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DEFICIENT DISCRETE QUARTIC SPLINE INTERPOLATION

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ABSTRACT. In the present paper, we have studied the convergence properties of deficient discrete quartic spline interpolants which match the given functional values at mesh points and mid points between successive mesh points.

1. Introduction. Let us consider a mesh on $[0,1]$ which is defined by

$$P : 0 = x_0 < x_1 < \cdots < x_n = 1.$$

For $i = 0, 1, \ldots, n-1$, p_i shall denote the length of the mesh interval $[x_i, x_{i+1}]$. Let $p = \max p_i$ and $p^* = \min p_i.p$ is said to be a uniform mesh if p_i is a constant for all i. Throughout, h will represent a given positive real number. Consider a real valued function $s(x,h)$ defined over $[0,1]$ which is such that its restriction s_i on $[x_i, x_{i+1}]$ is a polynomial of degree m or less, for $i = 0, 1, \ldots, n-1$. Then $s(x,h)$ defines a deficient discrete spline of degree m with deficiency r, if

$$D_h^{\{j\}} s_i(x_i, h) = D_h^{\{j\}} s_{i+1}(x_i, h), \quad j = 0, 1, \ldots, m-r-1 \tag{1.1}$$

where the difference operator $D_h^{\{j\}}$ for a function f is defined by

$$D_h^{\{0\}} f(x) = f(x), \quad D_h^{\{1\}} f(x) = (f(x+h) - f(x-h))/2h,$$
$$D_h^{\{2\}} f(x) = (f(x+h) - 2f(x) + f(x-h))/h^2$$

and

$$D_h^{\{m+n\}} = D_h^{\{m\}} D_h^{\{n\}} f(x), \quad m, n \geq 0.$$

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Taking $m = 4$ and $r = 1$ in (1.1), the class of all such deficient discrete quartic splines with deficiency 1 satisfying the boundary conditions

$$D_h^{\{1\}} s(x_0, h) = D_h^{\{1\}} f(x_0), \quad D_h^{\{1\}} s(x_n, h) = D_h^{\{1\}} f(x_n) \tag{1.2}$$

is denoted by $D(4, 1, P, h)$.

Discrete splines were introduced by Mangasarian and Schumaker [**9**] as the solution of certain minimization problems involving differences. They have a close connection with best summation formulae, see [**10**], which is a special case of the abstract theory of best approximation of linear functionals. Malcolm [**8**] used discrete splines to compute nonlinear splines iteratively. For some different constructive aspects of discrete splines, we refer to Schumaker [**13**], Astor and Duris [**2**] and Jia [**6**]. Existence, uniqueness and convergence properties of discrete cubic spline interpolant matching the given function at intermediate points for uniform mesh have been studied by Dikshit and Powar [**3**] and Rana [**11**]. These results were generalized by Dikshit and Rana [**4**] for nonuniform meshes and it has been shown that nonuniform meshes permit a wider choice for the points of interpolation than those possible for the case of uniform meshes. Rana and Dubey [**12**] have obtained an asymptotically precise estimate of the difference between discrete cubic spline interpolant and the function interpolated, which is sometimes used to smooth a histogram. Deficient splines are quite useful than usual splines as they require less continuity requirement at mesh points. The object of the present paper is to study the existence, uniqueness and convergence properties of deficient discrete quartic spline matching the given functional values at mesh points and mid points which are sometimes useful to achieve a prescribed accuracy with less data than are required by some lower order method, see [**5**]. The results obtained in this paper include in particular some earlier results due to Howell and Verma [**5**] for the continuous case.

Now writing $2z_{i+1} = x_i + x_{i+1}$, we introduce the following interpolatory conditions for a given function f,

$$\begin{aligned} s(x_i, h) &= f(x_i), \quad i = 0, 1, \dots, n, \\ s(z_{i+1}, h) &= f(z_{i+1}), \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{1.3}$$

and pose the following.

Problem A. Given $h > 0$, for what restrictions on p does there exist a unique $s(x,h) \in D(4,1,P,h)$ which satisfies the conditions (1.2) and (1.3) ?

