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# TOPOLOGICAL DESCRIPTION OF A NON-DIFFERENTIABLE BIOECONOMICS MODEL

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ABSTRACT. A predator-prey model with non-differentiable functional response and with a Cobb-Douglas type production function is considered. We show that the non-differentiability has a strong influence on the dynamics of the model, locally and globally. We prove that there is not a uniqueness of solutions for any initial conditions on the coordinate axis. We conclude that for any conditions of the parameters, the dynamics of the model does not contain a globally attracting singularity. Finally, in the parameters space, we prove the existence of an open set such that, for all values in this set, the model has at least two small amplitude limit cycles generated by Hopf bifurcations.

**1. Introduction.** Let us consider the family of vector fields $X_\mu^{\alpha,\beta}$ where

$$X_\mu^{\alpha,\beta}:\begin{cases} dx/dt = rx(1-(x/K)) - qx^\alpha y^\beta \\ dy/dt = b(pqx^\alpha y^\beta - cy)\end{cases} \tag{1}$$

This system describes the dynamics of an open access fishery model, where for each time $t > 0$, $x = x(t)$ is the size of the fishing resource and $y = y(t)$ the effort realized by the predator (man, industrial fisheries, etc.)

System (1) is defined on the region $\bar{\Omega} = \{(x,y) \in \mathbf{R}^2 \mid x \geq 0\ ,\ y \geq 0\}$, where $\mu = (r,K,p,q,b,c) \in \mathbf{R}_+^6$ and $0 < \alpha, \beta < 1$ denotes the bioeconomics parameters which have the following meanings:

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$r$ is the *intrinsic growth rates* or *biotic potential of the prey.*

$q$ is the *maximal harvest rate*, or *capture coefficient.*

$K$ is the *environment carrying capacity of prey.*

$p$ is the *price of landed biomass.*

$c$ is the *cost of fishing effort.*

$b$ is the *stiffness parameter.*

In this model we assume that the natural growth of a resource is given by a logistic equation, see [**11**], and the harvest rate or production function is given by $H(x,y) = qx^{\alpha}y^{\beta}$, known in economics as a Cobb-Douglas type function [**18**]. This function is more realistic than the function $H(x,y) = qxy$, resulting in the *capture by effort unity hypothesis* or Schaefer's hypothesis [**6, 15**]. Moreover, we consider that the fishing effort $y = y(t)$ is proportional to the gross profit [**16**].

The values $\alpha$ and $\beta$ reflect important characteristics of the fishery [**10**] and represent shape parameters to account for effects of non-random searching, competition among units of effort, saturation of the harvesting gear, and so forth [**18**].

In the function $H(x,y)$, the term $\rho(x) = qx^{\alpha}$, called *concentration profile*, expresses the relationship between the exploited density $\rho$ and population size $x$ and corresponds to type III described in [**6**], when a fish species that tends to aggregate as its total abundance decreases as the case of the Anchoveta (*Engraulis Ringens*) [**6**]. The parameter $\alpha < 1$ indicates the grade of aggregation of the fishing resource; this means that each unit of effort catches a greater proportion of the stock as the stock diminishes [**10**]. On the other hand, the parameter $\beta < 1$ weighs the congestion level of the ships dedicated to the fishery; this means that each unit effort catches less and less from a given stock the more effort is applied, which could be due to crowding effects [**10**]. The Cobb-Douglas functions have been used in discrete models [**1, 4, 12**], without analysis in the detail in the model. The first of these papers [**1**], numerical estimates of bionomic equilibrium, shows that open access is seen to cause stock extinction except for very high cost/price ratios, in the case of the northeast Atlantic Minke whale. In [**4**], the authors indicate that it is not possible to solve for explicit expressions for a positive equilibrium point and they solve numerically with dates of certain Norwegian fisheries (North Sea herring and Norwegian purse seine) and show by simulation that this point is local asymptotically stable. Finally, in the last paper, for the Northern Anchovy fishery, it is shown by simulation that the behavior of the system depends also on the cost/price ratio $c/p$ and of the stiffness or adjustment parameter $b$; it is illustrated that the open access system will converge to a unique equilibrium point, or the systems falls into a limit cycle around the open access equilibrium and the bifurcation value can be approximated with a desired level of accuracy using numerical methods.

When $\alpha = 1$ and $\beta = 1$, the system is known as Smith's model and the unique equilibrium point in the first quadrant is globally asymptotically stable [**6**]. This model is also suggested to describe the predator-prey interaction and according to [**3**] it can be classified as $F$, that is, prey-dependent functional response model. The case $\beta = 1$ is proposed by Rosenzweig [**13**]. In [**8**] Freedman proposes to study the situation $\alpha = 1$ to analyze the effect of group defense in the predator-prey system and he named $\alpha$ the *mutual interference constant*.

