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# TIME DEPENDENT STABILITY FOR FEYNMAN'S OPERATIONAL CALCULUS

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ABSTRACT. A stability theory for Feynman's operational calculus in the time dependent setting is presented. We also discuss differences between the results obtained here in the time dependent setting and similar results for the time independent setting.

**1. Introduction.** In what follows we investigate certain stability properties of Feynman's operational calculus. The setting of the operational calculus used in this paper is that developed by Jefferies and Johnson in the papers [**7–9**] and [**10**]. Briefly, the operational calculus described in the aforementioned papers is that of the formation of functions of several noncommuting bounded linear operators on a Banach space. The development of the operational calculus carried out in these papers is restricted to the time independent setting wherein the operators involved are fixed. The time dependent operational calculus, that is to say the version of the operational calculus where operator-valued functions replace the fixed operators of the time independent operational calculus, was developed in [**16**] and [**11**] and, in fact, a stability theorem for the time independent operational calculus is presented in [**14**]. Other stability theorems for the operational calculus can be found in [**16**] and [**17**]. The stability theorems presented below differ from the theorem presented in [**14**] in that here we work in the time-dependent setting. Also, while some theorems concerning stability of the operational calculus in the time dependent setting can be found in [**17**], they are confined to a functional calculus for the exponential function. In this paper we investigate stability properties of the operational calculus using functions other than the exponential function.

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At this time it may be prudent to briefly explain the approach to Feynman's operational calculus used in this paper. As indicated above, we are concerned with the formation of functions of several noncommuting linear operators. However, the formation of simple functions of only two noncommuting operators can present ambiguities. For example, if $f(x, y) = xy$, what is $f(A, B)$ where $A$ and $B$ are noncommuting operators on some Banach space $X$? Do we set $f(A, B)$ equal to $AB$, $BA$, $(AB+BA)/2$, or some other quantity? This problem was addressed by Richard Feynman [**5**] due to his interest in quantum theory where the observables one uses are self-adjoint operators which generally do not commute. In order to cope with the ambiguity arising from noncommutativity, Feynman introduced some 'rules' for the formation of functions of noncommuting operators. These are:

(R1) Attach time indices to operators to indicate the order in which they act in products. (Note: Operators sometimes come with time indices naturally attached. For example, one may have an operator of multiplication by a time-dependent potential function.) Feynman's time ordering convention was that an operator with a smaller, or earlier, time index should act before one with a larger, or later, index no matter how they are ordered on the page.

*Remark* 1.1. Found in [**2**], and earlier in [**12**], is the use of measures to attach time indices to operators, rule (R1). These measures give us the relative position of the operators in the final disentangled expressions. For example, if we have operators $A_1, \dots, A_k$ and associated measures $\mu_1, \dots, \mu_k$ for which the support of $\mu_j$ lies to the left of the support of $\mu_{j+1}$, then the operator $A_j$ always acts *prior to* $A_{j+1}$. Of course, if the ordering of these supports is changed, then the order of the operators is also changed. Hence an entire family of functional calculi is indexed by various time-ordering measures $\mu_1, \dots, \mu_k$. One can find a detailed exposition of this approach to the attachment of time indices in [**13**] as well as in the memoir [**12**].

(R2) With time indices attached, form functions of these operators as if they were commuting. (If one stops naively at this point, the 'equality' involved is usually false. For example, it might say that $e^{A+B} = e^A e^B$ even though $A$ and $B$ do not commute.)

(R3) After (R2) is completed, 'disentangle' the resulting expression; that is, restore the conventional ordering of the operators. In practice, this means that we need to manipulate the expression, if possible, until the ordering on the page coincides with the time ordering.

A much more extensive introduction to Feynman's operational calculus including a discussion of its connection with the Feynman path integral can be found in the book of Johnson and Lapidus [**13**, see especially Chapter 14].

The formalism outlined in the next section of the paper begins with the construction of two commutative Banach algebras of functions of several variables. The second of these algebras that will be defined, the *disentangling* algebra, is the commutative setting in which the time-ordering calculations called for in Feynman's rules can be carried out in a mathematically rigorous fashion. (When one does the calculations called for by Feynman's rules in the noncommutative setting of $\mathcal{L}(X)$, the steps one goes through on their way to the final result are heuristic in nature.) Once the disentangling calculation is finished in the disentangling algebra, the result is then mapped to the noncommutative setting of $\mathcal{L}(X)$ via the *disentangling map*, see Definition 2.3 and equation (8). The operator obtained in this way gives us the required "disentangled" operator in $\mathcal{L}(X)$. (See (R3) above.)

It is the disentangling map that we investigate in this paper. Specifically, we consider the behavior of the disentangling map in the following situations:

(1) Given time-ordering measures $\mu_1, \dots, \mu_k$ and fixed operator-valued functions $A_1(\cdot), \dots, A_k(\cdot)$, choose sequences $\{\mu_{1n}\}_{n=1}^{\infty}, \dots, \{\mu_{kn}\}_{n=1}^{\infty}$ converging weakly to $\mu_1, \dots, \mu_k$, respectively. We consider the behavior of the disentangling map as $n \to \infty$. For each $n \in \mathbf{N}$ we obtain a particular functional calculus indexed by the time-ordering measures $\mu_{1n}, \dots, \mu_{kn}$. Hence the limit on $n$ is the limit of a family of functional calculi and the problem is to determine the limiting functional calculus.

(2) Fix time-ordering measures $\mu_1, \dots, \mu_k$. If we have operator-valued functions $A_1(s), \dots, A_k(s)$, we choose sequences $\{A_{1n}(s)\}_{n=1}^{\infty}, \dots, \{A_{kn}(s)\}_{n=1}^{\infty}$ of operator-valued functions converging to $A_1(s), \dots, A_k(s)$ respectively, in some specified way, we again want to determine the limit on $n$. Since the time-ordering measures are fixed, we are

working in one particular functional calculus in this case and are determining the limit of a sequence of operators obtained from this functional calculus.

Future work on the stability properties of the operational calculus includes consideration of the stability of the disentangling map defined in the presence of a $(C_0)$ semi-group, see [**11**]. Also to be addressed is the construction of a stability theory for Feynman's operational calculus where time-ordering measures with nonzero discrete parts are used. Work in this direction is in progress.

**2. The time-dependent disentangling map.** Before presenting the stability theorems for the operational calculus in the time dependent setting, we have to define the disentangling map in the time dependent setting. In doing this we follow the initial definitions set out in [**16**] and [**11**].