2. Existence and uniqueness. Let $P(z)$ be a discrete quartic polynomial on $[0,1]$. Then, we can show that

$$\begin{aligned} P(z) &= P(0)\,Q_1(z) + P(1/2)\,Q_2(z) + P(1)\,Q_3(z) \\ &\quad + D_h^{\{1\}}P(0)\,Q_4(z) + D_h^{\{1\}}P(1)\,Q_5(z), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} Q_1(z) &= \big[1 + z\{z(-11 + 18z - 8z^2) - 2h^2(8z^3 - 24z^2 + 12z + 9) \\ &\qquad + 16h^4(2z-3)\}A\big], \\ Q_2(z) &= 16z(1-z)\big[z(1-z) - 2h^2(z^2 - z - 1) + 4h^2\big]A, \\ Q_3(z) &= -z(1-2z)\big[z(5-4z) - 2h^2(4z^2 - 2z - 7) + 16h^4\big]A, \\ Q_4(z) &= z\big[1 - \{(2z^2 - 5z + 4)z + h^2(4z^3 - 16z^2 + 17z + 5) + 16h^4\}A\big], \\ Q_5(z) &= z(1-z)(1-2z)\big[z + (3+2z)h^2\big]A, \end{aligned}$$

and $A = 1/(1 + 10h^2 + 16h^4)$.

Now we are set to answer Problem A in the following:

Theorem 2.1. *Suppose $p^* \geq \sqrt{7}h$. Then, there exists a unique deficient discrete quartic spline $s(x,h) \in D(4,1,P,h)$ which satisfies the conditions* (1.2) *and* (1.3).

Proof of Theorem 2.1. Denoting $(x - x_i)/p_i$ by t, $0 \leq t \leq 1$, we can write (2.1) in the form of the restriction $s_i(x,h)$ of the quartic spline $s(x,h)$ on $[x_i, x_{i+1}]$ as follows

$$\begin{aligned} s_i(x,h) &= f(x_i)\,Q_1(t) + f(z_{i+1})\,Q_2(t) + f(x_{i+1})\,Q_3(t) \\ &\quad + p_i\,Q_4(t)D_h^{\{1\}}s(x_i,h) + p_i\,Q_5(t)D_h^{\{1\}}s(x_{i+1},h). \end{aligned} \tag{2.2}$$

In view of (2.1), it may be seen that $s_i(x,h)$ is quartic on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$ and satisfies (1.2)–(1.3). Now, applying the continuity of second difference of $s_i(x,h)$ at x_i given by (1.1), we get the following system of equations:

$$\begin{aligned}(2.3)\quad & p_{i-1}D_h^{\{2\}}Q_4(1)D_h^{\{1\}}s(x_{i-1},h)\\ &\quad + \left[p_{i-1}D_h^{\{2\}}Q_5(1) - p_iD_h^{\{2\}}Q_4(0)\right]D_h^{\{1\}}s(x_i,h)\\ &\quad - p_iD_h^{\{2\}}Q_5(0)D_h^{\{1\}}s(x_{i+1},h) = F_i(h), \quad i=1,2,\dots,n\end{aligned}$$

where

$$\begin{aligned}F_i(h) &= \left[D_h^{\{2\}}Q_1(0) - D_h^{\{2\}}Q_3(1)\right]f(x_i) + D_h^{\{2\}}Q_2(0)f(z_{i+1})\\ &\quad - D_h^{\{2\}}Q_2(1)f(z_i) + D_h^{\{2\}}Q_3(0)f(x_{i+1}) - D_h^{\{2\}}Q_1(1)f(x_{i-1}).\end{aligned}$$

Write $D_h^{\{1\}}s(x_i,h) = M_i(h) = M_i$ (say), for all i. We can easily see that excess of the absolute value of the coefficient of M_i over the sum of the absolute value of the coefficients of M_{i-1} and M_{i+1} in (2.3) under the conditions of Theorem 2.1 is given by

$$(2.4)\qquad C_i(h) = 2A(3+24h^2)(1/p_i + 1/p_{i-1}),$$

which is clearly positive. Therefore, the coefficient matrix of the system of equations (2.3) is diagonally dominant and hence invertible. Thus, the system of equations (2.3) has a unique solution. This completes the proof of the Theorem 2.1. □

