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SEMIUMBILICS AND 2-REGULAR IMMERSIONS OF SURFACES IN EUCLIDEAN SPACES

S.M. MORAES AND M.C. ROMERO-FUSTER

ABSTRACT. The semiumbilics are points at which the curvature ellipse of a surface degenerates to a segment. We characterize them here as critical points of the principal configurations associated to essential normal fields on the surface. This allows us to show that orientable closed surfaces with nonvanishing Euler number must have semiumbilics when immersed in 4-space. We also obtain as a consequence some conclusions relating the existence of 2-regular embeddings of surfaces in $\mathbf{R}^5$ in the sense of Feldman with that of globally defined essential normal fields.

1. Introduction. Feldman introduced in [**4**] the concept of regular immersion of order 2 of a submanifold in $\mathbf{R}^n$. He showed that, in the case of a surface, the most relevant dimensions to study this property are $n = 5, 6$, for there are no regular immersions of order 2 for $n \leq 4$ and, on the other hand, such immersions become generic when $n > 6$. A natural question in this context is that of the existence of 2-regular immersions for a given surface in $\mathbf{R}^n$, $n = 5, 6$. The answer is nontrivial, it is not at all easy to obtain 2-regular immersions of, for instance, a 2-sphere in $\mathbf{R}^5$. A well-known one is the Veronesse surface [**3**], but you can see that a small perturbation of this introduces 2-singular points [**9**]. It was observed in [**9**] that for $n = 5$ this problem can be put in terms of the contacts of the surface with hyperplanes. We point out that in a first stage these contacts are governed by the behavior of the second fundamental form of the surface, which is also related to that of the principal configurations associated to normal fields on the surface. Our purpose here is to use these tools in order to obtain further information on the possibility of obtaining 2-regular immersions of surfaces.

We find in the way that one obstruction for the 2-regularity of the surface is the existence of semiumbilics. Such points are defined in

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terms of the curvature ellipses and have a special interest by themselves. The curvature ellipse at a point p of a surface M immersed in $\mathbf{R}^n$ was introduced in [**11**] as the locus of the end points of the curvature vectors of the normal sections along all the tangent directions to M at p. This ellipse lies in the normal subspace at p and is completely determined by the second fundamental form. At certain points of M this ellipse may degenerate becoming a segment or even a point. The first are known as semiumbilic points of M, whereas the second are the umbilic. The semiumbilic points are said to be of radial or of ordinary type according to whether the ellipse is a radial or a nonradial segment in the normal subspace to M at the considered point. It is not difficult to see that every point of any surface in 3-space is a semiumbilic of radial type (for N_pM is just a line in this case) or, exceptionally, an umbilic. Conversely, it can be deduced from some results contained in [**15**] that a compact surface immersed in $\mathbf{R}^n$ exclusively made of semiumbilic points of radial type lies in some 3-space. The semiumbilic points of generic surfaces in $\mathbf{R}^4$ were introduced and studied from the viewpoint of the contacts of the surface with hyperspheres (singularities of distance squared functions on M) in [**10**]. It was shown there that, under adequate genericity assumptions, they form regularly embedded closed curves. Those of radial type, known as *inflection points*, are isolated on these curves. The surfaces contained in a 3-sphere have the property that all their points are of semiumbilic type, see [**16**]. Surfaces generically immersed in $\mathbf{R}^5$ may only have isolated semiumbilic points, whereas surfaces in higher dimensional Euclidean spaces do not present such points in a generic way.

It was proven in [**5**] that a compact surface with nonvanishing Euler number generically immersed in $\mathbf{R}^4$ as a convex surface always has inflection points. It follows from this that it must have at least a closed curve made of semiumbilic points. We provide here a characterization of the semiumbilic points of a surface immersed in 4-space as critical points of the principal configurations associated to the normal fields on the surface, Theorem 1. This leads us to obtain the following, Corollary 1.

Corollary 1. *Any orientable closed, i.e., compact without boundary, surface with nonvanishing Euler number immersed in* 4*-space has semiumbilic points.*

In the case of a surface immersed in a higher dimensional space, we see (Theorem 2) that the semiumbilic points can also be characterized as critical points of the principal configurations associated to special normal fields on the surface. In fact, these fields, called essential, are special in the sense that they contain all the relevant information with respect to the set of all the possible principal configurations on the surface [**13**].

Finally, we see that since the semiumbilic are a particular case of singular points of order two of the immersion in the sense of Feldman, the above characterization provides a relation between the existence of regular immersions of order two and that of essential normal fields globally defined over the surfaces, Theorem 4. In particular, it follows that the classical immersion of the projective plane into a 4-sphere of $\mathbf{R}^5$, known as the Veronesse surface, [**6**], being a 2-regular surface in $\mathbf{R}^5$, does not admit globally defined essential normal fields, which implies the non existence of globally defined normal fields whose tangent component to 4-sphere is nowhere vanishing (Corollary 3).

2. Curvature ellipses and semiumbilics. Let M be a surface immersed in $\mathbf{R}^n$, $n \geq 4$, and let $\overline{\nabla}$ denote the Riemannian connection of $\mathbf{R}^n$. Given vector fields, X, Y, locally defined along M, we can choose local extensions $\overline{X}, \overline{Y}$ over $\mathbf{R}^n$, and define the Riemannian connection on M as $\nabla_X Y = \big(\overline{\nabla}_{\overline{X}} \overline{Y}\big)^{\top}$, that is, the tangent component of $\overline{\nabla}_{\overline{X}}$ on M.