*Remark* 2.1. As the reader will note, the algebras of functions defined below are referred to as Banach algebras. We will not prove this assertion here but will instead refer the reader to the paper [**7**] where the proof is carried out for the time independent setting. However, as noted in [**16**] and [**11**], the proof of this fact for the time dependent setting is the same.

*Remark* 2.2. We will assume throughout that the Banach space $X$ is separable.

**Definition 2.1.** Fix $T > 0$. For $i = 1, \dots, k$ let $A_i : [0,T] \to \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in $[0,T]$ for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite continuous Borel measure $\mu_i$ on $[0,T]$ and we require that, for each $i$,

$$r_i = \int_{[0,T]} \|A_i(s)\|_{\mathcal{L}(X)} \mu_i(ds) < \infty. \tag{1}$$

We define, as in [**7, 11**] and [**16**], the commutative Banach algebra $\mathbf{A}_T(r_1, \dots, r_k)$ of functions $f$ of $k$ complex variables that are analytic on the open polydisk $\{(z_1, \dots, z_k) : |z_i| < r_i,\ i = 1, \dots, k\}$ and continuous on its boundary. (We emphasize that the weights we are using here depend on the operator-valued functions as well as on $T$ and on the measures.) The norm for this Banach algebra is defined to be

$$\|f\|_{\mathbf{A}_T} = \sum_{n_1,\ldots,n_k=0}^{\infty} |a_{n_1,\ldots,n_k}|\, r_1^{n_1}\cdots r_k^{n_k}. \tag{2}$$

**Definition 2.2.** To the algebra $\mathbf{A}_T$ we associate as in [**7, 11**] and [**16**] a disentangling algebra by creating formal commuting objects $(A_i(\cdot),\mu_i)^\sim$, $i = 1,\ldots,k$. (These objects play the role of the indeterminants $z_1,\ldots,z_k$.) We define the disentangling algebra $\mathbf{D}_T\left((A_1(\cdot),\mu_1)^\sim,\ldots,(A_k(\cdot),\mu_k)^\sim\right)$ to be the collection of functions of the new indeterminants with the same properties as the elements of the algebra defined in Definition 2.1. However, rather than using the notation $(A_i(\cdot),\mu_i)^\sim$ below, we will often abbreviate to $A_i(\cdot)^\sim$, especially when carrying out calculations. The norm for $\mathbf{D}_T$ is the same as defined in (2) for the Banach algebra $\mathbf{A}_T$ though we will refer to it as $\|\cdot\|_{\mathbf{D}_T}$ if a distinction needs to be made.

It is not hard to show that $\mathbf{A}_T$ and $\mathbf{D}_T$ are commutative Banach algebras which are isomorphic to one another, see Propositions 1.1–1.3 in [**7**].

*Remark* 2.3. We will often write $\mathbf{D}_T$ in place of $\mathbf{D}_T(A_1(\cdot)^\sim,\ldots, A_k(\cdot)^\sim)$ or $\mathbf{D}_T\left((A_1(\cdot),\mu_1)^\sim,\ldots,(A_k(\cdot),\mu_k)^\sim\right)$.

For each $t \in [0,T]$ we define the disentangling map

$$\mathcal{T}^t_{\mu_1,\ldots,\mu_k} : \mathbf{D}_T\left((A_1(\cdot),\mu_1)^\sim,\ldots,(A_k(\cdot),\mu_k)^\sim\right) \longrightarrow \mathcal{L}(X) \tag{3}$$

as in [**7, 11**] and [**16**]. In order to state the following definition, which gives us the action of the disentangling map, we must first introduce some notation. (This notation is essentially the same as used in [**7, 11**], and [**16**].) For a nonnegative integer $n$ and a permutation $\pi \in S_n$, the set of all permutations of $\{1,\ldots,n\}$, we define subsets $\Delta_n^t(\pi)$ of $[0,t]^n$ by

$$\Delta_n^t(\pi) = \left\{(s_1,\ldots,s_n) \in [0,t]^n : 0 < s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(n)} < t\right\}. \tag{4}$$

We next define, for nonnegative integers $n_1, \dots, n_k$ and a permutation $\pi \in S_n$ with $n := n_1 + \cdots + n_k$,

(5)
$$\tilde{C}_{\pi(i)}(s_{\pi(i)}) = \begin{cases} A_1(s_{\pi(i)})^{\sim}, & \text{if } \pi(i) \in \{1, \dots, n_1\}, \\ A_2(s_{\pi(i)})^{\sim}, & \text{if } \pi(i) \in \{n_1, \dots, n_1 + n_2\}, \\ \vdots & \\ A_k(s_{\pi(i)})^{\sim}, & \text{if } \pi(i) \in \{n_1 + \cdots + n_{k-1} + 1, \dots, n\}. \end{cases}$$

Now, for every $t \in [0, T]$, we define the action of the disentangling map on monomials.

**Definition 2.3.** Let $P_t^{n_1, \dots, n_k}(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}) = (A_1(\cdot)^{\sim})^{n_1} \cdots (A_k(\cdot)^{\sim})^{n_k}$. We define the action of the disentangling map on this monomial by

(6)
$$\begin{aligned} &\mathcal{T}^t_{\mu_1, \dots, \mu_k} P_t^{n_1, \dots, n_k}(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}) \\ &\qquad = \mathcal{T}^t_{\mu_1, \dots, \mu_k}\left((A_1(\cdot)^{\sim})^{n_1} \cdots (A_k(\cdot)^{\sim})^{n_k}\right) \\ &\qquad := \sum_{\pi \in S_n} \int_{\Delta_n^t(\pi)} C_{\pi(n)}(s_{\pi(n)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\ &\qquad\quad \times (\mu_1^{n_1} \times \cdots \times \mu_k^{n_k})(ds_1, \dots, ds_n) \end{aligned}$$

where the notation is as defined in (5) except that here we omit the tildes and so obtain the appropriate operator-valued functions in place of the formal commuting objects.

