ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2004

RING DECOMPOSITIONS INDUCED BY CERTAIN LIE IDEALS

LEONARD CASCIOTTI

ABSTRACT. This paper studies a decomposition of a semiprime ring R with involution $*$ containing a subring U which is both a self-adjoint Lie ideal of R and contains a fixed power of each element of R. These results are applied to the case where U is the subring generated by the symmetric elements S and the norm elements $\{xx^* \mid x \in R\}$.

1. Introduction. This paper will investigate the procedure by which the Lie ideals of a ring R are used to determine certain characterizations of R itself. Conditions of "self-adjoint" and "simple Jordan" are imposed on the Lie ideals, from which the exact structure of the ring R is determined.

R is an associative ring with *involution* denoted by $*$. The involution $*$ is defined for each $x, y \in R$ such that

$$(x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*.$$

Let $S = \{x \in R \mid x = x^*\}$ denote the set of symmetric elements of the ring R. Then $\overline{S} = \{\sum s_1 s_2 s_3 \cdots s_n \mid s_i \in S\}$ is the subring generated by the symmetric elements S. A Lie multiplication is defined for the ring R as follows $[u, r] = (ur - ru)$, $u, r \in R$. An additive group U of R is said to be a *Lie ideal* of R if $[U, R] \subseteq U$, that is, $[u, r] \in U$, for all $u \in U$, $r \in R$.

For any arbitrary subsets A, B of R, $[A, B]$ denotes the additive subgroup generated by finite sums of products of the form $\pm[a, b]$, i.e., $\{\sum \pm[a, b] \mid a \in A, b \in B\}$.

R is 2-*torsion free* if $2x = 0$ implies $x = 0$. Therefore, R is 2-torsion free implies R is not of characteristic 2.

2. *-simplicity. A set L is *self-adjoint* if $L = L^*$. A self-adjoint ideal I of R is called a *-ideal. The notation $I \oplus K$ denotes an ideal

1991 *Mathematics Subject Classification.* 16N60, 16N99.
Received by the editors on July 26, 2000, and in revised form on June 16, 2001.

direct sum for any ideals I, K of R. Recall that R is a simple ring if $R^2 \neq (0)$ and the only 2-sided ideals of R are (0) and R. Analogously R is $*$-*simple* if $R^2 \neq (0)$ and the only $*$-ideals are (0) and R. $Z(R)$ denotes the center of the ring R.

The following technical lemmas are needed to carry out the main ring decomposition theorem. The first lemma concerning $*$-simple rings appears in the literature in several places, Aburawash [**1**] and Birkenmeier [**3**].

Definition 2.1. An idea I of R is said to be a *simple ideal* if and only if $I^2 \neq (0)$ and there exists no proper nonzero ideals J, of I. That is, $(0) \neq J \subseteq I$ implies $J = I$.

Lemma 2.2. *If R is not a simple ring, then R is $*$-simple if and only if $R = I \oplus I^*$ for I a simple ideal.*

Proof. Assume R is $*$-simple but not simple. There exists an ideal I, $(0) \neq I \subsetneqq R$. Clearly $I + I^*$ and $I \cap I^*$ are $*$-ideals and hence $R = I \oplus I^*$. Let K be a 2-sided ideal of I, i.e., $IKI \subseteq K$, then $RKR = (I \oplus I^*)K(I \oplus I^*) \subseteq K$ and so K is an R-ideal. Now $K + K^*$ is a $*$-ideal of R implies $K + K^* = R$ or (0). $K + K^* = (0)$ implies $K = (0)$. If $K + K^* = R$, then $I \subseteq K$, hence I Is a simple ideal. If $I^2 = (0)$, then $I(I \oplus I^*) = (0)$. $I \subseteq$ annihilator R^a, which is a $*$-ideal of R, and therefore $R^a = R$ or (0). Since $R^2 \neq (0)$, then $R^a = (0)$ which implies $I = (0)$.

Now the other direction. Let M be $*$-ideal of R. Then $I \cap M \neq (0)$ implies $I \subseteq M$ in which case $I^* \subseteq M$ and therefore $M = R$. If $I \cap M = (0)$, then a simple argument shows $M = (0)$. $\square$

Example 2.3. Let R be a subset of the 2×2 matrices of the form $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ with entries from a division ring D. The operations are $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b+d \\ a+c & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & db \\ ac & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Clearly R is not commutative and the transpose $*$ is an involution. To show R is $*$-simple, let $M \neq (0)$ be a $*$-ideal where

$\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \in M$ and either a or $b \neq (0)$. Now M is closed under all the operations and certainly contains an element of the form $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \in M$ with $a \neq (0)$. Since D is a division ring, then $aD = Da = D$. Hence $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \in M$. This shows $R \subseteq M$. Let I be the ideal $\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$. Properties of a division ring imply I is a simple ideal and maximal. So $R = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}^*$. □

Lemma 2.4. *If R is $*$-simple, then one of the following holds.*

(i) *For R not simple, then $Z(R) = Z(I) \oplus Z(I^*)$ for a simple ideal, I, of R.*

(ii) *For R simple and $Z(R) \neq (0)$, then R has an identity and $Z(R)$ is a field.*

(iii) *If R is simple, 2-torsion free and $Z(R) = (0)$, then R and (0) are the only subrings which are Lie ideals of R.*

Proof. Parts (i) and (ii) can be found in most standard texts on rings (Jacobson [**8**]) and part (iii) in Hestein [**6**, Theorem 1.2]. □

The next lemma first stated by Zuev [**12**] utilizes Lie ideals in a fundamental way.