If we denote by $\mathcal{X}(M)$ and $\mathcal{N}(M)$ respectively the spaces of tangent and normal fields on M, the second fundamental form on M is defined as follows:

$$\begin{aligned} \alpha : \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow \mathcal{N}(M) \\ \big(X, Y\big) &\longmapsto \overline{\nabla}_{\overline{X}} \overline{Y} - \nabla_X Y, \end{aligned}$$

This is a well-defined bilinear symmetric map and induces, for each $p \in M$ and $\nu \in N_pM$, $\nu \neq 0$, a bilinear form on the tangent space T_pM given by $H_\nu(v, w) = \big\langle \alpha(v, w), \nu \big\rangle$. The corresponding quadratic form $II_\nu(v) = H_\nu(v, v) = \big\langle \alpha(v, v), \nu \big\rangle$ is known as the *second fundamental form in the direction* ν.

Given $p \in M$, consider the unit circle in T_pM parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ the curve obtained by intersecting M

with the hyperplane at p composed by the direct sum of the normal subspace N_pM and the straight line in the tangent direction represented by θ. The curvature vector $\eta(\theta)$ of γ_θ in p lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the *curvature ellipse* of M at p, see [**6**] and [**11**].

We can take M locally as the image of an embedding $\phi : \mathbf{R}^2 \to \mathbf{R}^n$. Let $\{x, y\}$ be isothermic coordinates and $\{e_1, e_2, \dots, e_n\}$ an orthonormal frame in a neighborhood of a point $p = \phi(0,0) \in M$, in such a way that $\{e_1, e_2\}$ is the tangent frame determined by these coordinates and $\{e_3, \dots, e_n\}$ is a normal frame. Then the second fundamental form of M is determined by the parameters a_i, b_i, c_i, $i = 1, \dots, n-2$, at p is given by

$$\alpha_\phi(p) = \begin{bmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_{n-2} & b_{n-2} & c_{n-2} \end{bmatrix},$$

where

$$\begin{aligned} a_i &= \alpha_\phi(e_1, e_1) \cdot e_{i+2} = \frac{1}{E}\frac{\partial^2 \phi}{\partial x^2}(p) \cdot e_{i+2}, \\ b_i &= \alpha_\phi(e_1, e_2) \cdot e_{i+2} = \frac{1}{E\sqrt{EG - F^2}}\left(E\frac{\partial^2 \phi}{\partial x \partial y}(p) - F\frac{\partial^2 \phi}{\partial y^2}(p)\right) \cdot e_{i+2} \\ &= \frac{1}{E}\frac{\partial^2 \phi}{\partial x^2}(p) \cdot e_{i+2} \end{aligned}$$

and

$$\begin{aligned} c_i &= \alpha_\phi(e_2, e_2) \cdot e_{i+2} \\ &= \frac{1}{E(EG - F^2)}\left(E^2\frac{\partial^2 \phi}{\partial y^2}(p) - 2EF\frac{\partial^2 \phi}{\partial x \partial y}(p) + F^2\frac{\partial^2 \phi}{\partial x^2}(p)\right) \cdot e_{i+2} \\ &= \frac{1}{E}\frac{\partial^2 \phi}{\partial y^2}(p) \cdot e_{i+2}, \end{aligned}$$

for $i = 1, \dots, n-2$, and $ds^2 = E(dx^2 + dy^2)$ is the first fundamental form.

The curvature ellipse can be seen as the image of the affine map

$$\eta : S^1 \subset T_pM \longrightarrow N_pM,$$

given by

$$\eta(\theta) = H + B\cos 2\theta + C\sin 2\theta,$$

where $H = (\sum_{i=1}^{n-2}/2)(a_i + c_i)\cdot e_{i+2}$, $B = (\sum_{i=1}^{n-2}/2)(a_i - c_i)\cdot e_{i+2}$ and $C = \sum_{i=1}^{n-2} b_i \cdot e_{i+2}$.

A point $p \in M$ at which the curvature ellipse is degenerate (a segment or a point) is said to be *semiumbilic*. An *inflection point* is a semiumbilic point for which the curvature ellipse is a radial segment, and a semiumbilic point at which the curvature ellipse becomes a unique point shall be called *umbilic*. In case this ellipse coincides with the origin p of N_pM, then p is said to be a *flat umbilic*. Semiumbilics form closed curves with isolated inflection points at generic surfaces in $\mathbf{R}^4$ [**9**]. On the other hand, they appear isolated on generic surfaces in $\mathbf{R}^5$ [**13**] and do not appear at all over generic surfaces in $\mathbf{R}^n$, $n \geq 6$. Moreover, umbilic and flat inflection points may be avoided over generic surfaces in both $\mathbf{R}^n$, $n \geq 4$.

The affine span of the curvature ellipse at a point p is an affine subspace, $H_p \subset N_pM$. We denote E_p the vector subspace of N_pM parallel to H_p. For surfaces in 4-space, the subspace $H_p = E_p$ coincides with the whole normal plane at a non semiumbilic points. If p is a semiumbilic point, H_p is an affine line in N_pM.