Let $\Omega = \bar{\Omega} - \{(x, y) \mid xy = 0\}$ be the maximal open set in $\bar{\Omega}$. It is clear that $X_\mu^{\alpha,\beta}$ is a $C^\infty$-vector field in $\Omega$ and nondifferentiable vector field in $\bar{\Omega}$, that is,

$$X_\mu^{\alpha,\beta} \in \mathcal{X}^0(\bar{\Omega}) - \mathcal{X}^1(\bar{\Omega}) \quad \text{and} \quad X_\mu^{\alpha,\beta} \in \mathcal{X}^\infty(\Omega).$$

In order to describe the dynamics of $X_\mu^{\alpha,\beta}$ we will consider a vector field qualitatively equivalent by the following change of coordinates:

$$\zeta : \mathbf{R}^2 \times \mathbf{R}_0^+ \longrightarrow \mathbf{R}^2 \times \mathbf{R}_0^+ \tag{2}$$

such that

$$\zeta(u, v, \tau) = \left( Ku, \left( \frac{rK^{1-\alpha}}{q} \right)^{1/\beta} v, \ \frac{\tau}{r} \right) = (x, y, t)$$

where

$$\det D\zeta(u, v, \tau) = \left( \frac{K^{1-\alpha+\beta} r^{1-\beta}}{q} \right)^{1/\beta} > 0.$$

Let us consider $B$ and $C$ new parameters such that

$$B = pb \left( \frac{q}{rK^{1-\alpha-\beta}} \right)^{1/\beta} > 0 \qquad C = \frac{bc}{rB} > 0. \tag{3}$$

Then in the new coordinates the vector field is given by $Y_\eta^{\alpha,\beta} = \zeta_* X_\mu^{\alpha,\beta}$, where

$$Y_\eta^{\alpha,\beta} : \begin{cases} du/d\tau = u(1-u) - u^\alpha v^\beta \\ dv/d\tau = B(u^\alpha v^\beta - Cv), \end{cases} \tag{4}$$

and it is defined in the region $\bar{\Gamma} = \zeta^{-1}(\bar{\Omega} \times \{0\})$ with $\eta = (B, C) \in \mathbf{R}_+^2$ and $0 < \alpha,\ \beta < 1$.

If $\Gamma = \zeta^{-1}(\Omega \times \{0\})$, then it is clear that

$$Y_\eta^{\alpha,\beta} \in \mathcal{X}^0(\bar{\Gamma}) - \mathcal{X}^1(\bar{\Gamma}) \quad \text{and} \quad Y_\eta^{\alpha,\beta} \in \mathcal{X}^\infty(\Gamma).$$

Note that both coordinate-axes $u = 0$ and $v = 0$ are invariant sets for (4), the equilibrium points are on the curve $v = (1/C)u(1-u)$ and the only singularity in the positive semi-axis $v = 0$ is $(1, 0)$. In the parameters, in order to obtain a simpler description of the bifurcation diagram of (4) we define the sets

$$\begin{aligned} \Delta &= \{(\alpha, \beta, B, C) \in \mathbf{R}_+^4 \mid \alpha, \beta < 1\} \\ \Upsilon^{sg(\varepsilon)} &= \{(\alpha, \beta, B, C) \in \Delta \mid \alpha + \beta - 1 = sg(\varepsilon)\} \end{aligned}$$

and

$$\begin{aligned} \Lambda^{sg(\varepsilon)} = \{(\alpha, \beta, B, C) \in \Delta \mid (1-\beta)^{1-\beta}(1-\alpha-\beta)^{1-\alpha-\beta} C^\beta \\ - (2-\alpha-2\beta)^{2-\alpha-2\beta} = sg(\varepsilon),\ 1-\alpha-\beta > 0\} \end{aligned}$$

**Definition.** We say that a saddle point $p$ of $Y_\eta^{\alpha,\beta}$ is non Lipschitzian if and only if there does not exist uniqueness of solutions in at least one point of $W^s(p)$ and/or $W^u(p)$. Furthermore, we say that an elliptical sector of $Y_\eta^{\alpha,\beta}$ is non Lipschitzian if and only if there does not exist uniqueness of solutions in at least one point of its separatrices.

## 2. Main results.

**Theorem 1.** *For the vector field $Y_\eta^{\alpha,\beta}$ we have:*

i) *There is not uniqueness of solutions at points in the positive u-axis, different from the singularity* $(1, 0)$. (*The vector field* (4) *is not Lipschitzian.*)

ii) *The singularity* $(1,0)$ *is a non Lipschitzian saddle point.*

iii) *There is not uniqueness of solutions at points in the positive v-axis.* (*The vector field* (4) *is not Lipschitzian.*)

**Lemma 1.** *If* $\xi = (\alpha, \beta, B, C) \in \Delta$, *the existence of singularities of the vector field* (4) *in* $\Gamma$ *is given by*:

1. *Either if* $\xi \in \Lambda^-$ *or if* $\xi \in \Lambda^+ \cup \Lambda^0 \cup \Upsilon^0$ *with* $C \leq 1$, *there are not singularities.*