Lemma 2.5. *Let U be a Lie ideal of R. Then $W(U) = \{w \in U \mid wR \subseteq U$ for U a Lie ideal of $R\}$ is a 2-sided ideal of R.*

Proof. Let $W = W(U)$. Clearly, W is a right ideal of R. To reach the desired conclusion, one need only show $RW \subseteq W$. One first notes $[w, r] = wr - rw \in U$ for all $w \in W$, $r \in R$. By the definition of W, $rw \in U$ for all $r \in R$. Therefore, $rw \in U$. Clearly, for all $r' \in R$, $(rw)r' - r'(rw) = [rw, r'] \in U$. Regrouping $(rw)r' - r'(rw) = (rw)r' - (r'r)w)$. One has from the previous statement $(r'r)w \in U$. Therefore, $(rw)r' \in U$ for all $r' \in R$. Hence for all $r \in R$, $rw \in W$ and so W is a left ideal of R. □

Corollary 2.6. *If U is self-adjoint and a Lie ideal of R, then $W(U)$ is a self-adjoint ideal of R.*

Proof. $W(U) \subseteq U$ implies $W^*(U) \subseteq U$ since U is self-adjoint. One notes that $w^*R \subseteq (R^*w)^* \subseteq (Rw)^* \subseteq W^*(U) \subseteq U$. Hence $w^* \in W(U)$ for $w \in W(U)$. □

Corollary 2.7. *If U is a Lie ideal and a subring, then $[U,U] \subseteq W(U)$.*

Proof. Consider $[u,v]r$ for u and $v \in U$, $r \in R$. One notes: $[u,v]r = uvr - vur = (u(vr) - (vr)u) + (vru - vur) = [u,vr] - v[u,r]$. Since U is both a Lie ideal and a subring of R, the latter summands are in U. Hence $[u,v]R \subseteq U$ and so $[u,v] \in W(U)$. Thus one concludes $[U,U] \subseteq W(U)$. □

One defines in an obvious manner the condition for a ring R with involution to be R $*$-*semi-prime*; namely, if A is a $*$-ideal and $A^2 = (0)$ implies $A = (0)$. It is known that R $*$-semi-prime is equivalent to R semi-prime, [**1**]. One notes that if R is a semi-prime ring with involution, then it is obvious that R is $*$-semi-prime. Suppose R is $*$-semi-prime and I is an ideal of R with $I^2 = (0)$. Clearly $(I^*)^2 = (0)$. $(I + I^*)^2 = II^* + I^*I \subseteq I$ implies $((I + I^*)^2)^2 \subseteq I^2 = (0)$. Since R is $*$-semi-prime, then $(I + I^*)^2 = (0)$ and in addition $(I + I^*) = (0)$. Hence $I = -I^*$ implies I is a $*$ ideal and consequently $I^2 = (0)$ implies $I = (0)$. Thus R is semi-prime.

Lemma 2.8. *Let U be both a Lie ideal and a subring in a semi-prime ring R. If U is 2-torsion free, then U is a semi-prime ring and $Z(U) = Z(R) \cap U$.*

Proof. To show that U, as a subring, is semi-prime, one needs to show $K^2 = (0)$ for an ideal K of U implies $K = (0)$. By Lemma 2.5, $W = W(U) \subseteq U$ is a 2-sided ideal of R. Since $WKW \subseteq K$, then $(WK)^2 \subseteq (WK)(WK) \subseteq (WKW)K \subseteq KK \subseteq K^2 = (0)$. Thus, WK is a nilpotent left ideal of R. Since R is semi-prime, $WK = (0)$. Thus, $K \subseteq R(W)$, the right annihilator of W. Now $(K \cap W) \subseteq (R(W) \cap W)$

and $(R(W) \cap W)^2 = (0)$. From this, one concludes $(R(W) \cap W) = (0)$.

By Corollary 2.7, $[U, U] \subseteq W$. Let $x \in R$ and $t \in K$. Since U is a Lie ideal of R and, by the above, one concludes $-2txt = [t, [t, x]] \in K \cap W = (0)$. In a semi-prime ring R, $(2t)R(2t) = (0)$ implies $2t = 0$. Since U is 2-torsion-free, then $2t = (0)$ implies $t = (0)$. Thus, $K = (0)$ and U is semi-prime.