Finally, for $f \in \mathbf{D}_T((A_1(\cdot), \mu_1)^{\sim}, \dots, (A_k(\cdot), \mu_k)^{\sim})$ written as

(7)
$$f(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}) = \sum_{n_1, \dots, n_k = 0}^{\infty} c_{n_1, \dots, n_k} (A_1(\cdot)^{\sim})^{n_1} \cdots (A_k(\cdot)^{\sim})^{n_k},$$

we define the action of the disentangling map on $f$ by

(8)
$$\begin{aligned} &\mathcal{T}^t_{\mu_1, \dots, \mu_k} f(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}) \\ &\quad = f_{t; \mu_1, \dots, \mu_k}(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}) \\ &\quad = \sum_{n_1, \dots, n_k = 0}^{\infty} c_{n_1, \dots, n_k} \mathcal{T}^t_{\mu_1, \dots, \mu_k} P_t^{n_1, \dots, n_k}(A_1(\cdot)^{\sim}, \dots, A_k(\cdot)^{\sim}). \end{aligned}$$

*Remark* 2.4. As is shown in [**16**] and [**11**] the disentangling map is a linear contraction from the disentangling algebra to the noncommutative Banach algebra of bounded linear operators on the Banach space $X$. This differs somewhat from the time independent setting of [**7**] where the disentangling map, defined exactly as above, is shown to be a norm one contraction. As remarked in [**16**] and [**11**], it is the presence of time dependent $\mathcal{L}(X)$-valued functions that causes the map to be a contraction not necessarily of norm one.

*Remark* 2.5. It is worth pointing out that equation (6) is where the calculations involved in applying Feynman's 'rules' are carried out. In fact, this equation is stated as part of Proposition 2.2 of [**7**] when continuous Borel probability measures are used for time-ordering. It is shown in [**8**] that the same disentangling results when finite continuous Borel measures are used in place of probability measures.

## 3. Stability of the time-dependent operational calculus.

**3.1 Introductory remarks.** We wish to comment briefly here on what we mean by stability in the context of this paper. Below we will consider stability of the calculus in two different situations. First we will consider stability of the calculus with respect to the time-ordering measures. Specifically, given operator-valued functions $A_1(\cdot), \dots, A_k(\cdot)$ and the associated time-ordering measures $\mu_1, \dots, \mu_k$ (which will, for convenience, be taken to be continuous probability measures) on the interval $[0,T]$, we will choose sequences $\{\mu_{1n}\}_{n=1}^{\infty}, \dots, \{\mu_{kn}\}_{n=1}^{\infty}$ of continuous probability measures on $[0,T]$ that converge weakly to the given continuous probability measures $\mu_1, \dots, \mu_k$ and determine that the sequence $\{\mathcal{T}^T_{\mu_{1n},\dots,\mu_{kn}}\}_{n=1}^{\infty}$ converges in an appropriate topology to $\mathcal{T}^T_{\mu_1,\dots,\mu_k}$. The reader will note that for each $n \in \mathbf{N}$ we have a particular functional calculus and consequently our stability result in this setting states that we have a family of functional calculi converging to a certain functional calculus. A theorem along the same lines as that under discussion here can be found in [**14**] although that theorem is given in the time independent setting and the proof, while similar to the one given below, is different in some significant respects. The second stability result we will establish below is that of stability with respect to the operator-valued functions. The result obtained below is quite nice in that we only need choose sequences $\{A_{1n}(\cdot)\}_{n=1}^{\infty}, \dots, \{A_{kn}(\cdot)\}_{n=1}^{\infty}$ of operator-valued functions converging $\mu_i$ almost everywhere for each $i = 1, \dots, k$. A vector-valued version of Egorov's theorem is then used to establish the stability result. It is striking that we obtain convergence in the strong topology here with only almost everywhere convergence of the sequences of operator-valued functions.

**3.2 Stability with respect to the measures.** Let $A_i : [0,T] \to \mathcal{L}(X)$, $i = 1, \dots, k$, be measurable in the sense of Definition 2.1. Associate with each $A_i$ a continuous Borel probability measure $\mu_i$ on $[0,T]$. Also as in the introduction define

$$r_i := \int_{[0,T]} \|A_i(s)\| \, \mu_i(ds) \tag{9}$$

for $i = 1, \dots, k$, and assume that each of these quantities is finite. Construct the commutative Banach algebra $\mathbf{A}_T(r_1, \dots, r_k)$ and the disentangling algebra $\mathbf{D}_T\left((A_1(\cdot), \mu_1)^{\sim}, \dots, (A_k(\cdot), \mu_k)^{\sim}\right)$, also a commutative Banach algebra.

Now suppose that $\{\mu_{1n}\}_{n=1}^{\infty}, \dots, \{\mu_{kn}\}_{n=1}^{\infty}$ are sequences of continuous Borel probability measures on $[0,T]$ such that $\mu_{in} \rightharpoonup \mu_i$ for $i = 1, \dots, k$.

*Remark* 3.1. The reader will recall that a sequence $\{\nu_n\}_{n=1}^{\infty}$ of probability measures on a metric space $S$ converges weakly to the probability measure $\nu$ on $S$, denoted by $\nu_n \rightharpoonup \nu$, if $\int_S f \, d\nu_n \to \int_S f \, d\nu$ for every bounded continuous function $f$ on $S$.

We assume now that each $A_i(\cdot)$ is continuous with respect to the usual topology on $[0,T]$ and the norm topology on $\mathcal{L}(X)$. Define sequences $\{r_{in}\}_{n=1}^{\infty}$, $i = 1, \dots, k$ of real numbers by

$$r_{in} = \int_{[0,T]} \|A_i(s)\| \, \mu_{in}(ds). \tag{10}$$

Note that $r_{in} \to r_i$ for each $i = 1, \dots, k$. For each $n \in \mathbf{N}$ we obtain the commutative Banach algebras $\mathbf{A}_T(r_{1n}, \dots, r_{kn})$ and

$\mathbf{D}_T((A_1(\cdot),\mu_{1n})^{\sim},\dots,(A_k(\cdot),\mu_{kn})^{\sim})$. Construct the Banach algebra $\mathcal{U}_{\mathbf{D}}$ by defining

$$\mathcal{U}_{\mathbf{D}} := \sum_{n\in\mathbf{N}\cup\{0\}} \bigoplus \mathbf{D}_T((A_1(\cdot),\mu_{1n})^{\sim},\dots,(A_k(\cdot),\mu_{kn})^{\sim}). \tag{11}$$

where for $n=0$ the summand is $\mathbf{D}_T((A_1(\cdot),\mu_1)^{\sim},\dots,(A_k(\cdot),\mu_k)^{\sim})$. (The author is indebted to R. Crist for a suggestion that led to the use of this direct sum Banach algebra in this paper.) The norm on $\mathcal{U}_{\mathbf{D}}$ is

$$\|\{f_n\}\|_{\mathcal{U}_{\mathbf{D}}} = \sup\{\|f_n\|_{\mathbf{D}_T}((A_1(\cdot),\mu_{1n})^{\sim},\dots,(A_k(\cdot),\mu_{kn})^{\sim}) : n=0,1,2,\dots\}. \tag{12}$$

(See, for example, the book [**15**] for the details of direct sum Banach algebras.) Let $\pi_n$ be the canonical projection from $\mathcal{U}_{\mathbf{D}}$ to the $n$th summand for $n=0,1,2,\dots$.