The vector valued quadratic form α_ϕ induces, for each $p \in M$, a linear map A_p from the normal space, N_pM, of M at p to the space Q of quadratic forms in the variables x and y. If we represent a vector $v \in N_pM$ by its coordinates $(v_3, \dots, v_n)$ with respect to the basis $\{e_3, \dots, e_n\}$, we have

$$A_p(v_3, \dots, v_n) = v_3(d^2\phi \cdot e_3) + \cdots + v_n(d^2\phi \cdot e_n).$$

Now, by using the natural identifications (through the basis induced by the above frame) of N_pM with $\mathbf{R}^{n-2}$ and of Q with $\mathbf{R}^3$, we can view this as the linear map $A_p : \mathbf{R}^{n-2} \to \mathbf{R}^3$, whose matrix is the transpose of that of $\alpha_\phi(p)$.

We classify a point $p \in M$ into the type M_i, $i = 3, 2, 1, 0$, according to rank$(A_p) = 3, 2, 1, 0$.

Lemma 1. *Given a surface M in $\mathbf{R}^n$, $n \geq 4$,*

i) *If $p \in M$ is semiumbilic, then* rank$(A_p) < 3$.

ii) $p \in M_1$ *if and only if* p *is either an inflection point or a* (*non flat*) *umbilic.*

iii) p *is a flat umbilic if and only if* $p \in M_0$.

Proof. i) If p is semiumbilic so $\eta(\theta)$ degenerates to a segment, we then have that the vectors $H = (\sum_{i=1}^{n-2}/2)(a_i + c_i) \cdot e_{i+2}$, $B = (\sum_{i=1}^{n-2}/2)(a_i - c_i) \cdot e_{i+2}$ and $C = \sum_{i=1}^{n-2} b_i \cdot e_{i+2}$ must be linearly dependent. But this implies that $\alpha_\phi(p)$ has not maximal rank.

ii) We observe that $p \in M_1$ if and only if rank $\alpha_\phi(p) = 1$ which is equivalent to say that each two of vectors H, B and C are linearly dependent, but this means that the curvature ellipse is either a radial segment ($B \neq 0$ or $C \neq 0$) or reduces a point ($B = 0$ and $C = 0$). Since $\operatorname{rank} \alpha_\phi(p) = 1$, it follows in this case that $H \neq 0$ and thus this point is a (non flat) umbilic.

iii) Notice that the curvature ellipse at p reduces to origin p of N_pM if and only if H, B and C vanish at p, but this means that rank $\alpha_\phi(p) = 0$. $\square$

Proposition 1. *Given a surface* M *in* $\mathbf{R}^n$, $n \geq 4$,

i) $p \in M_3$ *if and only if* H_p *is a plane and* $p \notin H_p$.

ii) $p \in M_2$ *is non semiumbilic if and only if* H_p *is a plane and* $p \in H_p = E_p$.

iii) $p \in M_2$ *is semiumbilic if and only if* H_p *is a line and* $p \notin H_p$.

iv) $p \in M_1$ *is an inflection point if and only if* H_p *is a line and* $p \in H_p$.

v) $p \in M_1$ *is an umbilic if and only if* H_p *is a point different from* p.

vi) $p \in M_0$ *is a flat umbilic if and only if* $H_p = p$.

Proof. First of all we observe that, in the expression of the curvature ellipse $\eta(\theta) = H + B\cos 2\theta + \sin 2\theta$, the mean curvature vector H is the vector from the origin p of N_pM to the center of curvature ellipse.

Suppose now that p is a non semiumbilic point. We then have that the vectors B and C are linearly independent and H_p, the affine span of the curvature ellipse at p, is the plane spanned by them. Moreover, it follows from Lemma 1 that $p \in M_3 \cup M_2$.

i) We have that $p \in M_3$ if and only if $\alpha_\phi(p)$ has maximal rank, which is equivalent to saying that the vectors H, B and C must be linearly independent. But this implies that the vector H does not lie in the plane H_p. Therefore, $p \notin H_p$. Conversely, if H_p is a plane and $p \notin H_p$, then the vectors B and C are linearly independent and H does not belong to the plane H_p. Thus the vectors H, B and C must be linearly independent, this mean that $p \in M_3$.

ii) If p is non semiumbilic and $p \in M_2$, then the vectors H, B and C must be linearly dependent. So, the vector H lies in the plane H_p. Consequently, $p \in H_p$. The converse follows that, if H_p is a plane and $p \in H_p$, then the curvature ellipse is non degenerate and $p \notin M_3$ (by i)). Therefore, $p \in M_2$.

In the case that p is a semiumbilic point we have that the vectors B and C are linearly dependent and H_p is the line spanned by them. Furthermore, it follows from Lemma 1 that in this case $p \in M_1 \cup M_2$.

iii) If $p \in M_2$ and is semiumbilic, it follows from Lemma 1 that $\eta(\theta)$ is a non radial segment. So H does not lie over the line H_p. And thus $p \notin H_p$. The converse follows easily that, if H_p is a line and $p \notin H_p$, then p is a non radial semiumbilic. Thus, it follows from Lemma 1 that $p \in M_2$.

iv) We know that, if $p \in M_1$ is an inflection point, then the curvature ellipse is a radial segment. Moreover, each two of vectors H, B and C are linearly dependent. But this implies that the vector H lies over the line H_p. And thus $p \in H_p$. Conversely, if H_p is a line and $p \in H_p$, then $\eta(\theta)$ is a radial segment. Thus, it follows from Lemma 1 that $p \in M_1$.