Next $Z(U) = Z(R) \cap U$. One need only show $Z(U) \subseteq Z(R) \cap U$. Let $h \in Z(U)$ and $r \in R$. Then $[h, r] \in U$. Hence $[h, [h, r]] = 0$. $2[h, a][h, b] = 0$ for $a, b \in R$ follows from Herstein [**6**, Lemma 1.3]. Since U is 2-torsion free, a Lie ideal, and a subring, then $[h, a][h, b] = 0$. The Herstein lemma then shows in a semi-prime R that $[h, a] = 0$ for $a \in R$. Hence $h \in Z(R)$. □

Corollary 2.9. *If R is a semi-prime ring and $\overline{S}$ is 2-torsion free, then $\overline{S}$ is a semi-prime ring.*

Proof. The proof will follow if $\overline{S}$ satisfies the conditions imposed on U in Lemma 2.8. One need only show that the subring $\overline{S}$ is a Lie ideal. For $s \in S$, $r \in R$, $[s, r] = sr - rs = ((sr + r^*s) - (r + r^*)s) \in \overline{S}$. Assuming the induction hypothesis on the generators $s_1 s_2 s_3 \cdots s_n$ in $\overline{S}$, $[s_1 s_2 s_3 \cdots s_n, r] \in \overline{S}$. One has $[s_1 s_2 s_3 \cdots s_{n+1}, r] = [s_1 s_2 s_3 \cdots s_n, s_{n+1} r] + [s_{n+1}, r(s_1 s_2 s_3 \cdots s_n)] \in \overline{S}$. Thus by induction and the distributive rule, $\overline{S}$ is a Lie ideal. □

One now states and proves the main structure theorem whose motivation can in part be found in [**10**, Theorem 3.8]. Recall R is $*$-*prime* if, for $*$-ideals A, B and $AB = (0)$, then either $A = (0)$ or $B = (0)$. This follows the well-known characterization for a prime ring R.

Theorem 2.10. *Let U be a $*$-simple subring and a self-adjoint Lie ideal of a semi-prime ring R. In addition, let $x^n \in U$ for $x \in R$ where n is a fixed positive integer.*

(i) *If $Z(R) \neq 0$ and $[U, U] \neq (0)$, then $R = U$ is either a $*$-simple ring or a simple ring with unit.*

(ii) *If $Z(R) \neq 0$ and $[U, U] = (0)$, then R is a commutative ring and $U = F$ or $U = F \oplus F$ for a subfield $F \subset R$.*

(iii) *If* $Z(R) = (0)$ *and* $[U, U] \neq (0)$*, then* R *is a* $*$*-prime ring and* U *is a unique minimal* $*$*-ideal of* R*.*

(iv) *If* $Z(R) = (0)$ *and* $[U, U] = (0)$*, then* $2R = (0)$*.*

Proof. (i) The $*$-simple subring U satisfies the hypothesis of Lemma 2.4 and thus either $Z(U) = Z(I) \oplus Z(I^*)$ for U $*$-simple or $Z(U)$ is a field for U simple. For U simple, then $Z(U) \neq (0)$. Suppose otherwise then $z^n \in (Z(R) \cap U) \subseteq Z(U) = (0)$ for all $z \in Z(R)$. Therefore $Z(R) = (0)$ leads to a contradiction since the semi-prime ring R would contain nilpotent elements.

In the case $Z(U) = Z(I) \oplus Z(I^*)$ either $Z(I)$ or $Z(I^*) \neq (0)$. Assume $Z(I) \neq (0)$, then $Z(I)$ is the center of a simple ring I and therefore a field. Let e be the identity in I and set $h = e + e^*$ then $h^2 = h \neq 0$. For $(i + j^*) \in U = I \oplus I^*$, then $h(i + j^*) = (e + e^*)(i + j^*) = ei + e^* j^* = i + j^* = ie + j^* e^* = (i + j^*)(e + e^*) = (i + j^*)h$. It follows that $h \in Z(U)$ and $hu = u$ for $u \in U$. This, together with the fact that U is both a Lie ideal and a subring of R, implies $h[h, x] = [h, x] = [h, x]h$ for $x \in R$. This results in $h \in Z(R)$.

Therefore h is a central idempotent of R and R has the ideal decomposition $R = Rh \oplus R(1 - h)$. Applying Corollary 2.6, together with U is $*$-simple, leads to $U = W(U)$ and U is a $*$-ideal of R. This, together with Corollary 2.7, implies $(0) \neq [U, U] \subseteq W(U) = U = Uh \subseteq Rh \subseteq U$ and therefore $Rh = U$.

Let $x \in R(1 - h)$, then $x^n \in U \cap R(1 - h) = (0)$. Thus $R(1 - h)$ is a nil ideal of a bounded index in a semi-prime ring R and, by [**6**], $R(1 - h) = (0)$. Hence $R = Rh = U$ with identity h, thus disposing of the case U is $*$-simple.

For the case where U is simple, $Z(U) \neq (0)$ contains a central idempotent h and the above argument, repeated verbatim, results in the same conclusions.

(ii) Under the hypothesis, U is a Lie ideal and subring. By [**6**, Lemma 1.3] either $U \subseteq Z(R)$ or $2R = (0)$. For the case $U \subseteq Z(R)$ with R semi-prime and $x^n \in Z(R)$ for all $x \in R$, R is commutative, see [**8**, page 218]. If U is simple, then U is a subfield of R and if U is $*$-simple, then $U = I \oplus I^*$ where I, I^* are subfields.