2. *If* $\xi \in \Lambda^0$ *and* $C > 1$, *there exists only one singularity which is a saddle-node point.*

3. *If* $\xi \in \Lambda^+$ *and* $C > 1$, *only two singularities exist, a saddle point and a hyperbolic focus or a node.*

4. *Either if* $\xi \in \Upsilon^0$ *with* $C > 1$ *or if* $\xi \in \Upsilon^+$, *there exists only one singularity which is a center focus.*

**Lemma 2.** 1. *If* $\xi \in \Lambda^+$ *and* $C > 1$, *in part* 3 *of Lemma* 1, *the abscissa of the saddle point is the smallest to the abscissas of the singularities of* (4). *The focus is an attracting hyperbolic singularity.*

2. *If* $\xi \in \Upsilon^0$ *with* $C > 1$, *the center focus singularity in part* 4 *of Lemma* 1 *is an attracting hyperbolic focus.*

3. *If* $\xi \in \Upsilon^+$, *there exists* $0 < C^* < 1$ *such that, if* $C > C^*$ *the center focus singularity in part 4 of Lemma 1 is a hyperbolic attracting focus. Moreover, for each* $0 < C < C^*$ *and for each* $B > 0$, *arbitrary but fixed, then the following functions exist*

$$\begin{cases} F : [0,1] \times [0,1] \to \mathbf{R} \quad \textit{and} \\ \ell : [0,1] \times [0,1] \to \mathbf{R} \end{cases}$$

*such that*

$$\begin{cases} F(\alpha,\beta) = [1-\alpha - BC(1-\beta)]^{\alpha+\beta-1} \\ \quad -C^{\beta}(2-\alpha)^{\alpha+2\beta-2}[1+BC(1-\beta)]^{1-\beta} \\ \ell(\alpha,\beta) = \alpha + 2\beta - 2. \end{cases}$$

*In the parameter space,* $H = F^{-1}(0)$ *is a Hopf bifurcation curve.*

*Moreover, if* $F^{-1}(0) \cap l^{-1}(0) = \{(\bar{\alpha}, \bar{\beta})\}$ *in the parameter space* $\alpha\beta$, *there exists a neighborhood* $\mho$ *of* $(\bar{\alpha}, \bar{\beta})$ *such that the center focus singularity in part* 4 *of Lemma* 1 *is*:

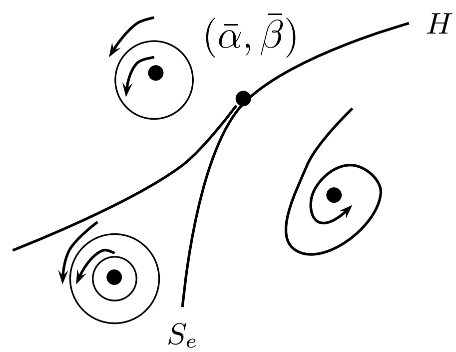


FIGURE 1.

i) *An attracting weak focus of order two if* $(\alpha,\beta) = (\bar{\alpha},\bar{\beta})$.

ii) *An attracting weak focus of order one if* $(\alpha,\beta) \in F^{-1}(0) \cap \ell^{-1}(0,\infty) \cap \mho$.

iii) *A repelling weak focus of order one if* $(\alpha,\beta) \in F^{-1}(0) \cap \ell^{-1}(-\infty,0) \cap \mho$.

iv) *A hyperbolic attracting focus if* $(\alpha,\beta) \in F^{-1}(-\infty,0) \cap \mho$.

v) *A hyperbolic repelling focus if* $(\alpha,\beta) \in F^{-1}(0,\infty) \cap \mho$.

**Theorem 2.** *If* $\xi \in \Upsilon^+$, *there exists* $0 < C^* < 1$ *such that, for each* $0 < C < C^*$, *in the parameter space* $\alpha\beta$, *there exists a neighborhood* $\mho_c$ *of* $(\bar{\alpha},\bar{\beta})$ *such that the diagram of bifurcations at the center focus singularity of* (4) *in part* 4 *of Lemma* 1 *is shown in Figure* 1, *where* $H$ *and* $S_e$ *denote the Hopf and the semi-stable limit cycles bifurcation curve.*

**Theorem 3.** *The vector field* $Y_\eta^{\alpha,\beta}$ *at the origin has*:

i) *A non Lipschitzian elliptical sector in the following cases*:

$*$ *If* $\xi \in \Lambda^-$,

$*$ *If* $\xi \in \Lambda^+ \cup \Lambda^0 \cup \Upsilon^0$ *and* $C \leq 1$

$*$ *If* $\xi \in \Lambda^+ \cup \Lambda^0$ *and* $C > 1$.

ii) *An hyperbolic sector in the following cases*:

$*$ *If* $\xi \in \Upsilon^0$ *and* $C > 1$

$*$ *If* $\xi \in \Upsilon^+$.