We observe that, if the curvature ellipse reduces a point, we then have that $B = 0$ and $C = 0$, and in this case H_p is the end point of the vector H.

v) If $p \in M_1$, we clearly have that $H \neq 0$ and thus $p \notin H_p$. Conversely, if p is a semiumbilic point and $p \notin H_p$, then $H \neq 0$; thus, it follows from Lemma 1 that $p \in M_1$.

vi) Finally, $p \in M_0$ if and only if H, B and C vanish at p which is equivalent to saying that $\eta(\theta)$ coincides with the origin p of N_pM, that is, if and only if $H = 0$, and thus $p \in H_p$. $\square$

The family of height functions on M associated to an immersion of a surface M in n-space, locally given by $\phi : \mathbf{R}^2 \to \mathbf{R}^n$ is defined as

$$\begin{aligned} \lambda(\phi) : M \times S^{n-1} &\longrightarrow \mathbf{R}^5 \\ (p, v) &\longmapsto \phi_v(p) = \langle \phi(p), v \rangle. \end{aligned}$$

Clearly, ϕ_v has a singularity at $p \in M$ if and only if v is normal to M at p. Generically this is a Morse singularity in the sense that, for most normal directions $v \in N_pM$, ϕ_v has a nondegenerate singularity at p. Nevertheless, for $n \geq 5$, we can always find some normal directions inducing degenerate height functions at each point, see [**9**] for the case of surfaces in $\mathbf{R}^5$, the arguments for $n > 5$ are similar. Such directions are called *binormal*.

Lemma 2. *Given a point p in a surface M immersed in $\mathbf{R}^n$, $n \geq 4$, and a nontrivial vector $v \in N_pM$, the quadratic forms $II_v(p)$ and* $\operatorname{Hess}(\phi_v)(p)$ *are equivalent, in the sense that it is possible to find some local coordinate system in which their expressions coincide.*

Proof. Given $p \in M$ take an orthonormal frame at $p = (0, 0)$, $\{e_1, e_2\}$ for TM and $\{e_3, \dots, e_n\}$ for NM as above. We can always suppose that the local embedding of M, $\phi : \mathbf{R}^2 \to \mathbf{R}^n$, is given in the Monge form. Then, for any normal vector $v \in N_pM$, we can write $v = v_1e_3 + \cdots + v_{n-2}e_n$. And thus the height function in the direction v is given by

$$\begin{aligned} \phi_v : \mathbf{R}^2 &\longrightarrow \mathbf{R}^n \\ (x, y) &\longmapsto \phi_v(x, y) = v_1\phi_1(x, y) + \cdots + v_{n-2}\phi_{n-2}(x, y). \end{aligned}$$

We then have

$$\frac{\partial^2 \phi_v}{\partial x^2}(p) = \sum_{i=1}^{n-2} a_i v_i, \quad \frac{\partial^2 \phi_v}{\partial x \partial y}(p) = \sum_{i=1}^{n-2} b_i v_i, \quad \frac{\partial^2 \phi_v}{\partial y^2}(p) = \sum_{i=1}^{n-2} c_i v_i.$$

Therefore

$$\operatorname{Hess}\big(\phi_v(p)\big) = \begin{bmatrix} \sum_{i=1}^{n-2} a_i v_i & \sum_{i=1}^{n-2} b_i v_i \\ \sum_{i=1}^{n-2} b_i v_i & \sum_{i=1}^{n-2} c_i v_i \end{bmatrix}.$$

Now observe that the coefficients of the second fundamental form with respect to the normal direction v at p are given by

$$e_v(p) = \phi_{xx}(p)\cdot v, \quad f_v(p) = \phi_{xy}(p)\cdot v, \quad g_v(p) = \phi_{yy}(p)\cdot v.$$

So $e_v(p) = \sum_{i=1}^{n-2} a_i v_i$, $f_v(p) = \sum_{i=1}^{n-2} b_i v_i$ and $g_v(p) = \sum_{i=1}^{n-2} c_i v_i$, and hence

$$II_v(p) = \begin{bmatrix} \sum_{i=1}^{n-2} a_i v_i & \sum_{i=1}^{n-2} b_i v_i \\ \sum_{i=1}^{n-2} b_i v_i & \sum_{i=1}^{n-2} c_i v_i \end{bmatrix} = \operatorname{Hess}\big(\phi_v(p)\big). \qquad \square$$

Remark 1. Notice that given normal direction $v \in N_pM$, $v \in \ker A_p$, if and only if p is a singularity of corank 2 of ϕ_v.

Lemma 3. *Given a point p in a surface M immersed in $\mathbf{R}^n$, $n \geq 4$, and a nontrivial binormal direction $v \in E_p \subset N_pM$, the point p is a singularity of corank 1 of ϕ_v. On the other hand, if $v \in E_p^{\perp}$ is a binormal direction p is a singularity of corank 2 of ϕ_v.*