In case $2R = (0)$, one notes for all $x \in R$, $[u, x] \in U$ and $[u, [u, x]] \in$

$[U, U] = (0)$. Using the identity $[u^2, x] = [u, [u, x]] + (uxu + uxu) - (xu^2 + xu^2) = 0$, one concludes that $u^2 \in Z(R)$ for $u \in U$. Recalling $x^n \in U$, then $(x^n)^2 \in Z(R)$ for $x \in r$. Using Jacobson's result once again, R is commutative. Therefore, as above, the same conclusions follow.

(iii) Corollaries 2.6 and 2.7 imply $(0) \neq [U, U] \subseteq W(U) = U$ and hence U is a $*$-ideal of R. Let $L \neq (0)$ be a $*$-ideal of R, the $*$-simplicity of U implies either $L \cap U = (0)$ or $U \subseteq L$. The case $L \cap U = (0)$ implies L is a nil ideal of bounded index in a semi-prime ring R and therefore (0). Otherwise U is a minimal $*$-ideal contained in every $*$-ideal of R. Let A, B be nonzero $*$-ideals of R such that $AB = (0)$. Clearly $U^2 = (0)$. However R is semi-prime. Therefore $U = (0)$, a contradiction. Hence, either $A = (0)$ or $B = (0)$. That is, R is $*$-prime.

(iv) As shown in Case (ii) above, the Herstein results yield $2U \subseteq Z(R) = (0)$. $2(x^n) = (2x)^n = (0)$ for all $x \in R$ and so $(2R)^n = (0)$. Therefore, $2R$ is a nil ideal of bounded index in the semi-prime ring R; thus $2R = (0)$. □

Corollary 2.11. *Let U be a simple subring and a Lie ideal of a semi-prime ring R and further for n a fixed positive integer, $x^n \in U$ for all $x \in R$.*

(i) *If $Z(R) \neq 0$ and $[U, U] \neq (0)$, then $R = U$ is a simple ring with unit.*

(ii) *If $Z(R) \neq (0)$ and $[U, U] = (0)$, then R is a commutative ring and U is a subfield of R.*

(iii) *If $Z(R) = (0)$ and $[U, U] \neq (0)$, then R is a prime ring and U is a unique minimal ideal of R.*

(iv) *If $Z(R) = (0)$ and $[U, U] = (0)$, then $2R = (0)$.*

Proof. The corollary follows directly from the theorem by replacing U is $*$-simple with U is simple. □

The second structure theorem shows the relation of the Lie ideal U in which the nil elements in Theorem 2.10 are replaced with a type of symmetric elements of R.

Theorem 2.12. *Let U be a $*$-simple subring and a self-adjoint Lie ideal of a semi-prime ring R. Further, assume both $x + x^*$, $xx^* \in U$ for all $x \in R$.*

(i) *If $Z(R) \neq (0)$ and $[U, U] \neq (0)$, then $R = U$ is either a $*$-simple ring or a simple ring with unit.*

(ii) *If $Z(R) \neq (0)$ and $[U, U] = (0)$, then either $R = Z(R)$ a field or $[R : Z(R)] = 4$ or $2R = (0)$.*

(iii) *If $Z(R) = (0)$ and $[U, U] \neq (0)$, then R is a $*$-prime ring and U is a unique minimal $*$-ideal of R.*

(iv) *If $Z(R) = (0)$ and $[U, U] = (0)$, then $2R = (0)$.*

Proof. (i) The proof models case (i) of Theorem 2.10 to the point where $U = Rh$ in the ideal direct sum $R = Rh \oplus R(1-h) = U \oplus R(1-h)$ for h a central idempotent. Let $y \in R(1-h)$ and, since U and $R(1-h)$ are ideals, then yy^*, $y(y + y^*) \in U \cap R(1-h) = (0)$. It follows that $y^2 = y(y + y^*) - yy^* = 0$ and so $R(1-h)$ is a nil ideal of bounded index. Now the proof picks up again in case (i) of Theorem 2.10.

(ii) Utilizing [**6**, Lemma 1.3], either $U \subseteq Z(R)$ or $2R = (0)$. For $U \subseteq Z(R)$, the polynomial identity $x^2 - x(x + x^*) + xx^* = 0$ holds for $x \in R$. If R is simple, then by a theorem of Kaplansky, R is primitive and $[R : Z(R)] \leq 4$. This can be sharpened using [**8**, Theorem 2, p. 122] to $[R : Z(R)] = 1$ or $[R : Z(R)] = 4$. Hence R is a field or is four-dimensional over its center. The remaining case is $2R = (0)$.

(iii) The proof models case (iii) of Theorem 2.10 to the point where $L \cap U = (0)$. For $y \in L$ and $yy^*, y(y + y^*) \in (L \cap U) = (0)$ implies $y^2 = y(y + y^*) - yy^* = 0$. L is a nil ideal of bounded index in a semi-prime ring R and therefore (0). The proof continues as in case (iii) of Theorem 2.10.

