ab^2)(4+a^2+2b+b^2+2a(1+b)))$ is clearly nonnegative. We conclude that for each space $V = \operatorname{span}\{(1,a,0),\,(0,b,1)\} = \operatorname{span}\{(1,a+b,1),\,(-1,b-a,1)\}$ there is associated the unconditional space

$$V_u = \operatorname{span}\{(1,a+b,1),(-1,0,1)\}$$

with $\|\tilde{T}\| \le \|\tilde{T}_u\|$, provided $d_2 > 0$ and $|d_1-d_2|$ is sufficiently small. But $\|\tilde{T}_u\| \le f_D(2,3)$ by Lemma 1 of [**6**]. We summarize this discussion in the following. (Note that we also use the well-known fact that L^1 is a *maximal* overspace for any 2-dimensional real subspace.)

Theorem 5. *Let V be a two-dimensional subspace of $l_1^{(3)}$. Then*

$$E(D,V) \le f_D(2,3) \le \sqrt{d_1^2+d_2^2}$$

if $d_2 > 0$ and $|d_1-d_2|$ is sufficiently small.

Remark 3. Although, for the purposes of this paper, we have shown the above result only for $d_2 > 0$ and $|d_1-d_2|$ sufficiently small ($\tilde{T}$ was represented in the above form for precisely this reason), this assumption is not necessary, as further analysis, analogous to that found in [**6**], will show.

Remark 4. By use of the theory of [**3**], the operators $\tilde{T}$ and $\tilde{T}_u$ above can be shown to be minimal with respect to the prescribed action on V and V_u respectively. (See [**5**] for example.)

Remark 5. The proof of the theorem above uses the basis $(w_1,\, w_2) = (1,a+b,1),\,(-1,b-a,1)$ for V to obtain an associated unconditional basis $(z_1,\, z_2) = (1, a+b, 1)$, $(-1, 0, 1)$ for the associated space V_u. This association can be described in a systematic way as follows. First, V is isometric to the subspace $[1,\, t]$ of $L_1(\mu)$, where

$$\mu(t) = \delta_{-1}(t) + (a+b)\delta_{b-a/a+b}(t) + \delta_1(t).$$

Now "symmetrize" μ as follows by defining

$$\mu_s(t) := (\mu(t) + \mu(-t))/2 = \delta_{-1} + \frac{a+b}{2}\delta_{\frac{a-b}{a+b}} + \frac{a+b}{2}\delta_{\frac{b-a}{a+b}} + \delta_1.$$

Then let V_s be the subspace $[1,\, t]$ of $L_1(\mu_s)$ Next, following the procedure in [**6**], we construct the unconditional space V_u from the unconditional space V_s and obtain that $E(D, V_s) \le E(D, V_u)$. We conjecture that in fact $E(D, V) \le E(D, V_s)$. What we have shown here in this paper (see the theorem and note above) is that $E(D, V) \le E(D, V_u)$ directly. The process that leads from V to V_s above and the accompanying conjecture that $E(D, V) \le E(D, V_s)$ indicates that we might want to say that $(1, a+b, 1)$, $(-1, b-a, 1)$ is a *near-unconditional* basis for V and generalize the notion of a near-unconditional basis first to any 2-dimensional space and then to an arbitrary k-dimensional space. (Note that any 2-dimensional space is isometric to a subspace $[1,\, t]$ of $L_1(\mu)$ for some measure μ.)

Now we shall disprove Conjecture 2 from [**8**, p. 75]. The conjecture states that for each $\delta > 0$ there exists $N \in \mathbf{N}$ such that for any $n > N$ and any n-dimensional matrix A with spectral radius 1 we have

$$\inf_{T \in L_A(l_\infty^n)} e(T) \le 1 + \delta.$$

We refer to [**8**] for the background material that led to this conjecture.

A counterexample. Consider the matrix

$$B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

It is easy to see that

$$B^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows, in particular, from here that $\rho(B) = 1$, the spectral radius.

Let $A_n = B \oplus I_{n-2}$, the direct sum of B and the identity matrix of the size $(n-2) \times (n-2)$. It is clear that $\rho(A_n) = 1$ for every $n \geq 2$.

It is well known that $||T|| = e(T)$ for $T \in L(l_\infty^n)$. Hence we need to show that there exists $\delta > 0$ such that $||T|| > 1 + \delta$ for every n and any $T \in L_{A_n}(l_\infty^n)$.

Assume the contrary. Then for any $\delta > 0$ there exist $n \in \mathbf{N}$ and an operator $T_n \in L_{A_n}(l_\infty^n)$ satisfying $||T_n|| \leq 1 + \delta$. Let $V \subset l_\infty^n$ be the linear span of the first two vectors of the corresponding basis. Then

$$P = \frac{I_n - T_n^4}{2},$$

where I_n is the identity operator on l_∞^n, is a projection onto V and its norm is $\leq 1 + (1+\delta)^4/2$. By the result of Zippin [**43**], [**44**] (see also [**1**]) if $\delta > 0$ is small enough, this estimate implies the following estimate for the Banach-Mazur distance: $d(V, l_\infty^2) \leq f(\delta)$, where f is a function defined for small $\delta > 0$ and such that $f(\delta) \to 1$ as $\delta \to 0$.

In [**8**, Theorem 2] it was proved that any operator on l_∞^2 whose matrix, in some basis, is B has norm $\geq \sqrt{2}$. Therefore, any operator on V whose matrix is B has norm $\geq \sqrt{2}/d(V, l_\infty^2)$. Choose δ in such a way that $\sqrt{2}/f(\delta) > 1 + \delta$. We get a contradiction. $\square$

4. Open problems. It seems that the asymptotic behavior of the quantitative relations between different basic constants and the asymmetry constant has not been studied so far. In particular, we suggest to estimate the asymptotic, as $n \to \infty$, growth of the numbers

$$D_n = \sup\left\{\frac{\mathrm{dbc}\,(V)}{s(V)} \;:\; \dim V = n\right\}$$

and

$$d_n = \sup\left\{\frac{s(V)}{\mathrm{dbc}\,(V)} \;:\; \dim V = n\right\},$$

where $s(V)$ is the asymmetry constant (as it was defined in [**13**, p. 349], see also [**41**, p. 133]).

It is clear that $1 \leq d_n \leq \sqrt{n}$, $1 \leq D_n \leq \sqrt{n}$. The results of [**13**, pp. 353–354] imply that $d_n \geq cn^{1/4}$, but it is difficult to believe that this estimate is the best possible, see [**29**], [**39**] for relevant results. Our Theorem 1 is a rather 'isometric' result. It does not provide any asymptotic estimates for D_n.

We also would like to mention here that Conjecture 1 from [**8**] (we reproduced the most important case of it as the equality (9)), and the problem on the generalization of the Kadets-Snobar theorem (see the inequality (8) above) remain open.

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