3. Norm of difference between two splines. In this section we compute the distance between two spline interpolants $s(x,u)$ and $s(x,v)$ for $h=u,v$ interpolating the same data on P of Theorem 2.1 to see that this distance only depends on the data which is computable. Also, we shall show that the discrete quartic spline converges to the C^2 quartic spline as h goes to zero. The system of equations (2.3) may be written as

$$(3.1)\qquad A(h)M(h) = F(h),$$

where $A(h)$ is the coefficient matrix and $M(h)$ and $F(h)$ are column vectors $(M_i(h))$ and $(F_i(h))$ respectively defined in Section 2. Unless stated otherwise $\| \, . \, \|$ will denote the sup norm throughout the present paper. Thus, it may be observed that the row max norm of coefficient matrix $A(h)$, cf. [**1**, p. 21],

$$\| A^{-1}(h) \| \leq C(h), \tag{3.2}$$

where

$$C(h) = \max_i \{C_i^{-1}(h)\}.$$

Setting

$$M_i(u,v) = M_i(u) - M_i(v)$$

and

$$M_i^*(u,v) = u^2 M_i(u) - v^2 M_i(v),$$

we denote the single column vectors $(M_i(u,v))$, or $M_i^*(u,v)$, by $M(u,v)$, or $M^*(u,v)$. We shall first prove the following lemma.

Lemma 3.1. *Let $s(x,h)$ be the unique deficient discrete quartic spline interpolant of f under the assumptions of Theorem* 2.1. *Then, we have*

$$\| M(u,v) \| \leq K_1 \, |u^2 - v^2|, \tag{3.3}$$

where K_1 is a positive function of p and h and

$$\| M^*(u,v) \| \leq K_2 \, |u^2 - v^2|, \tag{3.4}$$

where $K_2 = u^2 K_1 + \| M(v) \|$.

Proof of Lemma 3.1. To prove Lemma 3.1, we see from (3.1) that, for $h = u, v$,

$$A(u)M(u,v) = (A(v) - A(u))M(v) + K_3(u^2 - v^2), \tag{3.5}$$

where $K_3 = \|F\| \, \|32(3p^2 + 2) + (u^2 + v^2)(4 + 8p^2)/p^4\|$ and

$$\|F\| = \max_i |f(x_i)|.$$

However, as already shown in the proof of Theorem 2.1 that $A(h)$ is invertible, we get the following from (3.1) and (3.2),

$$\|M(v)\| \le C(v)\|F(v)\|. \tag{3.6}$$

Now using (3.6) in (3.5) we get,

$$\|M(u,v)\| \le C(u)\big[C(v)\|A(v)-A(u)\|\,\|F(v)\| + K_3|u^2-v^2|\,\big]. \tag{3.7}$$

It may also be seen easily that

$$\|A(v)-A(u)\| \le K_4|u^2-v^2|, \tag{3.8}$$

where

$K_4 = 24[1+6(u^2+v^2)+64u^2v^2]/[p^*(1+10v^2+16v^4)(1+10u^2+16u^4)]$.

Combining (3.6)–(3.8), we prove (3.3) of Lemma 3.1. By a parallel reasoning, we can also prove (3.4) of Lemma 3.1.

Thus, we are now set to prove the following:

Theorem 3.1. *For a given function f, let $s(x,h)$ be the deficient discrete quartic spline interpolant of Theorem* 2.1. *Then, for $h = u, v >$ 0, we have*

$$\|s(x,u)-s(x,v)\| \le p\,K_5|u^2-v^2|, \tag{3.9}$$

where $K_5 = (9K_1+26K_2)/(1+10u^2+16u^4)(1+10v^2+16v^4)$.

Proof of Theorem 3.1. In order to prove Theorem 3.1, we see from (2.2) that, for $h = u, v$, we have

$$\|s(x,u)-s(x,v)\| \le p\,\big[9\,\|M(u,v)\| + 26\|M^*(u,v)\|\,\big]/U \tag{3.10}$$

where

$$U = (1+10u^2+16u^4)(1+10v^2+16v^4).$$

Thus, we get the following when we appeal to Lemma 3.1 in (3.10),

$$\|s(x,u)-s(x,v)\| \le K_5\,p\,|u^2-v^2|, \tag{3.11}$$

where

$$K_5 = (9K_1 + 26K_2)/(1 + 10u^2 + 16u^4)(1 + 10v^2 + 16v^4).$$

This completes the proof of Theorem 3.1. If we allow $u \longrightarrow 0$, then Theorem 3.1 gives a comparison of continuous and discrete interpolating splines.