(iv) Under the hypothesis that U is a Lie ideal and subring, then from [**6**, Lemma 1.3] either $U \subseteq Z(R)$ or $2R = (0)$. For $U \subseteq Z(R)$, the polynomial identity $x^2 - x(x + x^*) + xx^* = 0$ holds for $x \in R$. Clearly, $x + x^*$, $xx^* = 0$ and so $x^2 = 0$ for $x \in R$. Hence $(2x)^2 = 4(x^2) = (0)$ for $x \in R$ and so $(2R)^2 = (0)$ which implies $2R$ is a nil ideal of bounded index in R semi-prime, thus $2R = 0$. □

The above ring decompositions were achieved without the assumptions of chain conditions or idempotents. The presence of Lie ideals is a structural property of any associative ring, and the next example shows that requiring Lie ideals is not a trivial condition.

Example 2.13. Let R be a subset of the 2×2 matrices with entries from a ring. The operations are $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) + \left(\begin{smallmatrix} x' & y' \\ z' & w' \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+x' & y+y' \\ z+z' & w+w' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \left(\begin{smallmatrix} x' & y' \\ z' & w' \end{smallmatrix}\right) = \left(\begin{smallmatrix} xx'-yz' & xy'+yw' \\ zx'+wz' & zy'+ww' \end{smallmatrix}\right)$. Let $U = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\right\}$, then U is a subring under the usual addition and the multiplication $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} x & c \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax & ac+bx \\ 0 & ax \end{smallmatrix}\right)$. U is a Lie ideal of R under the Lie multiplication $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) - \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} -bz & bw-xb \\ 0 & -bz \end{smallmatrix}\right)$. However, U is not an ideal of R as seen by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax-bz & ay+bw \\ az & aw \end{smallmatrix}\right)$. $\square$

In general, every two-sided ideal of R is a Lie ideal, but a Lie ideal may not be an ideal of R. The Lie ideal structure was essential for the above results.

3. Jordan simplicity and applications. A Jordan ideal structure on S induces additional decompositions on the ring R. In Osborn and Lanski's work (Lanski[**9**], Osborn**11**]), the elements of S were either nil or invertible; however, for this paper only the ideal properties play a role and so provide a "global" versus a "local" approach.

All relevant assumptions in the Introduction remain intact. A Jordan structure is now imposed on the ring R by defining a multiplication as follows. $x \circ y = xy + yx$, $x, y \in R$. If A and B are subsets of R, then the additive subgroup $A \circ B = \{\sum \pm(a \circ b) \mid a \in A, b \in B\}$. An additive group J is a *Jordan ideal* of S if $J \circ S \subseteq J$, i.e., $j \circ s \in J$ for $j \in J$, $s \in S$. S is *Jordan simple* if (0) and S are the only Jordan ideals of S.

One now continues the investigation into the characterizations of R by considering a special subset $\overline{N} \subseteq \overline{S}$. These sets are partially dealt with in Lanski [**10**] and his results will be generalized. The *norm* N is defined as the additive subgroup $N = \{\sum \pm x_i x_i^* \mid x_i \in R\}$ generated by finite sums of products of $x_i x_i^*$. Clearly $N \subseteq S$. $\overline{N} = \{\sum n_1 n_2 n_3 \cdots n_k \mid n_i \in N\}$ is the subring generated by finite sums of products of $n_i \in N$.

The following lemmas are needed to prove the next structure theorem for a semi-prime ring R.

Lemma 3.1. *N is a Jordan ideal of S.*

Proof. Clearly $(\sum x_i x_i^*) \circ s = \sum (x_i x_i^*) \circ s$. For $x_i x_i^* \in N$ the following holds. $x_i x_i^* \circ s = x_i x_i^* s + s x_i x_i^* = (x_i + s x_i)(x_i + s x_i)^* - x_i x_i^* - (s x_i)(s x_i)^* \in N$. Hence, $N \circ S \subseteq N$ and therefore N is a Jordan ideal of S. ◻

Lemma 3.2. *If $R = R^2$, then $\overline{N}$ is a self-adjoint Lie ideal of R.*

Proof. Clearly $\overline{N}$ is self-adjoint. Now $R = R^2$ implies that, for any $x \in R$, $x = \sum z_i y_i$ where $z_i, y_i \in R$.

One observes that $zy + y^* z^* = (z + y^*)(z + y^*)^* - zz^* - y^*(y^*)^* \in N$ for all $z, y \in R$. Since, as above, $x = \sum z_i y_i$, then $[n, x] = [n, \sum z_i y_i] = \sum (n z_i y_i - z_i y_i n) = \sum (n(z_i y_i + y_i^* z_i^*) - ((z_i y_i) n + n^*(z_i y_i)^*)) \in \overline{N}$ for $n \in N$. Hence $[n, x] \in \overline{N}$. Using induction, assume $[n_1 n_2 \ldots n_k, x] \in \overline{N}$ for all $x \in R$. Then $[n_1 n_2 \cdots n_k n_{k+1}, x] = (n_1 n_2 \cdots n_k)[n_{k+1}, x] + [n_1 n_2 \cdots n_k, x] n_{k+1} \in \overline{N}$ completes the induction argument. Finally, one observes $[(\sum n_1 n_2 \cdots n_k), x] = \sum [n_1 n_2 \cdots n_k, x]$. Thus, $\overline{N}$ is a Lie ideal. ◻

Lemma 3.3. *If $R = R^2$ and I an ideal of R such that $N \subseteq I$, then $2R \subseteq I$.*