## 3. Proofs of the main results.

*Proof of Theorem* 1. In order to study the vector field $Y_\eta^{\alpha,\beta}$ we consider the $t$-scaling and the change of coordinates given by

$$\varphi : \mathbf{R}_+^2 \times \mathbf{R}_0^+ \longrightarrow \Gamma \times \mathbf{R}_0^+ \tag{5}$$

such that

$$\varphi(x,y,t) = \Big(x+1, y^{2/(1-\beta)}, ty\Big) = (u,v,\tau)$$

where

$$\det D\varphi(x,y,t) = \frac{2}{1-\beta}\, y^{2/(1-\beta)} > 0.$$

The obtained vector field is given by $\bar{Y}_\eta^{\alpha,\beta} = \varphi_* Y_\eta^{\alpha,\beta}$, where in the region $\varphi^{-1}(\Gamma \times \{0\})$ has the following form

$$\bar{Y}_\eta^{\alpha,\beta} : \begin{cases} dx/dt = -x(x+1)y - (x+1)^\alpha y^{(1+\beta)/(1-\beta)} \\ dy/dt = B(1-\beta/2)((x+1)^\alpha - Cy^{(2/1-\beta)}). \end{cases} \tag{6}$$

*Proof of* i). We observe that (6) can be continuously extended to the semi-axis $x \geq -1$ and $\bar{Y}_\eta^{\alpha,\beta}(x,0) = B(1-\beta/2)(x+1)^\alpha(\partial/\partial y)$. Then, for $x > -1$ the vector field (6) is orthogonal to the $x$-axis, see Figure 2. If $-1 < x_0$ and $\gamma$ is the orbit of vector field (6) with initial condition in the point $(x_0, 0)$, then $\gamma^* = \gamma - \{(x_0,0)\}$ is also an orbit to the vector field (6). As $\varphi$ is a homeomorphism, systems (4) and (6) are $C^0$-*equivalent* in the first quadrant $\mathbf{R}_+ \times \mathbf{R}_+$, hence $\varphi(\gamma^*)$ is an orbit of (4). By continuity $\varphi(\gamma)$ is an orbit of (4) that is tangent to the vector field $Y_\eta^{\alpha,\beta}$ at the point $(x_0+1, 0)$. But the $u$-axis, $v = 0$, is clearly an invariant set and $Y_\eta^{\alpha,\beta}(x_0+1,0) = (x_0+1)x_0(\partial/\partial u)$. Thus, for the point $(x_0+1, 0)$, at least two orbits exist. Therefore system (4) is not Lipschitzian, since no uniqueness of solutions exists.

In order to prove ii), if $x_0 = 0$ with the above argument, we have $\gamma^* = \gamma - \{(0,0)\}$ and $\varphi(\gamma^*)$ is an orbit of (4) whose $\alpha$-limit is $(1,0)$. On

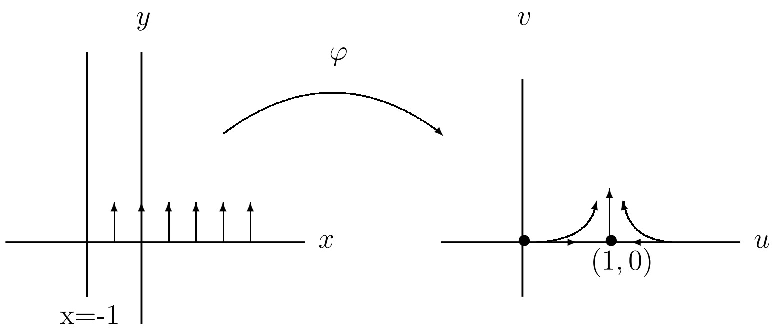


FIGURE 2.

the other hand, $Y_\eta^{\alpha,\beta}(u,0) = u(1-u)(\partial/\partial u)$. Then, in a neighborhood of the point $(1,0)$, the $u$-axis is contained in the stable manifold of $(1,0)$ and is it formed by points that have tangent regular orbits to the straight line $v = 0$ whose $\omega$-limit is contained in $\Gamma$. Then the singularity $(1,0)$ is a non Lipschitzian saddle point, see Figure 2.

To prove iii), we must consider the following change of coordinates in (4) that includes a scaling of the $u$-axis and the $v$-axis and a scaling of the time:

$$\varphi : \mathbf{R}_+^2 \times \mathbf{R}_0^+ \longrightarrow \Gamma \times \mathbf{R}_0^+ \tag{7}$$

such that

$$\varphi(x,y,t) = (x^{2/(1-\alpha)}, y^{2/(1-\beta)}, xyt) = (u,v,\tau)$$

where

$$\det D\varphi(x,y,t) = \frac{4}{(1-\alpha)(1-\beta)}\, x^{2/(1-\alpha)} y^{2/(1-\beta)} > 0.$$