With these preliminaries finished we state the following theorem.

**Theorem 3.1.** *Let $A_i:[0,T]\to\mathcal{L}(X)$, $i=1,\dots,k$, be continuous with respect to the usual topology on $[0,T]$ and the norm topology on $\mathcal{L}(X)$. Associate to each $A_i(\cdot)$ a continuous Borel probability measure $\mu_i$ on $[0,T]$. Let $\{\mu_{in}\}_{n=1}^{\infty}$, $i=1,\dots,k$, be sequences of continuous Borel probability measures on $[0,T]$ such that, for each $i=1,\dots,k$, $\mu_{in}\rightharpoonup\mu_i$. Construct the direct sum Banach algebra $\mathcal{U}_{\mathbf{D}}$ as above, see equation* (11)*, with norm $\|\cdot\|_{\mathcal{U}_{\mathbf{D}}}$ given by* (12)*. Then*

$$\lim_{n\to\infty}\left|\Lambda\left(\mathcal{T}^T_{\mu_{1n},\dots,\mu_{kn}}(\pi_n(\theta_f))\right)-\Lambda\left(\mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_0(\theta_f))\right)\right|=0 \tag{13}$$

*for all $\Lambda\in\mathcal{L}(X)^*$ and all $\theta_f:=(f,f,f,\dots)\in\mathcal{U}_{\mathbf{D}}$. Note that equation* (13) *can be restated as*

$$\lim_{n\to\infty}\left|\Lambda\left(\mathcal{T}^T_{\mu_{1n},\dots,\mu_{kn}}(f)-\mathcal{T}^T_{\mu_1,\dots,\mu_k}(f)\right)\right|=0. \tag{14}$$

*Proof.* To accomplish the proof we need to first establish that we can interchange the limit on $n$ and the sum over nonnegative integers $m_1,\dots,m_k$ in the series expansion of $f$. (Recall the definition of the disentangling map given in equation (8).) Once this is done, we show that

$$\lim_{n\to\infty}\left|\int_{\Delta_m(\pi)}\Lambda\left(C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\right)\right.$$
$$\times\left(\mu_{1n}^{m_1}\times\cdots\times\mu_{kn}^{m_k}\right)(ds_1,\dots,ds_m)$$
$$(15)\qquad -\int_{\Delta_m(\pi)}\Lambda\left(C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\right)$$
$$\left.\times\left(\mu_1^{m_1}\times\cdots\times\mu_k^{m_k}\right)(ds_1,\dots,ds_m)\right|$$
$$=0$$

for $m_1,\dots,m_k\in\mathbf{N}\cup\{0\}$ and each $\pi\in S_m$.

For the first step in the proof, we calculate as follows. Given $\Lambda\in\mathcal{L}(X)^*$ and $\theta_f:=(f,f,f,\dots)\in\mathcal{U}_{\mathbf{D}}$, we have

(16)

$$\left|\Lambda\left(\mathcal{T}^T_{\mu_{1n},\dots,\mu_{kn}}(\pi_n(\theta_f))\right)-\Lambda\left(\mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_0(\theta_f))\right)\right|$$
$$\le\|\Lambda\|\left\|\sum_{m_1,\dots,m_k=0}^{\infty}c_{m_1,\dots,m_k}\sum_{\pi\in S_m}\int_{\Delta_m(\pi)}C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\right.$$
$$\times\left(\mu_{1n}^{m_1}\times\cdots\times\mu_{kn}^{m_k}\right)(ds_1,\dots,ds_m)$$
$$-\sum_{m_1,\dots,m_k=0}^{\infty}c_{m_1,\dots,m_k}\sum_{\pi\in S_m}d\int_{\Delta_m(\pi)}C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})$$
$$\left.\times\left(\mu_1^{m_1}\times\cdots\times\mu_k^{m_k}\right)(ds_1,\dots,ds_m)\right\|$$
$$\le\|\Lambda\|\left\{\sum_{m_1,\dots,m_k=0}^{\infty}|c_{m_1,\dots,m_k}|\sum_{\pi\in S_m}\right.$$
$$\times\int_{\Delta_m(\pi)}\|C_{\pi(m)}(s_{\pi(m)})\|\cdots\|C_{\pi(1)}(s_{\pi(1)})\|\cdot$$
$$\times\left(\mu_{1n}^{m_1}\times\cdots\times\mu_{kn}^{m_k}\right)(ds_1,\dots,ds_m)$$

$$
\begin{aligned}
&+ \sum_{m_1,\dots,m_k=0}^{\infty} |c_{m_1,\dots,m_k}| \sum_{\pi \in S_m} \\
&\times \int_{\Delta_m(\pi)} \|C_{\pi(m)}(s_{\pi(m)})\| \cdots \|C_{\pi(1)}(s_{\pi(1)})\| \\
&\times (\mu_1^{m_1} \times \cdots \times \mu_k^{m_k})(ds_1,\dots,ds_m)\Big\} \\
\le \|\Lambda\| & \left\{ \|f\|_{\mathbf{D}_T((A_1(\cdot),\mu_{1n})^{\sim},\dots,(A_k(\cdot),\mu_{kn})^{\sim})} \right. \\
&\left. + \|f\|_{\mathbf{D}_T((A_1(\cdot),\mu_1)^{\sim},\dots,(A_k(\cdot),\mu_k)^{\sim})} \right\} \\
\le \|\Lambda\| & \left\{ \|\theta_f\|_{\mathcal{U}_{\mathbf{D}}} + \|f\|_{\mathbf{D}_T((A_1(\cdot),\mu_1)^{\sim},\dots,(A_k(\cdot),\mu_k)^{\sim})} \right\}.
\end{aligned}
$$

Let $\varepsilon > 0$ be given. Using the definition of the norm on $\mathcal{U}_{\mathbf{D}}$ there is an $n_0 \in \mathbf{N}$ such that

$$
\|f\|_{\mathbf{D}_T\left((A_1(\cdot),\mu_{1n_0})^{\sim},\dots,(A_k(\cdot),\mu_{kn_0})^{\sim}\right)} + \varepsilon > \|\theta_f\|_{\mathcal{U}_{\mathbf{D}}}. \tag{17}
$$

We now consider the map

$$
\begin{aligned}
(m_1,\dots,m_k) \longmapsto \|\Lambda\| \big(&|c_{m_1,\dots,m_k}| r_{1n_0}^{m_1} \cdots r_{kn_0}^{m_k} + \varepsilon/2^m \\
&+ |c_{m_1,\dots,m_k}| r_1^{m_1} \cdots r_k^{m_k}\big).
\end{aligned} \tag{18}
$$