4. Error bounds. For convenience, we assume in this section that $b = a + Nh$ where N is a positive integer. It is also assumed that the mesh points $\{x_i\}$ are such that $x_i \in [a, b]_h$ for $i = 1, 2, \dots, n$ where the discrete interval $[a, b]_h$ is the set of points $\{a, a + h, \dots, a + Nh\}$. For a function f and two distinct points x_1, x_2 in its domain, the first divided difference is defined by

$$[x_1, x_2]f = \{f(x_1) - f(x_2)\}/(x_1 - x_2).$$

For convenience we write $f^{\{1\}}$ for $D_h^{\{1\}} f$, $f_i^{\{1\}}$ for $D_h^{\{1\}} f(x_i)$ and $w(f, p)$ for the modulus of continuity of f. The discrete norm of a function f over the interval $[0, 1]_h$ is defined by

$$\|f\| = \max_{[0,1]_h} |f(x)|.$$

Without assuming any smoothness condition on the data f, we shall obtain in the following the bounds for the error function $e(x) = s(x, h) - f(x)$ over the discrete interval $[0, 1]_h$.

Theorem 4.1. *Suppose $s(x, h)$ is the deficient discrete quartic spline interpolant of Theorem* 2.1*. Then*

$$\|(e_i^{\{1\}})\| \le C(h)\, K(p, h)\, w(f, p) \tag{4.1}$$

and

$$\|(e(x))\| \le p\, K^*(p, h)\, w(f, p). \tag{4.2}$$

Proof of Theorem 4.1. To obtain the error estimate (4.1) first we replace $M_i(h)$ by $e^{\{1\}}(x_i) = D_h^{\{1\}} s(x_i, h) - f_i^{\{1\}}$ in (3.1) and get

$$A(h)(e^{\{1\}}(x_i)) = (F_i(h)) - A(h)(f_i^{\{1\}}) = (L_i), \quad \text{say.} \tag{4.3}$$

To estimate the row max norm of the matrix (L_i) in (4.3), we shall need the following result due to Lyche [**7**].

Lemma 4.1. *Let* $\{a_i\}_{i=1}^m$ *and* $\{b_j\}_{j=1}^n$ *be given sequences of nonnegative real numbers such that* $\sum a_i = \sum b_j$. *Then, for any real valued function* f *defined on a discrete interval* $[0,1]_h$, *we have*

$$\left|\sum_{i=1}^m a_i[x_{i0},x_{i1},\dots,x_{ik}]\,f - \sum_{j=1}^n b_j[y_{j0},y_{j1},\dots,y_{jk}]\,f\right| \le w\left(f^{\{k\}},\,|1-kh|\right)\sum a_i/k!, \tag{4.4}$$

where $x_{ik},\ y_{jk}\in[0,1]_h$ *for relevant values of* i,j *and* k.

It may be observed that the ith row of the righthand side of (4.3) is written as

$$(L_i) = \sum_{i=1}^8 a_i[x_{i0},x_{i1}]\,f - \sum_{j=1}^5 b_j[y_{j0},y_{j1}]\,f, \tag{4.5}$$

where

$$\begin{aligned}
&a_1 = 16h^4/p_{i-1},\quad a_2 = (h^2(4+8h^2)/p_{i-1}^3)+11/2p_{i-1},\\
&a_3 = (h^2(4+8h^2)/p_i^3)+11/2p_i,\quad a_4 = 16h^4/p_i,\quad a_5 = 24h^2/p_{i-1},\\
&a_6 = 24h^2/p_i,\quad a_7 = \left[(1-7h^2)p_{i-1}^2+(2+4h^2)h^2\right]/p_{i-1}^3,\\
&a_8 = \left[(1-7h^2)p_i^2+(2+4h^2)h^2\right]/p_i^3,\\
&b_1 = \left[h^2(4+8h^2)+(5/2)p_{i-1}^2\right]/p_{i-1}^3,\\
&b_2 = 16h^4/p_i,\quad b_3 = 16h^4/p_{i-1},\\
&b_4 = \left[h^2(4+8h^2)+(5/2)p_i^2\right]/p_i^3,\\
&b_5 = (4+17h^2)(1/p_i+1/p_{i-1})+(2+4h^2)h^2(1/p_i^3+1/p_{i-1}^3),
\end{aligned}$$

and

$$\begin{aligned}
&x_{21}=y_{30}=x_{30}=y_{21}=x_{51}=x_{60}=x_i,\\
&x_{11}=y_{11}=x_{20}=y_{20}=z_i,\quad x_{10}=y_{10}=x_{50}=x_{i-1},\\
&x_{31}=y_{31}=x_{40}=y_{40}=z_{i+1},\quad x_{41}=y_{41}=x_{61}=x_{i+1},
\end{aligned}$$

$$x_{70} = x_{i-1} - h, \quad x_{71} = x_{i-1} + h, \quad y_{50} = x_i - h, \quad y_{51} = x_i + h,$$
$$x_{80} = x_{i+1} - h, \quad x_{81} = x_{i+1} + h.$$