Proof. Given $v \in N_pM$, we have that p is a degenerate singularity of corank 2 of f_v if and only if the matrix,

$$\operatorname{Hess}\big(\phi_v(p)\big) = \begin{bmatrix} \sum_{i=1}^{n-2} a_i v_i & \sum_{i=1}^{n-2} b_i v_i \\ \sum_{i=1}^{n-2} b_i v_i & \sum_{i=1}^{n-2} c_i v_i \end{bmatrix}$$

has null entries. Now, since $E_p = \Big\langle \sum_{i=1}^{n-2}(a_i - c_i)\cdot e_{i+2}, \sum_{i=1}^{n-2} b_i \cdot e_{i+2}\Big\rangle$, it follows that v must be orthogonal to E_p. Therefore, if $v \in E_p$ is a nontrivial binormal direction, we have that ϕ_v has a singularity of corank 1 at p. On the other hand, if $v \in E_p^{\perp}$ is a binormal direction, we have that $\sum_{i=1}^{n-2} b_i v_i = 0$, $\sum_{i=1}^{n-2} a_i v_i = \sum_{i=1}^{n-2} c_i v_i$ and $\det\big(\operatorname{Hess}(\phi_v(p))\big) = 0$. But this implies that

$$\operatorname{Hess}\big(\phi_v(p)\big) = \begin{bmatrix} \sum_{i=1}^{n-2} a_i v_i & 0 \\ 0 & \sum_{i=1}^{n-2} a_i v_i \end{bmatrix},$$

and thus $(\sum_{i=1}^{n-2} a_i v_i)^2 = 0$ which is equivalent to saying that $\sum_{i=1}^{n-2} a_i v_i = 0$. We then have that p is a singularity of corank 2 of ϕ_v. $\square$

3. Critical points of principal configurations. Given any normal field ν on M, we can consider its associated shape operator defined by

$$\begin{aligned} S_\nu : T_pM &\longrightarrow T_pM \\ X &\longmapsto S_\nu(X) = -\big(\overline{\nabla}_{\overline{X}}\bar{\nu}\big)^\top, \end{aligned}$$

where $\bar{\nu}$ is a local extension of ν over a neighborhood of p in $\mathbf{R}^n$ and $\top$ denotes the tangent component of the normal connection $\overline{\nabla}$.

This is a self-adjoint operator and is related as follows with the symmetric bilinear map H_ν,

$$\big\langle S_\nu(X), Y\big\rangle = H_\nu(X,Y), \quad \forall\, X, Y \in T_pM.$$

The second fundamental form in the direction ν can thus be given as

$$II_\nu(X) = \big\langle S_\nu(X), X\big\rangle.$$

Consequently, for each $p \in M$, there is an orthonormal basis in T_pM made of eigenvectors of S_ν (*ν-principal directions*) at which the second fundamental form reaches its maximum and minimum values. The corresponding eigenvalues, k_1 and k_2 shall be called *maximal* and *minimal ν-principal curvature*, respectively. The point p shall be said to be *ν-umbilic* if both ν- principal curvatures coincide at it.

Denote by $\mathcal{U}_\nu$ the subset of all the ν-umbilic points of M. The ν-principal directions define two, mutually orthogonal tangent fields all over the region $M - \mathcal{U}_\nu$, whose critical points are the ν-umbilics. The corresponding integral curves shall be called ν-curvature lines. The two foliations, together with their critical points form the *ν-principal configuration* of M.

The differential equation of the ν-curvature lines is given by

$$S_\nu\big(X(p)\big) = \lambda(p)X(p). \tag{1}$$

Suppose that $U \subset M$ is a local chart with coordinates (x, y), and let E, F, G denote the corresponding coefficients of the first fundamental form. The coefficients of the second fundamental form in the direction ν are then given by

$$\begin{aligned} e_\nu &= II_\nu(\partial x) = \langle \alpha(\partial x, \partial x), \nu \rangle, \\ f_\nu &= \langle \alpha(\partial x, \partial y), \nu \rangle = \langle \alpha(\partial y, \partial x), \nu \rangle \\ g_\nu &= II_\nu(\partial y) = \langle \alpha(\partial y, \partial y), \nu \rangle, \end{aligned}$$

where we denote $\partial x = (\partial / \partial x)$ and $\partial y = (\partial / \partial y)$.

Then equation (1) has the following local expression in these coordinates (see [**14**]),

$$(f_\nu E - e_\nu F)\, dx^2 + (g_\nu E - e_\nu G)\, dx\, dy + (g_\nu F - f_\nu G)\, dy^2 = 0.$$

Moreover, if we take isothermic coordinates, i.e., $E = G > 0$ and $F = 0$, this equation becomes

$$f_\nu\, dx^2 + (g_\nu - e_\nu)\, dx\, dy + f_\nu\, dy^2 = 0.$$

We shall see next that the semiumbilics of surfaces in 4-space can be characterized as critical points of principal configurations.

Theorem 1. *Let $\phi : M \to \mathbf{R}^4$ be an embedding of a surface M in $\mathbf{R}^4$. A point $p \in M$ is ν-umbilic for some* (*locally defined*) *normal field ν on M if and only if p is a semiumbilic point of M.*

Proof. We can assume without loss of generality that ν is a unit normal field all over M and that p is a ν-umbilic. Let $\nu^\perp$ be another unit normal field orthogonal to ν at every point. Now, the area of the curvature ellipse at p can be given in terms of the coefficients of the second fundamental forms in the directions ν and $\nu^\perp$ evaluated at p by the following expression [**6**, p. 266],

$$\text{Area}\left(\eta(\theta)\right) = \frac{\pi}{2} \left| (e_\nu - g_\nu) f_{\nu^\perp} + (e_{\nu^\perp} - g_{\nu^\perp}) f_\nu \right|.$$

We have now that $p \in M$ is semiumbilic if and only if the ellipse at p degenerates into a segment, which occurs if and only if this area is zero.