In the expression on the right above, the product $|c_{m_1,\dots,m_k}| r_{1n_0}^{m_1} \cdots r_{kn_0}^{m_k}$ is the summand for the norm in the disentangling algebra

$$
\mathbf{D}_T\left((A_1(\cdot),\mu_{1n_0})^{\sim},\dots,(A_k(\cdot),\mu_{kn_0})^{\sim}\right).
$$

The last term on the right above is the summand for the norm on the disentangling algebra $\mathbf{D}_T\left((A_1(\cdot),\mu_1)^{\sim},\dots,(A_k(\cdot),\mu_k)^{\sim}\right)$. Finally, the term $\varepsilon/2^m$ above enters in via equation (17) where $m := m_1 + \cdots + m_k$. Using equation (17) the map defined in equation (18) is clearly a summable dominating function for

$$
\begin{aligned}
\sum_{m_1,\dots,m_k=0}^{\infty} |c_{m_1,\dots,m_k}| \sum_{\pi \in S_m} \Bigg| &\int_{\Delta_m(\pi)} \Lambda\left\{C_{\pi(m)}(s_{\pi(m)}) \cdots \right. \\
&\left. C_{\pi(1)}(s_{\pi(1)})\right\} (\mu_{1n}^{m_1} \times \cdots \times \mu_{kn}^{m_k})(ds_1,\dots,ds_m) \\
&- \int_{\Delta_m(\pi)} \Lambda\left\{C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})\right\} \\
&\times (\mu_1^{m_1} \times \cdots \times \mu_k^{m_k})(ds_1,\dots,ds_m)\Bigg|.
\end{aligned} \tag{19}
$$

The dominated convergence theorem applies and consequently we are allowed to apply the limit on $n$ to

$$
\begin{aligned}
&\bigg| \int_{\Delta_m(\pi)} \Lambda \left\{ C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \right\} \\
&\quad \times \left(\mu_{1n}^{m_1} \times \cdots \times \mu_{kn}^{m_k}\right)(ds_1, \ldots, ds_m) \\
&\quad - \int_{\Delta_m(\pi)} \Lambda \left\{ C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \right\} \\
&\quad \times \left(\mu_1^{m_1} \times \cdots \times \mu_k^{m_k}\right)(ds_1, \ldots, ds_m) \bigg|
\end{aligned} \tag{20}
$$

for fixed $\pi \in S_m$ and fixed $m_1, \ldots, m_k \in \mathbf{N}$.

In order to calculate the limit of the expression above, we need to appeal to the continuity assumption on the operator-valued functions $A_1(\cdot), \ldots, A_k(\cdot)$ as well as to properties of weak convergence of probability measures. To this end we rewrite the expression given in (20) as

$$
\begin{aligned}
&\bigg| \int_{[0,T]^m} \chi_{\Delta_m(\pi)} \Lambda \left\{ C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \right\} \\
&\quad \times \left(\mu_{1n}^{m_1} \times \cdots \times \mu_{kn}^{m_k}\right)(ds_1, \ldots, ds_m) \\
&\quad - \int_{[0,T]^m} \chi_{\Delta_m(\pi)} \Lambda \left\{ C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \right\} \\
&\quad \times \left(\mu_1^{m_1} \times \cdots \times \mu_k^{m_k}\right)(ds_1, \ldots, ds_m) \bigg|.
\end{aligned} \tag{21}
$$

Note that $\chi_{\Delta_m(\pi)}$ is continuous except on a set of $\mu_1^{m_1} \times \cdots \times \mu_k^{m_k}$-measure zero since our measures are all continuous. Next, since $[0,T]^m$ is a separable metric space in the product topology, Theorem 3.2 of [**1**] tells us that $\mu_{1n}^{m_1} \times \cdots \times \mu_{kn}^{m_k} \rightharpoonup \mu_1^{m_1} \times \cdots \times \mu_k^{m_k}$. Finally, since the set of discontinuities of the integrands in equation (21) have $\mu_1^{m_1} \times \cdots \times \mu_k^{m_k}$-measure zero, Theorem 5.2 of [**1**] asserts that the limit on $n$ of the expression in (21) vanishes. This finishes the proof. □

*Remark* 3.2. (i) We note that the conclusion of Theorem 3.1 is not as strong as the convergence in the operator norm topology on $\mathcal{L}(X)$ that was obtained in the corresponding theorem in [**14**]. (Weak convergence of sequences of measures is used in both theorems.) This difference is of course due to the change to the time dependent setting and the requirements of weak convergence of sequences of measures.

(ii) The reader should note that a simple corollary to the theorem just proved is the observation that we can replace equation (14) with a corresponding equation concerning convergence in the weak operator topology on $\mathcal{L}(X)$.

**3.3 Stability with respect to the operator-valued functions.** To establish a stability theorem for the disentangling map with respect to the operator-valued functions in the time dependent setting, we need some preliminaries. Suppose $A_i : [0, T] \to \mathcal{L}(X)$, $i = 1, \dots, k$ are given and associate to each $A_i(\cdot)$ a continuous Borel probability measure $\mu_i$ on $[0, T]$. In order to obtain the proof of Theorem 3.5 below, we need to change somewhat the definition of measurability that we will use here. The reason for this is to accommodate the use of the vector-valued version of Egorov's theorem on almost uniform convergence of sequences of functions. The definition found immediately below is that given by Dunford and Schwartz in [**4**, p. 106].

**Definition 3.1.** Let $(S, \mathcal{M}, \mu)$ be a finite measure space where $\mu$ is a countably additive real or complex measure on $S$ and $\mathcal{M}$ is the $\sigma$-algebra of $\mu$-measurable sets. For $X$ a Banach space and $f : S \to X$, we say that $f$ is $\mu$-measurable if there is a sequence $\{f_n\}$ of $X$-valued simple functions on $S$ that converge to $f$ in $X$-norm $\mu$-almost everywhere.

Two characterizations of our definition of $\mu$-measurability are the following (Theorems 10 and 11, respectively, pp. 148 and 149 of [**4**]).

**Theorem 3.2.** *Let $(S, \mathcal{M}, \mu)$ be a finite measure space with $\mu$ a countably additive measure on $S$. Let $(S, \mathcal{M}^*, \mu)$ be the completion of the measure space $(S, \mathcal{M}, \mu)$. Then, given a Banach space $X$, $f : S \to X$ is $\mu$-measurable if and only if, for every $F \in \mathcal{M}$,*

(i) *there is a $\mu$-null set $N \subset F$ and a countable set $H \subset X$ such that $f(F - N) \subset \mathrm{cl}\,(H)$, and*

(ii) *$F \cap f^{-1}(G) \in \mathcal{M}^*$ for each open $G \subset X$.*

*Remark* 3.3. The set $G$ in (ii) above can be taken as a Borel set in $X$.