Then, in the new coordinates the obtained vector field defined in $\varphi^{-1}(\Gamma \times \{0\})$ is $\bar{Y}_\eta^{\alpha,\beta} = \varphi_* Y_\eta^{\alpha,\beta}$, where

(8)

$$\bar{Y}_\eta^{\alpha,\beta} : \begin{cases} dx/dt = (1-\alpha/2)[(1-x^{2/1-\alpha})x^{2/1-\alpha}y - y^{(1+\beta)/(1-\beta)}] \\ dy/dt = B(1-\beta)/2[x^{(1+\alpha)/(1-\alpha)} - Cxy^{2/(1-\beta)}]. \end{cases}$$

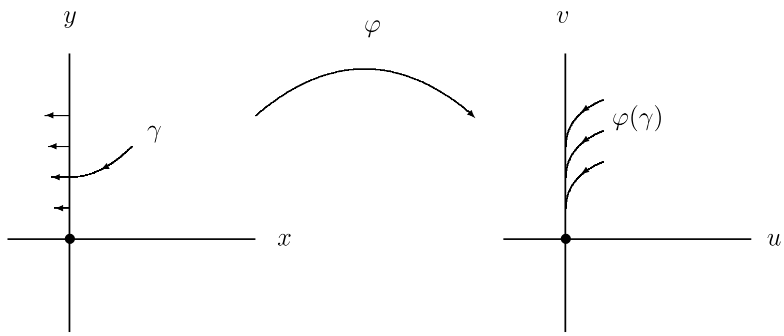


FIGURE 3.

Since $\bar{Y}_\eta^{\alpha,\beta}(0,y) = -(1-\alpha/2)y^{(1+\beta)/(1-\beta)}(\partial/\partial x)$, then for $y>0$ the vector field (8) is orthogonal to the $y$-axis. For $y_0>0$, let $(0,y_0)$ be an initial condition and let $\gamma$ be the orbit of vector field $\bar{Y}_\eta^{\alpha,\beta}$ through this point. If $\gamma^* = \gamma - \{(0,y_0)\}$, then $\varphi(\gamma^*)$ is an orbit of (4). By continuity, $\varphi(\gamma)$ is an orbit of (4) tangent to the vector field $Y_\eta^{\alpha,\beta}$ at the point $(0, y_0^{2/(1-\beta)})$. On the other hand, since $Y_\eta^{\alpha,\beta}(0,v) = -BCv(\partial/\partial v)$, the $v$-axis is an invariant set by the vector field (4). This proves that, for the point $(0, y_0^{2/(1-\beta)})$, at least two orbits of the vector field $Y_\eta^{\alpha,\beta}$ exist and this proves the theorem, see Figure 3. $\square$

*Proof of Lemma* 1. Let $P(u,v) = u(1-u) - u^\alpha v^\beta$ and $Q(u,v) = B(u^\alpha v^\beta - Cv)$ be the component of (4), and let

$$\begin{cases} P_1(u,v) = u(1-u) - Cv \\ Q_1(u,v) = u^\alpha v^\beta - Cv \end{cases} \quad \text{where} \quad Q = BQ_1,\ P_1 = Q_1 + P.$$

The set of the singularities of (4) in $\bar{\Gamma}$ is

$$\text{(9)} \qquad \operatorname{Sing}(Y_\eta^{\alpha,\beta}) = P^{-1}(0) \cap Q^{-1}(0) = P_1^{-1}(0) \cap Q_1^{-1}(0).$$

In $\Gamma$ the zeros of $Q_1$ define the implicit function

$$v = f(u) = \frac{1}{C^{1/(1-\beta)}}\, u^{\alpha/(1-\beta)}.$$

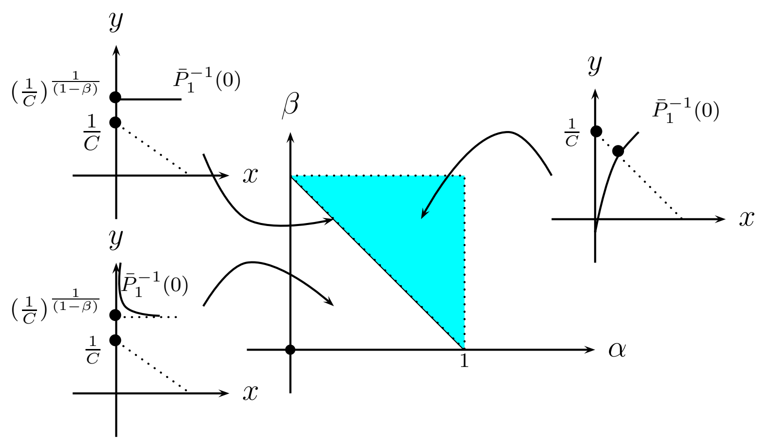


FIGURE 4. ($C < 1$).