On the other hand, it follows from the above differential equation that p is ν-umbilic if and only if the coefficients of the second fundamental form in the direction ν (in isothermic coordinates) satisfy the relations $e_\nu = g_\nu, f_\nu = 0$, but this implies that the area of the ellipse vanishes at p and hence p must be semiumbilic.

Conversely, suppose that the ellipse at p is a segment, and taking some direction orthogonal to the segment, we can extend this in order to obtain a locally defined normal field ν whose image at p lies in the subspace $E_p^\perp$. We then have that $\sum_{i=1}^{2}(a_i - c_i)\nu_i = 0$ and $\sum_{i=1}^{2} b_i\nu_i = 0$, which is equivalent to saying that the Hessian matrix of the height function in the direction ν at p is a diagonal matrix with repeated entries in the diagonal. But since, from Lemma 2 we have that this matrix coincides with that of the second fundamental form in the direction ν, which coincides in turn with that of the shape operator S_ν, we conclude that p is a ν-umbilic. $\square$

Now, from the fact that a surface immersed in $\mathbf{R}^4$ is orientable if and only if it admits some globally defined normal field [**1**], we obtain

Corollary 1. *Any orientable closed, i.e., compact without boundary, surface with nonvanishing Euler number immersed in* $\mathbf{R}^4$ *has semiumbilic points.*

We analyze following the case of surfaces immersed in higher dimensional spaces.

Lemma 4. *Given* $M \subset \mathbf{R}^n$, $n \geq 5$, *and a normal field* ν *on* M, *we have that a point* p *is* ν*-umbilic if and only if* $\nu(p) \in E_p^\perp$.

Proof. A point p is ν-umbilic if and only if $e_\nu(p) = g_\nu(p)$ and $f_\nu(p) = 0$, or in other words $\sum_{i=1}^{n-2}(a_i - c_i)\nu_i = 0$ and $\sum_{i=1}^{n-2} b_i\nu_i = 0$. But this amounts to saying that $\nu(p) \perp E_p$. $\square$

In the case of surfaces immersed in codimension higher than or equal to 3, the curvature ellipses induce a rank 2 subbundle EM' of NM over the non semiumbilic region M' of M. A normal field ν that satisfies $\nu(M') \subset EM'$ is said to be *essential*. An *essential principal configuration* for M is the one associated to any essential normal field. A consequence of the above lemma is the following:

Theorem 2. *Given an immersion $\phi : M \to \mathbf{R}^n$, $n \geq 5$, let η be an essential normal field on M. If point of M is η-umbilic, then it is semiumbilic.*

Proof. If p is η-umbilic, then it follows from the Lemma 4 that $\eta(p) \in E_p^{\perp}$. But by hypothesis η is an essential normal field, and $\eta(M') \subset EM'$, where M' is the non semiumbilic region of M. So, if p is a non semiumbilic, then $\eta(p) \in E_p \subset EM'$. Therefore p is semiumbilic. $\square$

Moreover, semiumbilic point of surfaces in $\mathbf{R}^n$, $n \geq 5$, can be characterized as follows:

Theorem 3. *Given an immersion $\phi : M \to \mathbf{R}^n$, $n \geq 5$, a point $p \in M$ is semiumbilic if and only if there exist $(n-3)$ linearly independent normal fields $\nu^1, \ldots, \nu^{n-3}$, locally defined at p, such that p is ν^i-umbilic, for all $i \in \{1, \ldots n-3\}$.*

Proof. Suppose that there are $(n-3)$ linearly independent normal fields $\nu^1, \ldots, \nu^{n-3}$, locally defined at p and such that M is ν^i-umbilic, for all $i \in \{1, \ldots n-3\}$, so in isothermic coordinates we have

$$e_{\nu^i}(p) = g_{\nu^i}(p) \quad \text{and} \quad f_{\nu^i}(p) = 0, \quad \forall\, i = 1, \ldots n-3.$$

Writing the ν^i in terms of a normal frame $\{e_3, \ldots, e_n\}$,

$$\nu^i = \sum_{j=1}^{n-2} \nu_j^i e_{j+2},$$

the above expressions amount to

$$\sum_{j=1}^{n-2} a_j \nu_j^i = \sum_{j=1}^{n-2} c_j \nu_j^i \quad \text{and} \quad \sum_{j=1}^{n-2} b_j \nu_j^i = 0, \quad \forall\, i = 1, \ldots n-3.$$

That is,

$$\sum_{j=1}^{n-2}(a_j - c_j)\nu_j^i = 0 \quad \text{and} \quad \sum_{j=1}^{n-2} b_j\nu_j^i = 0, \quad \forall\, i \in \{1, \dots n-3\},$$

where $a_i = (\partial^2\phi/\partial x^2)(p) \cdot e_{i+2}$, $b_i = (\partial^2\phi/\partial x\partial y)(p) \cdot e_{i+2}$ and $c_i = (\partial^2\phi/\partial y^2)(p) \cdot e_{i+2}$, for $i = 1, \dots, n-2$. Therefore, all the ν^i are perpendicular to the vectors $B = (\sum_{i=1}^{n-2}/2)(a_i - c_i) \cdot e_{i+2}$ and $C = \sum_{i=1}^{n-2} b_i \cdot e_{i+2}$ that generate the subspace E_p. But this implies that $\dim E_p = 1$ and hence p must be semiumbilic. The converse follows easily by taking $(n-3)$ linearly independent normal vectors at p, all of them orthogonal to E_p, extending them locally at p and going backwards through the above considerations. $\square$