**Theorem 3.3.** *Let* $(S, \mathcal{M}, \mu)$ *be a finite measure space and* $\mu$ *a countably additive measure on* $\mathcal{M}$. *Let* $X$ *be a Banach space. Then* $f : S \to X$ *is* $\mu$*-measurable if and only if*

(i) $f$ *satisfies* (i) *above in Theorem* 3.2 *for* $F = S$, *and*

(ii) *for every* $x^* \in X^*$, *the scalar function* $x^* f$ *on* $S$ *is measurable.*

*Remark* 3.4. In view of Theorem 3.2, there appears to be a connection to the definition of measurability used in the discussion of the first stability theorem, it is not at this time entirely clear to the author what the precise connection is.

We will assume from this point on that each $A_i$ is $\mu_i$-measurable for each $i = 1, \dots, k$. We will, in addition, assume that

$$\sup\{\|A_i(s)\| : s \in [0, T]\} < \infty \tag{22}$$

for $i = 1, \dots, k$. It follows from the measurability assumption and the norm boundedness assumption above that each of the functions $A_i(\cdot)$ are Bochner integrable on $[0, T]$.

*Remark* 3.5. It should be noted that we really only need the boundedness assumption given in equation (22) to be true $\mu_i$ almost everywhere for each $i$. We make the stronger assumption above for convenience.

Next choose sequences $\{A_{i,n}(\cdot)\}_{n=1}^{\infty}$, $i = 1, \dots, k$, of $\mathcal{L}(X)$-valued functions which are measurable in the sense of Definition 3.1 and such that

$$\lim_{n\to\infty} \|A_{i,n}(s) - A_i(s)\|_{\mathcal{L}(X)} = 0 \tag{23}$$

$\mu_i$ almost everywhere in $[0, T]$ for each $i$. Assume further that

$$\sup\{\|A_{i,n}(s)\| : s \in [0, T], n = 1, 2, \dots\} < \infty$$

and that

$$\sup\{\|A_i(s)\| : s \in [0,T]\} < \infty$$

for $i = 1, \dots, k$.

We now state the vector-valued version of Egorov's theorem that will be used in the proof of Theorem 3.5 below. (See, for example, Theorem 12, p. 149, of [**4**].)

**Theorem 3.4.** *Let $(S, \mathcal{M}, \mu)$ be a finite measure space with a countably additive measure $\mu$. A sequence $\{f_n\}$ of $\mu$-measurable functions on $S$ with values in a Banach space $X$ converges $\mu$-uniformly to the $\mu$-measurable function $f$ if and only if $f_n \to f$ $\mu$ almost everywhere in $S$.*

*Remark* 3.6. Recall that $f_n$ converges to $f$ $\mu$-uniformly if for every $\varepsilon > 0$ there is a set $E \in \mathcal{M}$ such that $|\mu|(E) < \varepsilon$ and $f_n \to f$ uniformly on $S - E$. In the current setting of $X$-valued functions, convergence is with respect to the norm on $X$.

We now move on to the construction of the various Banach algebras needed for our stability theorem. Define

$$r_i := \int_{[0,T]} \|A_i(s)\| \, \mu_i(ds) \tag{24}$$

and

$$r_{i,n} := \int_{[0,T]} \|A_{i,n}(s)\| \, \mu_i(ds) \tag{25}$$

for every $i, n$. Note that, due to the norm finiteness assumptions above, all of these numbers are finite. We obtain a family $\{\mathbf{A}_T(r_{1,n}, \dots, r_{k,n}) : n \in \mathbf{N}\}$ of commutative Banach algebras. Associated with this family of Banach algebras is the corresponding family $\{\mathbf{D}_T(A_{1,n}(\cdot)^{\sim}, \dots, A_{k,n}(\cdot)^{\sim}) : n \in \mathbf{N}\}$ of disentangling algebras. Construct the Banach algebra

$$\mathcal{U}_{\mathbf{D}} = \sum_{n \in \mathbf{N} \cup \{0\}} \bigoplus \mathbf{D}_T(A_{1,n}(\cdot)^{\sim}, \dots, A_{k,n}(\cdot)^{\sim})$$

where the zeroth summand is the disentangling algebra $\mathbf{D}_T(A_1(\cdot)^\sim, \dots, A_k(\cdot)^\sim)$. The norm on this Banach algebra is that given above in (12). We can now state

**Theorem 3.5.** *Let* $A_i : [0,T] \to \mathcal{L}(X)$, $i = 1, \dots, k$, *be measurable in the sense of Definition* 3.1. *Associate to each* $A_i$ *a continuous Borel probability measure* $\mu_i$ *on* $[0,T]$. *Next choose sequences* $\{A_{i,n}(\cdot)\}_{n=1}^{\infty}$, $i = 1, \dots, k$, *of* $\mathcal{L}(X)$*-valued functions which are measurable in the sense of Definition* 3.1 *and such that*

$$\lim_{n\to\infty} \|A_{i,n}(s) - A_i(s)\|_{\mathcal{L}(X)} = 0 \tag{26}$$

$\mu_i$ *almost everywhere in* $[0,T]$ *for each* $i$. *Assume further that*

$$\sup\{\|A_{i,n}(s)\| : s \in [0,T], n = 1, 2, \dots\} < \infty$$

*and that*

$$\sup\{\|A_i(s)\| : s \in [0,T]\} < \infty$$

*for* $i = 1, \dots, k$. *Construct commutative Banach algebras* $\mathbf{A}_T(r_1, \dots, r_k)$ *and* $\mathbf{A}_T(r_{1,n}, \dots, r_{k,n})$ *with weights defined in equations* (24) *and* (25), *the associated disentangling algebras* $\mathbf{D}_T(A_1(\cdot)^\sim, \dots, A_k(\cdot)^\sim)$ *and* $\mathbf{D}_T(A_{1,n}(\cdot)^\sim, \dots, A_{k,n}(\cdot)^\sim)$, *and the direct sum Banach algebra* $\mathcal{U}_{\mathbf{D}}$. *It follows that*

$$\lim_{n\to\infty} \mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_n(\theta_f)) = \mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_0(\theta_f)) \tag{27}$$