In order to simplify the study of (9), we consider the horizontal blowing-up

$$\Psi : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \quad \text{such that} \quad \Psi(x, y) = (x, xy) = (u, v). \tag{10}$$

Then

$$\begin{cases} (P_1 \circ \Psi)(x, y) = -Cx[y - (1/C)^{1/(1-\beta)} x^{(\alpha+\beta-1)/(1-\beta)}] \\ (Q_1 \circ \Psi)(x, y) = -Cx[y + (1/C)x - (1/C)]. \end{cases}$$

Now, if we consider the functions

$$\begin{cases} \bar{P}_1(x, y) = y - (1/C)^{1/(1-\beta)} x^{(\alpha+\beta-1)/(1-\beta)} \\ \bar{Q}_1(x, y) = y + (1/C)x - (1/C). \end{cases} \tag{11}$$

In $\Psi^{-1}(\Gamma)$, (9) is reduced to the set

$$\Psi^{-1}(\mathrm{Sing}\,(Y_\eta^{\alpha,\beta})) = (\bar{P}_1)^{-1}(0) \cap (\overline{Q_1})^{-1}(0)$$

So, if $\xi \in \Upsilon^-$ and $C \leq 1$, from the graphics of (11) is it easy to see that $\Psi^{-1}(\mathrm{Sing}\,(Y_\eta^{\alpha,\beta})) = \phi$, see Figure 4 on the existence of singularities. This partially proves the first part of Lemma 1.

If $(x_0, y_0) \in \Psi^{-1}(\mathrm{Sing}\,(Y_\eta^{\alpha,\beta}))$ is a tangent point of $(\bar{P}_1)^{-1}(0)$ and $(\bar{Q}_1)^{-1}(0)$, the coordinates of this point satisfy the system

$$\begin{cases} \nabla\vec{\bar{P}}_1(x,y) = \lambda\nabla\vec{\bar{Q}}_1(x,y) \\ \bar{P}_1(x,y) = \bar{Q}_1(x,y) = 0 \end{cases} \tag{12}$$

By the former equation we have

$$-\left(\frac{1}{C}\right)^{1/(1-\beta)} \frac{\alpha+\beta-1}{1-\beta}\, x_0^{(\alpha+2\beta-2)/(1-\beta)}\, \frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \lambda\left[\frac{1}{C}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right],$$

and from the latter component of this equality it follows that $\lambda = 1$. Furthermore, from (12) we obtain

$$\left(\frac{1}{C}\right)^{\beta/(1-\beta)} x^{(\alpha+\beta-1)/(1-\beta)} = \frac{1-\beta}{2-\alpha-2\beta}. \tag{13}$$

Then

$$\begin{aligned} \alpha+\beta<1 \implies \frac{1-\alpha}{2-\alpha-2\beta} < 1 \implies C^{\beta/(1-\beta)}x^{(1-\alpha-\beta)/(1-\beta)} \\ >1 \implies x > \frac{1}{C^{\beta/(1-\alpha-\beta)}}. \end{aligned}$$

Now, since $x < 1$ we have $C > 1$. Therefore, the system (12) has a solution if $\xi \in \Upsilon^-$ and $C > 1$. Moreover, using the equality (13) we have that (12) has a unique solution, see Figure 5 on existence of singularities.

Now, due to $(x_0, y_0) \in \overline{Q_1}^{-1}(0)$, by directed calculus we have that the coordinates of this point are

$$(x_0(\xi), y_0(\xi)) = \left(\frac{\alpha+\beta-1}{\alpha+2\beta-2}, \frac{1-\beta}{C(2-\alpha-2\beta)}\right)$$

where in $\Delta$ the parameters satisfy the condition of tangency

$$(1-\beta)^{1-\beta}(1-\alpha-\beta)^{1-\alpha-\beta}C^{\beta} = (2-\alpha-2\beta)^{2-\alpha-2\beta} \tag{14}$$

obtained from (12). But (14) implies that $\xi \in \Lambda^0$ and $C > 1$. This proves the existence and uniqueness of the singularity of part 2

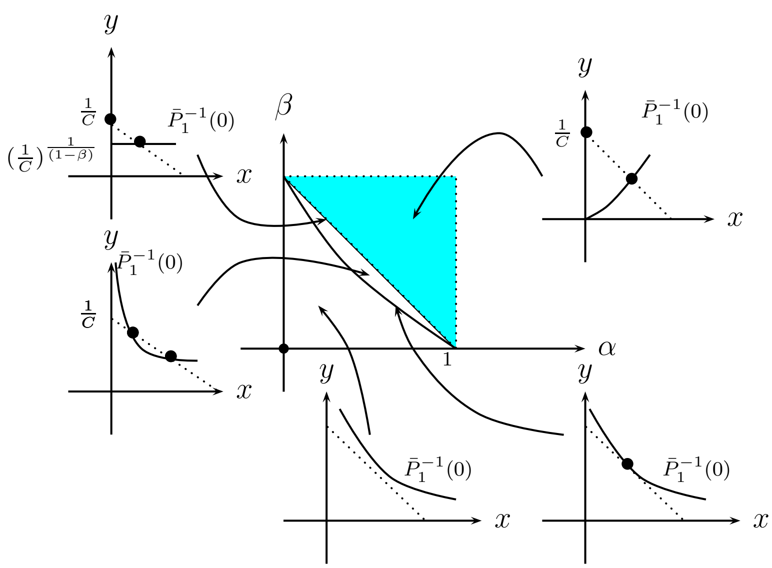


FIGURE 5. $(C > 1)$.