Remark 2. Suppose that ξ is a normal field on M. We can decompose it as $\xi = h_1\eta + h_2\eta^\perp$, where $\eta(p) \in E_p$, $\eta^\perp(p) \in E_p^\perp$, for all $p \in M$ and h_1, h_2 smooth functions on M. We can take η and $\eta^\perp$ to be non zero in a small enough neighborhood of a point p. Suppose that p is a non semiumbilic point. It follows from Lemma 4 that $h_1(p) = 0$ if and only if p is ξ-umbilic. But $\eta^\perp$ is an umbilic normal field and thus the ξ-principal lines coincide with the η-principal lines. Since p is non semiumbilic, it is not a critical point of the η-configuration and hence the ξ-principal lines do not present any special geometrical structure at p. So, although p is a ξ-umbilic it is not a critical point for the ξ-configuration.

We also observe that although ξ and η share the same principal configurations their principal curvatures differ. In fact, it is not difficult to see that $S_\xi(v) = h_1(p)S_\eta(v) + h_2(p)S_\eta^\perp(v)$. Therefore, we have the following relation between the principal curvatures:

$$k_\xi^i = h_1 k_\eta^i + h_1 k_{\eta^\perp}, \quad i = 1, 2,$$

where k_η^i are the η-principal curvatures and $k_{\eta^\perp}$ is the $\eta^\perp$-principal curvature, because $k_{\eta^\perp} = k_{\eta^\perp}^1 = k_{\eta^\perp}^2$. $\square$

A particular case of normal fields is given by the *binormal fields*, that is, those defining binormal directions at every point. Their corresponding principal configurations have a special geometrical meaning and lead to the asymptotic lines (corresponding to vanishing principal curvatures [**12**]).

Proposition 2. *If p is a ξ-umbilic and ξ is a binormal field, then $p \in M_2 \cup M_1 \cup M_0$.*

Proof. If p is a ξ-umbilic, Lemma 4 tells us that $p \in E_p^{\perp}$. It follows from Lemma 3, since $\xi(p)$ is a binormal direction at p, that the height function ϕ_ξ has a singularity of corank 2 at p. But, as we have observed in Remark 1, this implies that $\nu(p) \in \ker A_p$. Therefore, $p \in M_2 \cup M_1 \cup M_0$. $\square$

Remark 3. We can also distinguish among all the binormal fields the essential ones, whose critical points are semiumbilic. In the case of surfaces immersed in 4-space, all the binormal fields are essential and their critical points are the inflection points [**7**].

4. Normal fields on 2-regular surfaces. Following Feldman [**4**], we say that a point p of M is 2-*singular* provided the linear map $T_p^2\phi : T_p^2 M \to T_{\phi(p)}^2 \mathbf{R}^5$ is not injective. By choosing local coordinates $\{x, y\}$ at p in M, we have that the linear subspace $T_p^2 M$ is generated by the vectors

$$\left\{ \frac{\partial}{\partial x}\Big|_p, \frac{\partial \phi}{\partial y}\Big|_p, \frac{\partial^2 \phi}{\partial x^2}\Big|_p, \frac{\partial^2 \phi}{\partial x \partial y}\Big|_p, \frac{\partial^2 \phi}{\partial y^2}\Big|_p \right\}.$$

Thus the definition of 2-singular point amounts to saying that the vectors

$$\left\{ \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial x \partial y}, \frac{\partial^2 \phi}{\partial y^2} \right\}$$

are linearly dependent at p. An embedding $\phi : M \to \mathbf{R}^5$ is said to be *regular of order* 2 if there are no 2-singular points in M. It can be seen [**9**] that, for a surface in $\mathbf{R}^n$, $n \geq 5$, *a point $p \in M$ is* 2-*singular if and only if $p \in M_2 \cup M_1 \cup M_0$.*

Therefore, a 2-regular surface is strictly made of points of type M_3. Consequently, 2-regular surfaces do not have semiumbilic.

Theorem 4. *A* 2*-regular immersion of a compact surface with nonvanishing Euler number in* $\mathbf{R}^n$*,* $n \geq 5$*, does not admit globally defined essential normal fields.*

Proof. A globally defined essential normal field gives rise to globally defined principal configurations. Since $\chi(M) \neq 0$, the Poincaré-Hopf formula implies that these configurations must have critical points, but we have seen that the critical points of such configurations are semiumbilic points of M. ◻

On the other hand,

Theorem 5. 2*-regular immersions of compact surfaces with nonvanishing Euler number in* $\mathbf{R}^n$*,* $n \geq 5$*, do not admit globally defined binormal fields.*

Proof. A globally defined binormal field ν induces a globally defined ν-configuration on M (family of asymptotic lines). But Proposition 2 implies that the corresponding critical points are 2-singular points of M and thus, in virtue of the Poincaré-Hopf formula, the condition that $\chi(M) \neq 0$ would imply that M is not 2-regular. ◻

Remark 4. It can be shown that surfaces generically immersed in $\mathbf{R}^n$, $n \geq 6$, do not have semiumbilic points. Therefore, the property of having globally defined essential fields is non generic for the compact surfaces immersed in $\mathbf{R}^n$, $n \geq 6$. On the other hand, surfaces generically immersed in $\mathbf{R}^n$, $n \geq 7$, are 2-regular and, similarly, we can deduce that having a globally defined binormal field is a non generic property for compact surfaces in $\mathbf{R}^n$, $n \geq 7$.