*for all* $\theta_f \in \mathcal{U}_{\mathbf{D}}$ *where* $\theta_f$ *and* $\pi_n$ *are the same as in the previous subsection. Of course equation* (27) *can also be written as*

$$\lim_{n\to\infty} \mathcal{T}^T_{\mu_1,\dots,\mu_k}(f) = \mathcal{T}^T_{\mu_1,\dots,\mu_k}(f). \tag{28}$$

*Proof.* Let $m_1, \dots, m_k \in \mathbf{N}$. Then, by the definition of the disentangling map,

$$\begin{aligned} &P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_{1,n}(\cdot), \dots, A_{k,n}(\cdot)) \\ &\quad = \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} C_{\pi(m),n}(s_{\pi(m)}) \cdots C_{\pi(1),n}(s_{\pi(1)}) \\ &\qquad \times (\mu_1^{m_1} \times \cdots \times \mu_k^{m_k})(ds_1, \dots, ds_m) \end{aligned} \tag{29}$$

and so

(30)

$$
\begin{aligned}
&\left\| P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}\,(A_{1,n}(\cdot),\dots,A_{k,n}(\cdot)) - P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}\,(A_1(\cdot),\dots,A_k(\cdot)) \right\| \\
&\quad \le \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} \|C_{\pi(m),n}(s_{\pi(m)})\cdots C_{\pi(1),n}(s_{\pi(1)}) \\
&\qquad - C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})\|\,(\mu_1^{m_1}\times\cdots\times\mu_k^{m_k})\,(ds_1,\dots,ds_m) \\
&\quad \le \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} \sum_{j=1}^{m} \Big\{ \|C_{\pi(1),n}(s_{\pi(1)})\|\cdots\|C_{\pi(j-1),n}(s_{\pi(j-1)})\| \\
&\qquad \times \|C_{\pi(j),n}(s_{\pi(j)}) - C_{\pi(j)}(s_{\pi(j)})\| \\
&\qquad \times \|C_{\pi(j+1)}(s_{\pi(j+1)})\|\cdots\|C_{\pi(m)}(s_{\pi(m)})\| \Big\} \\
&\qquad \times (\mu_1^{m_1}\times\cdots\times\mu_k^{m_k})\,(ds_1,\dots,ds_m) \\
&\quad \le \sum_{\pi\in S_m} \sum_{j=1}^{m} \int_{[0,T]^m} \Big\{ \|C_{\pi(1),n}(s_{\pi(1)})\|\cdots\|C_{\pi(j-1),n}(s_{\pi(j-1)})\| \\
&\qquad \times \|C_{\pi(j),n}(s_{\pi(j)}) - C_{\pi(j)}(s_{\pi(j)})\|\,\|C_{\pi(j+1)}(s_{\pi(j+1)})\|\cdots \\
&\qquad \|C_{\pi(m)}(s_{\pi(m)})\| \Big\}\,(\mu_1^{m_1}\times\cdots\times\mu_k^{m_k})\,(ds_1,\dots,ds_m) \\
&\quad = \sum_{\pi\in S_m} \sum_{j=1}^{m} \Bigg[ \bigg\{ \int_{[0,T]} \|C_{\pi(1),n}(s_{\pi(1)})\|\,\mu_{\pi(1)}(ds_{\pi(1)}) \bigg\}\cdots \\
&\qquad \bigg\{ \int_{[0,T]} \|C_{\pi(j-1),n}(s_{\pi(j-1)})\|\,\mu_{\pi(j-1)}(ds_{\pi(j-1)}) \bigg\} \\
&\qquad \times \bigg\{ \int_{[0,T]} \|C_{\pi(j),n}(s_{\pi(j)}) - C_{\pi(j)}(s_{\pi(j)})\|\,\mu_{\pi(j)}(ds_{\pi(j)}) \bigg\} \\
&\qquad \times \bigg\{ \int_{[0,T]} \|C_{\pi(j+1)}(s_{\pi(j+1)})\|\,\mu_{\pi(j+1)}(ds_{\pi(j+1)}) \bigg\}\cdots \\
&\qquad\quad \bigg\{ \int_{[0,T]} \|C_{\pi(m)}(s_{\pi(m)})\|\,\mu_{\pi(m)}(ds_{\pi(m)}) \bigg\} \Bigg]
\end{aligned}
$$

where $\mu_{\pi(i)}$ denotes the measure associated with the operator-valued function $C_{\pi(i)}$. Let $\varepsilon > 0$ be given. Since $A_{j,n} \to A_j$ $\mu_j$-almost everywhere in $\mathcal{L}(X)$-norm for each $j = 1,\dots,k$ we can, using the vector-valued version of Egorov's theorem, choose Borel sets $E_j$, $j =$

$1, \dots, k$, in $[0, T]$ such that $\mu_j([0,T] \backslash E_j) < \varepsilon$ and $A_{j,n} \to A_j$ uniformly in $\mathcal{L}(X)$-norm on $E_j$, $j = 1, \dots, k$. We can therefore write, for each $j = 1, \dots, k$,

$$\begin{aligned}
&\int_{[0,T]} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds) \\
&\quad = \int_{[0,T]\backslash E_j} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds) + \int_{E_j} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds) \\
&\quad \leq \varepsilon \cdot \sup \left\{ \|A_{j,n}(s)\| + \|A_j(s)\| : s \in [0,T], n \in \mathbf{N} \right\} \\
&\qquad + \int_{E_j} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds) \\
&\quad =: \varepsilon R_j + \int_{E_j} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds).
\end{aligned}$$

By uniform convergence on $E_j$ there is an $N_j \in \mathbf{N}$ such that if $n \geq N_j$, then $\|A_{j,n}(s) - A_j(s)\| < \varepsilon$ for all $s \in [0, T]$. It follows that, for $n \geq N_j$,

$$\varepsilon R_j + \int_{E_j} \|A_{j,n}(s) - A_j(s)\| \, \mu_j(ds) < \varepsilon R_j + \varepsilon \mu_j(E_j) \tag{32}$$