Proof. Let $x = \sum zy$. Then $x + x^* = \sum (zy + y^* z^*) \in N$ follows from the observation in Lemma 3.2. From the hypothesis $N \subseteq I$ one concludes $x + x^*$, $xx^* \in I$ for $x \in R$. Now $x^2 = x(x + x^*) - xx^* \in I$ and so $x^2 \in I$ for $x \in R$. Substitute $x + y$ for x in x^2, called *linearizing*, one obtains $x \circ y = (xy + yx) = ((x + y)^2 - x^2 - y^2) \in I$ for $x, y \in R$. Using the preceding statement and the identity $2(xyz) = (xy) \circ z - (zx) \circ y + (yz) \circ x \in I$. Hence, $2(xyz) \in I$ for $x, y, z \in R$. Therefore, $2R^3 \subseteq I$ and from $R^2 = R$ and clearly $2R \subseteq I$. ◻

Lemma 3.4. *If $R = R^2$, characteristic $R \neq 4$, and N is simple Jordan, then $\overline{N}$ is a $*$-simple subring.*

Proof. First one establishes the fact that R is 2-torsion free. $T = \{x \in R \mid 2x = 0\}$ is clearly a $*$-ideal of R. Since $N \cap T$ is a Jordan ideal of S, then either $N \cap T = (0)$ or $N \subseteq T$.

Suppose $N \subseteq T$, Lemma 3.3 implies $2R \subseteq T$ and therefore $4R = (0)$ contrary to the hypothesis characteristic $R \neq 4$.

Otherwise $N \cap T = (0)$. Let $x \in T$; then $xx^* \in N \cap T$ and so $xx^* = 0$. Thus $x^3 = x(x+x^*)(x+x^*)^* = 0$. Hence T is a nil ideal in a semi-prime ring R and therefore (0).

Hence R is two-torsion free and certainly $\overline{N}$ is two-torsion free. Now, from Lemma 3.2, $\overline{N}$ is a self-adjoint Lie ideal and two-torsion free. Applying Lemma 2.8, one concludes that $\overline{N}$ is semi-prime.

One next proves that $\overline{N}$ is $*$-simple. Let I be a $*$-ideal of $\overline{N}$. Then either $I \cap N = (0)$ or $N \subseteq I$. Suppose $N \subseteq I$. Then $\sum n_1 n_2 n_3 \cdots n_k \in I$ and therefore $I = \overline{N}$. Using the exact argument above for the ideal T, the case $I \cap N = (0)$ implies $I = (0)$ since $\overline{N}$ is semi-prime.

Finally, one shows $\overline{N}^2 \neq (0)$. Assume $\overline{N}^2 = (0)$. Then $xx^*yy^* = 0$ for $x, y \in R$. Substituting $y + z$ for y in $xx^*yy^* = 0$ results in $xx^*(yy^* + yz^* + zy^* + zz^*) = xx^*yz^* + xx^*zy^* = 0$ for $x, y, z \in R$. Pre-multiply the last result by z and post-multiply by zxx^*z to obtain $z(xx^*yz^* + xx^*zy^*)zxx^*z = zxx^*y(z^*zxx^*)z + (zxx^*z)y^*(zxx^*z) = (zxx^*z)y^*(zxx^*z) = 0$. Since R is semi-prime, $zxx^*z = 0$. Post-multiplying by xx^*, one obtains $(zxx^*)^2 = 0$. Hence Rxx^* is a nil left ideal of bounded index 2 in a semi-prime ring R, and so $xx^* = 0$ for all $x \in R$. Substituting $x + y$ in $xx^* = 0$ one obtains $xy^* + yx^* = 0$. If one post-multiplies by $(x^*)^*$, one concludes that $xRx = (0)$. Since R is semi-prime, $x = 0$. Therefore $R = (0)$, which is false. Hence, $\overline{N}$ is $*$-simple. □

In the next structure theorem the norm now plays a major role. One uses arguments similar to those of Theorems 2.10 and 2.12. This is a generalization of Lanski's work on S.

Theorem 3.5. *If $R = R^2$, characteristic $R \neq 4$ and N is simple Jordan.*

(i) *If $Z(R) \neq (0)$ and $[\overline{N}, \overline{N}] \neq (0)$, then $R = \overline{N}$ is either a $*$-simple ring or a simple ring with unit.*

(ii) *If $Z(R) \neq (0)$ and $[\overline{N}, \overline{N}] = (0)$, then either $R = Z(R)$ a field, or $[R : Z(R)] = 4$ or $2R = (0)$.*

(iii) *If $Z(R) = (0)$ and $[\overline{N}, \overline{N}] \neq (0)$, then R is a $*$-prime ring and $\overline{N}$ is a unique minimal $*$-ideal of R.*

(iv) *If $Z(R) = (0)$ and $[\overline{N}, \overline{N}] = (0)$, then $2R = (0)$.*

Proof. (i), (iii). From Lemmas 3.2 and 3.4, $\overline{N}$ is a Lie ideal and $*$-simple subring of R. Replacing $\overline{N}$ for U in case (i) of Theorem 2.10 shows $\overline{N}$ to be a $*$-ideal of R. Now $xx^* \in \overline{N}$. Hence $x^3 = (x(x + x^*)(x + x^*)^* - x(xx^*) - (xx^*)x^* - (xx^*)x) \in \overline{N}$. The hypothesis of Theorem 2.10 (i) and (ii) is satisfied with $U = \overline{N}$. Hence, the resulting conclusions follow.