As a consequence of Theorem 3, we have

Corollary 2. *A* 2*-regular compact surface* $M \subset \mathbf{R}^n$*,* $n \geq 5$ *with nonvanishing Euler number can not admit globally defined normal distributions of dimension* $(n-3)$*.*

Proof. They may admit up to $(n-4)$ globally defined linearly independent normal fields all contained in $E^{\perp}$, but any further one would have nowhere vanishing projection on E and this would imply the existence of semiumbilics and thus the non regularity of M. $\square$

To illustrate the above results we consider the following example: Let V be a hyperspherical embedding of the projective plane in $\mathbf{R}^5$, known as the Veronese surface. This embedding is locally given by

$$V(x,y) = \left(\frac{y\sqrt{4-x^2-y^2}}{2}, \frac{x\sqrt{4-x^2-y^2}}{2}, \frac{xy}{2}, \frac{x^2-y^2}{4}, \frac{3x^2+3y^2-8}{4\sqrt{3}}\right),$$

and it can be seen that the image of V is completely homogeneous and is contained in a 4-sphere of radius $2/\sqrt{3}$. Moreover, the curvature ellipse is a circle at each point of V. Therefore it defines a 2-regular embedding of the projective plane in $\mathbf{R}^5$. Since $\chi(V\big(P^1(2)\big) \neq 0$, we can conclude

Corollary 3. *The only globally defined normal field on V is the radial field of the hypersphere $S^4(2/\sqrt{3})$ restricted to V.*

Moreover, if we consider the inverse of the stereographic projection $\psi : S^4(2/\sqrt{3}) \to \mathbf{R}^4$, we get an embedding of the projective plane in $\mathbf{R}^4$. Since stereographic projection takes semiumbilic into semiumbilic, we deduce that $\psi(V)$ is a surface with nonvanishing Euler characteristic embedded with no semiumbilics in 4-space. Therefore,

Corollary 4. *The surface $\psi(V)$ does not admit any globally defined normal field.*

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REFERENCES

1. T. Banchoff and F. Farris, *Tangential and normal Euler numbers, complex points and singularities of projections for oriented surfaces in four-space*, Pacific J. Math. **161** (1993), 1–24.

2. B.Y. Chen and K. Yano, *Integral formulas for submanifolds and their applications*, J. Differential Geom. **5** (1971), 467–477.

3. S.I.R. Costa, *Aplicações não singulares de ordem p*, Doctoral Thesis, University of Campinas, 1982.

4. E.A. Feldman, *Geometry of immersions* I, Trans. Amer. Math. Soc. **120** (1965), 185–224.

5. R.A. Garcia, D.K.H. Mochida, M.C. Romero-Fuster and M.A.S. Ruas, *Inflection points and topology of surfaces in* 4*-space*, Trans. Amer. Math. Soc. **352** (2000), 3029–3043.

6. J. Little, *On singularities of submanifolds of higher dimensional Euclidean space*, Ann. Mat. Pura Appl. **83** (1969), 261–336.

7. D.K.H. Mochida, M.C. Romero-Fuster and M.A.S. Ruas, *The geometry of surfaces in* 4*-space from a contact viewpoint*, Geom. Dedicata **54** (1995), 323–332.

8. ———, *Osculating hyperplanes and asymptotic directions of codimension 2 submanifolds of Euclidean spaces*, Geom. Dedicata **77** (1999), 305–315.

9. ———, *Inflection points and nonsingular embeddings of surfaces in* $\mathbf{R}^5$, Rocky Mountain J. Math. **33** (2003), 995–1009.

10. J.A. Montaldi, *Contact with applications to submanifolds of* $\mathbf{R}^n$, Ph.D. Thesis, University of Liverpool, 1983.

11. C.L.E. Moore and E.B. Wilson, *Differential geometry of two-dimensional surfaces in hyperspaces*, Proc. Amer. Acad. Arts Sci. **52** (1916), 267–368.

12. S.M. Moraes and M.C. Romero-Fuster, *Convexity and semiumbilicity of surfaces in* $\mathbf{R}^5$, in *Differential geometry* (Valencia, 2001), World Scientific Publ., River Edge, NJ, 2002, pp. 222–234.

13. S.M. Moraes, M.C. Romero-Fuster and F. Sánchez-Bringas, *Curvature lines and umbilicity of submanifolds in* $\mathbf{R}^N$, Bull. Belgian Math. Soc. Simon Stevin **11** (2004), 227–245.

14. A. Ramírez-Galarza and F. Sánchez-Bringas, *Lines of curvature near umbilical points on surfaces immersed in* $\mathbf{R}^4$, Ann. Global Anal. Geom. **13** (1995), 129–140.

15. L. Rodriguez and R. Tribuzy, *Reduction of codimension of regular immersion*, Math. Z. **185** (1984), 321–331.

16. M.C. Romero-Fuster and F. Sánchez-Bringas, *Umbilicity of surfaces with orthogonal asymptotic lines in* $\mathbf{R}^4$, Differential Geom. Appl. **16** (2002), 213–224.

Simone Maria de Moraes, Department of Mathematics, Federal University of Viçosa, 36571-000 Viçosa - MG, Brazil
E-mail address: smoraes2000@hotmail.com

María del Carmen Romero Fuster, Department of Geometry and Topology, Universitat de Valencia, 46100 Burjassot (Valencia), Spain
E-mail address: carmen.romero@uv.es