to Lemma 1. This singularity is a saddle-node, in fact, by the blowing-up (10) and the condition of the above tangency, the image $\Psi(x_0(\xi), y_0(\xi)) = p_\xi$ is a singularity with index zero of (4). To finish part 1 of Lemma 1, we consider $\xi^- = (\alpha, \beta^-, C, B) \in \Lambda^-$ and $\xi = (\alpha, \beta, C, B) \in \Lambda^0$. Then

$$\begin{aligned}
\beta^- < \beta \iff & (1-\beta^-)^{1-\beta^-}(1-\alpha-\beta^-)^{1-\alpha-\beta^-}C^{\beta^-} \\
& - (2-\alpha-2\beta^-)^{2-\alpha-2\beta^-} < 0 \\
\iff & \frac{1-\alpha-\beta}{2-\alpha-2\beta} < \frac{1-\alpha-\beta^-}{2-\alpha-2\beta^-} \wedge \frac{1-\beta}{2-\alpha-2\beta} < \frac{1-\beta^-}{2-\alpha-2\beta^-} \\
\iff & x_0(\xi) < x_0(\xi^-) \quad \wedge \quad y_0(\xi) < y_0(\xi^-).
\end{aligned}$$

Hence, since the graphics of $\overline{Q_1}(x,y) = 0$ is independent of the parameter $\beta$ it remains fixed whereas the graphics of $\overline{P_1}(x,y) = 0$ itself moves in the direction of the gradient. This proves that $\mathrm{Sing}\,(Y_\eta^{\alpha,\beta}) = \Phi$. Now, if $\xi \in \Upsilon^0$ and $C \leq 1$, from (11) we have

$\overline{P_1}(x,y) = y - (1/C)^{1/(1-\beta)}$, therefore

$$\overline{P_1}^{-1}(0) \cap \overline{Q_1}^{-1}(0) = \begin{cases} \Phi & \text{if } C < 1 \\ (0,1) & \text{if } C = 1 \end{cases}$$

because this corresponds to the intersection of two straight lines. Moreover, when $C = 1$, by a blowing-down (10), we have $\Psi(0,1) = (0,0)$. This concludes the proof of part 1 of this lemma.

In order to prove part 3 of Lemma 1, we consider $\xi^+ = (\alpha, \beta^+, C, B) \in \Lambda^+$ and $\xi = (\alpha, \beta, C, B) \in \Lambda^0$. As before, it is easy to see that

$$\beta^+ > \beta \iff x_0(\xi^+) > x_0(\xi) \ \wedge \ y_0(\xi^+) > y_0(\xi).$$

Then the graphics of $\overline{P_1}(x,y) = 0$ itself moves in the opposite sense to the direction of gradient. So, this proves that the point of tangency $(x_0(\xi), y_0(\xi))$ unfolds in two points and $\operatorname{Sing}(Y_\eta^{\alpha,\beta}) = \{(x_1,y_1),(x_2,y_2)\}$. By blowing-down (10), we have that $\Psi(x_1,y_1) = p_1$, $\Psi(x_2,y_2) = p_2$ are two different singularities of (4). Now, it is well known from bifurcation theory that one of these singularities is a hyperbolic saddle and the other point is a hyperbolic node. This concludes the proof of part 3 of Lemma 1.

In order to prove part 4 of this lemma, from (4) we have

$$DY_\eta^{\alpha,\beta}(u,v) = \begin{pmatrix} 1 - 2u - \alpha u^{\alpha-1}v^\beta & -\beta u^\alpha v^{\beta-1} \\ B\alpha u^{\alpha-1}v^\beta & B(\beta u^\alpha v^{\beta-1} - C) \end{pmatrix}. \tag{15}$$

If $\xi \in \Upsilon^0$ and $C > 1$, according to (11) it follows that $(\bar{P}_1)^{-1}(0)$ and $(\bar{Q}_1)^{-1}(0)$ are two straight lines whose intersection is the point $p(C) = (1 - (1/C)^{(1-\alpha)/\alpha}, (1/C)^{1/\alpha})$ and then $\Psi(p(C))$ is the only singularity of (4) in $\Gamma$. Now, using (15) we have that

$$\det DY_\eta^{\alpha,1-\alpha}(\Psi(p(C))) = \alpha BC\left(1 - \left(\frac{1}{C}\right)^{(1-\alpha)/\alpha}\right) > 0.$$

This proves that the singularity $\Psi(p(C))$ is a center-focus.