(ii), (iv). The proof of Lemma 3.3 shows $x + x^*$, $xx^* \in \overline{N}$ for $x \in R$. The hypothesis of Theorem 2.12 (ii) and (iv) is satisfied with $U = \overline{N}$. The resulting conclusions follow. □

The following theorem shows the scope of von Neumann's influence in algebra. A ring R is *von Neumann regular* if for $0 \neq x \in R$ there exists a $y \neq 0$ such that $xyx = x$. Clearly, $R^2 = R(x = y(x))$. See [**5**] for a discussion on von Neumann regular rings.

Theorem 3.6. *Let R be von Neumann regular, and let N be simple Jordan.*

(i) *If $[\overline{N}, \overline{N}] \neq 0$, then $R = \overline{N}$ is a $*$-simple ring.*

(ii) *If $[\overline{N}, \overline{N}] = (0)$, then $R = eRe \oplus (1 - e)Re \oplus eR(1 - e) \oplus (1 - e)R(1 - e)$ where e is a symmetric idempotent $(e \neq 0) \in \overline{N}$. Further, $\overline{N} \subseteq eRe$ and $(1 - e)R(1 - e)$ is a nil subring of bounded index 3.*

(iii) *If $[\overline{N}, \overline{N}] = (0)$, then either $R = Z(R)$ a field, or $[R : Z(R)] = 4$ or $2R = (0)$.*

Proof. (i) One first shows that R is $*$-simple. Let I be a $*$-ideal of R, then as in Lemma 3.4 either $I \cap N = (0)$ or $N \subseteq I$. If $N \subseteq I$, then $xx^* \in I$ for $x \in R$. Linearizing on x in xx^* one concludes $xy^* + yx^* \in I$. Post-multiply by x; then $xy^*x + y(x^*x) \in I$ and so $xRx \subseteq I$ for $x \in R$. Now R von Neumann regular implies $R \subseteq I$. The case $I \cap N = (0)$ implies $xx^* = 0$ for $x \in I$.

Since I is a $*$-ideal, if one linearizes on x in xx^*, then one obtains $xy + y^*x^* = 0$ for $x, y \in I$. Post-multiply by x and replace y by yr yield $xyrx = 0$ for $x, y \in I$ and $r \in R$. Hence xyR is a nil right ideal of bounded index 2 in a semi-prime ring R. Therefore, $xy = 0$. That is, $I^2 = (0)$. R semi-prime implies $I = (0)$. Hence, from the above, R is $*$-simple. Since R is von Neumann regular, $R^2 = R$, and thus from Lemma 3.2, $\overline{N}$ is a self-adjoint Lie ideal of R. By Corollaries 2.6 and 2.7, $[\overline{N}, \overline{N}] \subseteq W(\overline{N}) \subseteq \overline{N}$ where $W(\overline{N})$ is a $*$-ideal of R. Since R is $*$-simple, $[\overline{N}, \overline{N}] \neq (0)$ implies $R = \overline{N}$.

(ii) For $[\overline{N}, \overline{N}] = (0)$, then $\overline{N} \subseteq S$. Assume N is not simple, then $I \cap N = (0)$ where $(0) \neq I \subsetneqq \overline{N}$ for some ideal I. Define $Q = \{G \mid (0) \neq G \subsetneqq \overline{N}, \text{ and } G \cap N = (0)\}$ for ideals G contained in $\overline{N}$. $Q \neq \varnothing$ since $I \in Q$. Therefore, by Zorn's lemma, Q contains a maximal ideal M. Since $\overline{N} \subseteq S$, then for $x \in M$, $x^2 = xx^* \in M \cap N = (0)$. Hence M is a nil ideal of bounded index 2. Let $\overline{N}/M$ be a quotient ring and suppose $\overline{N}^2 \subseteq M$. Since M is of index 2, then $(xy)^2 = 0$ and in particular $x^4 = 0$ for all $x \in \overline{N}$. Since $\overline{N}$ is a Lie ideal of R, then $(yr)^5 = [y, r]^4(yr) = 0$ for all $y \in M$, $r \in R$. Hence yR is a nil right ideal of bounded index 5 which implies $y = 0$. Thus, $M = (0)$ which is false. Therefore, $\overline{N}^2 \nsubseteq M$ together with $[\overline{N}, \overline{N}] = (0)$ shows $\overline{N}/M \neq (0)$ and commutative. Since M is maximal, then $\overline{N}/M$ is a field.