Clearly in (4.5) $\{a_i\}$ and $\{b_j\}$ are sequences of nonnegative real numbers such that

$$\sum a_i = \sum b_j = (6 + 12h^2)\, h^2\, (1/p_i^3 + 1/p_{i-1}^3) + (1/p_i + 1/p_{i-1})\{16h^4 + 17h^2 + (13/2)\} = N(p,h), \text{ (say)}.$$

Thus, applying Lemma 4.1 in (4.5) for $i = 8$, $j = 5$ and $k = 1$, we get

$$|(L_i)| \le N(p,h)\, w(f^{\{1\}}, |1-p|\,). \tag{4.6}$$

Now, using the equations (3.2) and (4.6) in (4.3) we get

$$\|e^{\{1\}}(x_i)| \le C(h)\, K(p,h)\, w(f^{\{1\}}, p), \tag{4.7}$$

where $K(p,h)$ is some positive function of p and h.

We next proceed to obtain an upper bound for $e(x)$. Replacing $M_i(h)$ by $e_i^{\{1\}}$ in equation (2.2), we obtain

$$e(x,h) = p\left[Q_5(t)e_i^{\{1\}}(x_{i+1}) + Q_4(t)e_i^{\{1\}}(x_i)\right] + M_i(f). \tag{4.8}$$

Now we write the expression of $M_i(f)$ used in the righthand side of (4.8) in terms of the divided difference as following:

$$(M_i(f)) = \sum_{i=1}^{5} u_i\,[x_{i0}, x_{i1}]\, f - \sum_{j=1}^{5} v_j\,[y_{j0}, y_{j1}]\, f, \tag{4.9}$$

where $u_1 = \left[p_i\{11t^2 + (18 + 48h^2)h^2\, t + (8 + 16h^2)\, t^4\}A\right]/2$,

$$u_2 = \left[p_i\{32t^2h^4 + (24 + 48h^2)t^3\}A\right]/2, \quad u_3 = 24t^2h^2p_iA, \quad u_4 = p_i\, t,$$
$$u_5 = p_i\{t^2 + 3h^2t + (2 + 4h^2)t^4\}A,$$
$$v_1 = \left[p_i\{32t^2h^4 + (18 + 48h^2)t^3\}A\right]/2,$$
$$v_2 = \left[p_i\{5t^2 + (14 + 16h^2)h^2t + (8 + 16h^2)t^4\}A\right]/2,$$
$$v_3 = p_i\left[(4 + 17h^2)t^2 + (5 + 16h^2)h^2t + (2 + 4h^2)t^4\right]A,$$

$$v_4 = p_i\big[7h^2t^2 + 3t^3\big]A, \quad v_5 = p_i\, t$$

and

$$x_{10} = x_{30} = y_{10} = y_{50} = x_i, \quad x_{31} = y_{21} = x_{20} = x_{i+1},$$
$$x_{11} = x_{21} = y_{11} = y_{20} = z_{i+1},$$
$$x_{40} = y_{30} = x_i - h, \quad x_{41} = y_{31} = x_i + h,$$
$$x_{50} = y_{40} = x_{i+1} - h, \quad x_{51} = y_{41} = x_{i+1} + h, \quad y_{51} = x.$$

Observing the fact $\sum u_i = \sum v_j$, we again apply Lemma 4.1 suitably in (4.9) for $i = j = 5$ and $k = 1$ to see that

$$|M_i(f)| \le p\, N^*(p,h)\, w(f^{\{1\}}, |1-p|\,), \tag{4.10}$$

where $N^*(p,h) = p_i\{2t + (13 + 32h^4 + 48h^2)t^2 + (24 + 48h^2)t^3 + (24 + 48h^2)h^2t + (12 + 24h^2)t^4\}A]/2$. Thus, using (4.7) and (4.10) in (4.8), we get the following

$$\|e(x)\| \le p\, K^*(p,h)\, w(f^{\{1\}}, p\,), \tag{4.11}$$

where $K^*(p,h)$ is a positive constant of p and h. This completes the proof of Theorem 4.1. □

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