for $j = 1, \dots, k$. Therefore, for $n \geq \max(N_1, \dots, N_k)$, we have

$$\begin{aligned}
&\sum_{\pi \in S_m} \sum_{j=1}^{m} \Bigg[ \left\{ \int_{[0,T]} \|C_{\pi(1),n}(s_{\pi(1)})\| \, \mu_{\pi(1)}(ds_{\pi(1)}) \right\} \cdots \\
&\quad \left\{ \int_{[0,T]} \|C_{\pi(j-1),n}(s_{\pi(j-1)})\| \, \mu_{\pi(j-1)}(ds_{\pi(j-1)}) \right\} \\
&\quad \times \left\{ \int_{[0,T]} \|C_{\pi(j),n}(s_{\pi(j)}) - C_{\pi(j)}(s_{\pi(j)})\| \, \mu_{\pi(j)}(ds_{\pi(j)}) \right\} \\
&\quad \times \left\{ \int_{[0,T]} \|C_{\pi(j+1)}(s_{\pi(j+1)})\| \, \mu_{\pi(j+1)}(ds_{\pi(j+1)}) \right\} \cdots \\
&\quad\quad \left\{ \int_{[0,T]} \|C_{\pi(m)}(s_{\pi(m)})\| \, \mu_{\pi(m)}(ds_{\pi(m)}) \right\} \Bigg]
\end{aligned} \tag{33}$$

$$< \sum_{\pi \in S_m} \sum_{j=1}^{m} \Big\{ \big[\sup\{\|C_{\pi(1),n}(s_{\pi(1)})\| : s \in [0,T], n \in \mathbf{N}\}\big] \cdots \\ \big[\sup\{\|C_{\pi(j-1),n}(s_{\pi(j-1)})\| : s \in [0,T], n \in \mathbf{N}\}\big] \\ \times \varepsilon\{R_j + \mu_j(E_j)\} \cdot \big[\sup\{\|C_{\pi(j+1)}(s_{\pi(j+1)})\| : s \in [0,T]\}\big] \cdots \\ \big[\sup\{\|C_{\pi(m)}(s_{\pi(m)})\| : s \in [0,T]\}\big] \Big\} \\ = \varepsilon \ \{\text{finite sum of finite terms}\}.$$

Hence we have established that, as $n \to \infty$, $\|P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_{1,n}(\cdot),\dots, A_{k,n}(\cdot)) - P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_1(\cdot),\dots,A_k(\cdot))\| \to 0$.

To finish our proof, we choose $\theta_f = (f,f,\dots) \in \mathcal{U}_{\mathbf{D}}$. Then, proceeding as in our first stability result, Theorem 3.1,

$$\begin{aligned} &\big\|\mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_n(\theta_f)) - \mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_0(\theta_f))\big\| \\ &\quad \le \sum_{m_1,\dots,m_k=0}^{\infty} |c_{m_1,\dots,m_k}| \left(r^{m_1}_{1,n}\cdots r^{m_k}_{k,n} + r^{m_1}_1 \cdots r^{m_k}_k\right) \\ &\quad \le \|\theta_f\|_{\mathcal{U}_{\mathbf{D}}} + \|f\|_{\mathbf{D}_T((A_1(\cdot),\mu_1)^\sim,\dots,(A_k(\cdot),\mu_k)^\sim)}. \end{aligned} \tag{34}$$

Choose, given $\delta > 0$, an $n_0 \in \mathbf{N}$ such that

$$\|\theta_f\|_{\mathcal{U}_{\mathbf{D}}} \le \|f\|_{\mathbf{D}_T((A_{1,n_0}(\cdot),\mu_1)^\sim,\dots,(A_{k,n_0}(\cdot),\mu_k)^\sim)} + \delta. \tag{35}$$

The map

$$(m_1,\dots,m_k) \\ \longmapsto |c_{m_1,\dots,m_k}|\, r^{m_1}_{1,n_0}\cdots r^{m_k}_{k,n_0} + \frac{\delta}{2^m} + |c_{m_1,\dots,m_k}|\, r^{m_1}_1\cdots r^{m_k}_k \tag{36}$$

is therefore a summable dominating function for the sum over $m_1,\dots,m_k$ in equation (34). The dominated convergence theorem applies and we obtain

(37)

$$
\begin{aligned}
\lim_{n\to\infty} &\|\mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_n(\theta_f)) - \mathcal{T}^T_{\mu_1,\dots,\mu_k}(\pi_0(\theta_f))\| \\
&\leq \lim_{n\to\infty} \sum_{m_1,\dots,m_k=0}^{\infty} |c_{m_1,\dots,m_k}| \, \|P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_{1,n}(\cdot),\dots,A_{k,n}(\cdot)) \\
&\quad - P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_1(\cdot),\dots,A_k(\cdot))\| \\
&= \sum_{m_1,\dots,m_k=0}^{\infty} |c_{m_1,\dots,m_k}| \cdot \lim_{n\to\infty} \|P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_{1,n}(\cdot),\dots,A_{k,n}(\cdot)) \\
&\quad - P^{m_1,\dots,m_k}_{T;\mu_1,\dots,\mu_k}(A_1(\cdot),\dots,A_k(\cdot))\| \\
&= 0,
\end{aligned}
$$

finishing the proof of the theorem. □

**3.4 Discussion.** The stability theorems presented in the previous section give us a reasonable start to a stability theory for Feynman's operational calculus in the time dependent setting. The first theorem, Theorem 3.1, establishing stability with respect to the time-ordering measures, does not have as strong a conclusion as one would hope. However, because we are using weak convergence of sequences of probability measures, we needed to use a $\Lambda \in \mathcal{L}(X)^*$ in order to obtain a complex-valued function with which to use in conjunction with the weak convergence assumption. It may be possible to improve Theorem 3.1 and work on this is in progress.

The second stability theorem, Theorem 3.5, concerning stability of the calculus with respect to the operator-valued functions seems to be a fairly strong result. We are able to obtain strong operator convergence of the disentangled expressions by assuming only almost-everywhere convergence of the sequences of $\mathcal{L}(X)$-valued functions. Of course, the fact that we are working in a finite measure space is what enables us to apply Egorov's theorem and this is in fact the key to the proof.

Another point worth bringing out is that we can bring an unbounded operator into the calculus as a generator of a $(C_0)$ semi-group. Indeed, in [**11**] the disentangling algebra and the disentangling map is defined in the presence of a $(C_0)$ semi-group of operators. It is natural to ask about stability in this setting, and, in fact, one sees in [**16**] and [**17**] theorems concerning the stability of disentanglings of the exponential function of several noncommuting operator-valued functions in the presence of a generator of a $(C_0)$ semigroup. Similar theorems to some of those presented in [**14**] and [**17**] can certainly be stated and proved. Work on such theorems is under way. Also, in the presence of a semigroup generator one can also establish stability with respect to strong resolvant convergence of a sequence of semigroup generators and this theorem, too, will be presented in future work.

Finally, a version of the operational calculus using time-ordering measures with discrete and continuous parts is under development by the author and G.W. Johnson. While a stability theory in this more general setting is much more difficult to work out, it does appear that some stability theorems will be possible to establish. These results will also appear in future work.

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