Let $\bar{u} \in \overline{N}/M$ be the identity. Then $\bar{u}^2 = \bar{u}$. Thus $(u^2 - u) \in M$ for $u \in \overline{N}$. Since M is nil of index 2, then $u^4 - 2u^3 + u^2 = (u^2 - u)^2 = 0$. Using this relation and $\overline{N} \subseteq S$, consider the element $e = (3u^2 - 2u^3)$. Clearly $e \in \overline{N}$ is symmetric and $e^2 = e$. Furthermore, $e \neq 0$ since otherwise $3u^2 = 2u^3$ and $3\bar{u}^2 = 2\bar{u}^3$ in $\overline{N}/M$. Using $\bar{u}^2 = \bar{u}$, this reduces to $3\bar{u} = 2\bar{u}$ which implies $\bar{u} = 0$ contradicting $\bar{u}$ as the field identity.

The idempotent e induces in the ring R a two-sided Peirce decomposition where eRe and $(1 - e)R(1 - e)$ are $*$-subrings of R. Further,

$e \in (eRe \cap N) \neq (0)$ because $e = eee = ee^*$ and, since N is simple Jordan, then $N \subseteq eRe$. Now $N \cap (1-e)R(1-e) = (0)$. For $x \in (1-e)R(1-e)$, one then has $xx^*, x + x^* \in (1-e)R(1-e)$. However $xx^* \in N$ and therefore $xx^* = 0$. Thus $x^3 = x(x+x^*)(x+x^*) = 0$. Therefore, $(1-e)R(1-e)$ is a nil ring of bounded index 3.

(iii) Since R is von Neumann regular, for $x \in R$, $x = x(yx)$ for some $y \in R$. Assume $\overline{N}$ is simple. Let $z = yx$. Then $x + x^* = xz + z^*x^* = ((x+z^*)(x+z^*) - xx^* - zz*) \in N$ for all $x \in R$. Now $x + x^*$, $xx^* \in \overline{N}$. Setting $U = \overline{N}$, the result follows from Theorem 2.12 (ii). □

The next extension is to weaken the condition on N and see what further characterizations can be made on the ring R. The following material and relevant definitions can be found in [**4**]. Recalling the definition of prime ring, N is said to be *Jordan prime* if $A \cup_B = (0)$, under *quadratic multiplication*, then either $A = (0)$ or $B = (0)$ for Jordan ideals A, B of N.

Theorem 3.7. *If N is Jordan prime, then either R is a prime ring or R is a subdirect sum of two rings $R/I \oplus R/I^*$ for an ideal $(0) \neq I \nsubseteq R$.*

Proof. Assume R is not prime. $IJ = (0)$ for some proper, nonzero ideals I, J of R. Now $(I \cap N)\cup_{(J \cap N)} \subseteq IJ = (0)$. Since N is Jordan prime, then either $(I \cap N) = (0)$ or $(J \cap N) = (0)$. Clearly, $(I \cap I^*) \cap N = (0)$. $x^3 = x(x+x^*)(x+x^*) \in (I \cap I^*) \cap N = (0)$ for $x \in I \cap I^*$. Therefore, the $*$-ideal $I \cap I^* = (0)$ since it is of bounded index 3 in a semi-prime ring R. Hence R is a subdirect sum [**8**, page 14]. □

Theorem 3.8. *If N is Jordan prime and R is an involution ring. Further, let L be a semi-prime, self-adjoint subring of R; then L is a $*$-prime ring.*

Proof. Let I, J be $*$-ideal of L such that $IJ = (0)$. Then, as in Theorem 3.7, $(I \cap N)\cup_{(J \cap N)} = (0)$ for which one can assume $(I \cap N) = (0)$. Since $xx^* \in (I \cap N)$ for $x \in I$; then, as above, $x^3 = 0$. Hence I is a nil ideal of bounded index 3 in the semi-prime subring L. Therefore, $I = (0)$ implying L is $*$-prime. □

REFERENCES

1. U.A. Aburawash, *On $*$-simple involution rings with minimal $*$-biideals*, Studia Sci. Math. Hungar. **32** (1996), 455–458.

2. W.E. Baxter, *On the subring generated by symmetric elements*, Proc. Amer. Math. Soc. **19** (1968), 63–75.

3. G.F. Birkenmeier, N.J. Groenewald and H.E. Heatherly, *Minimal and maximal ideals in rings with involution*, Beiträge Algebra Geom. **38** (1997), 217–225.

4. L. Casciotti, *Relation of the prime radical and the radical of a subring generated by the symmetric elements*, Quaestiones Math. **22** (1999), 209–217.

5. K.R. Goodearl, *von Neumann regular rings*, Krierger Pub., 1991.

6. I.N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, 1969.

7. ———, *Rings with involution*, Univ. of Chicago Press, Chicago, 1977.

8. N. Jacobson, *Structures of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, AMS, Providence, RI, 1964.

9. C. Lanski, *Rings with involution whose symmetric elements are regular*, Proc. Amer. Math. Soc. **33** (1972), 264–270.

10. ———, *On the relationship of a ring with a subring generated by its symmetric elements*, Pacific J. Math. **44** (1973), 581–592.

11. J.M. Osborn, *Jordan and associative rings with nilpotent and invertible elements*, J. Algebra **15** (1970), 301–308.

12. I. Zuev, *Lie ideals of an associative ring*, Uspekhi Mat. Nauk **18** (1963), 155–158.

Mathematics and Computer Science Department, UNCP at Pembroke, One University Drive, Pembroke, NC 28372
E-mail address: leonard.casciotti@uncp.edu