If $\xi \in \Upsilon^+$ from (11) it is clear that $(\bar{P}_1)^{-1}(0)$ in the quadrant $\Psi^{-1}(\Gamma)$ is a strictly increasing curve, that it is extended continuously at the origin. Therefore, the intersection of this curve with the straight line

$(\bar{Q}_1)^{-1}(0)$ is the only point $p^*$. Then $\Psi(p^*) = (u^*, v^*)$ in $\Gamma$ is the only singularity of the vector field (4). Now, by the definition of the functions $P_1$ and $Q_1$, we have that $u^*$ satisfies the condition

$$u^{\alpha+\beta-1}(1-u)^{\beta-1} = C^{\beta}. \tag{16}$$

Replacing this condition in the determinant of (15) and using that $0 < u^* < 1$, we obtain

$$\det DY_\eta^{\alpha,\beta}(u^*, v^*) = BC[(1-\beta)u^* + (1-u^*)(\alpha+\beta-1)] > 0.$$

Then, $\Psi(p^*)$ is a center-focus of (4). This concludes the proof of Lemma 1. $\square$

*Proof of Lemma* 2. 1) In order to study the relative position of the saddle point and the focus point of (4) given by part 3 of Lemma 1, we consider $\xi \in \Lambda^+$ and $C > 1$. Let $p_1 = (u_1, v_1)$ and $p_2 = (u_2, v_2)$ be the respective singularities of the vector field, and let $p_0 = (u_0, v_0)$ be the saddle-node of (4) of part 2 of Lemma 1. Since $p_1, p_2 \in P_1^{-1}(0)$ are singularities of (4), they satisfy the condition (16). Replacing this condition in (15), we obtain

$$DY_\eta^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u) = \begin{pmatrix} 1-2u-\alpha(1-u) & -\beta C \\ B\alpha(1-u) & -BC(1-\beta) \end{pmatrix}.$$

This follows that

$$\begin{cases} \det DY_\eta^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u) = BC[(2u-1)(1-\beta) + \alpha(1-u)] \\ \operatorname{tr} DY_\eta^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u) = 1-\alpha-BC(1-\beta)-(2-\alpha)u. \end{cases} \tag{17}$$

Since $p_0$ is a saddle-node of (4), it is clear that $\det DY_\eta^{\alpha,\beta}|_{P_1^{-1}(0)}(u_0) = 0$. Then, if $u_1 < u_0$, respectively $u_2 > u_0$, we have that

$$\det DY_\eta^{\alpha,\beta}|_{P_1^{-1}(0)}(u_1) < 0,$$

respectively $\det DY_\eta^{\alpha,\beta}|_{P_1^{-1}(0)}(u_2) > 0$. Therefore, when $u_1 < u_2$, the singularities $p_1$ and $p_2$ are respectively saddle and focus of (4).

Evaluating the abscissa of the focus $p_2$ in (17) we have

$$\begin{cases} \det DY_\eta^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u_2) = BC[(2u_2-1)(1-\beta)+\alpha(1-u_2)] > 0 \\ \operatorname{tr} DY_\eta^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u_2) = 1-\alpha-BC(1-\beta)-(2-\alpha)u_2. \end{cases}$$

In order to prove the hyperbolicity of $p_2$, we analyze the condition of loss hyperbolicity ($\operatorname{tr} DY_\eta^{\alpha,\beta}(p_2) = 0$) in terms of the parameters of the system (4). Let us consider the blowing-up (10) and we assume that the coordinates of some singularity cancel the expression of the trace in this singularity. Then, from (17) and using $\Psi^{-1}((\operatorname{tr} DY_\eta^{\alpha,\beta})^{-1}(0))$ and (11) we obtain the following consistent system

$$\begin{gathered} Cy^{1-\beta} - x^{\alpha+\beta-1} = 0 \\ y + \frac{1}{C}x - \frac{1}{C} = 0 \\ 1-\alpha-BC(1-\beta)-(2-\alpha)x = 0. \end{gathered} \tag{18}$$

By (18) we have, in terms of variable $x$, the only value

$$x = \frac{1-\alpha-BC(1-\beta)}{2-\alpha}.$$

Hence, $x = 0$ if and only if $1-\alpha-BC(1-\beta) = 0$. As the abscissa of the focus is greater than the abscissa of the saddle, this implies that the singularity canceling the trace is the saddle of vector field. Therefore, the coordinates of $\Psi^{-1}(p_2)$ do not cancel the expression of the trace, and this proves that $p_2$ is a hyperbolic focus. Now, in order to know the kind of stability of this focus, it is enough to determine the sign of the trace in this singularity for any condition of the parameters $\xi \in \Lambda^+$ with $C > 1$, because this focus is hyperbolic. For example, if $1-\alpha-BC(1-\beta) = 0$, we have

$$\operatorname{tr} DY^{\alpha,\beta}\Big|_{P_1^{-1}(0)}(u_2) = -(2-\alpha)u_2 < 0.$$

This finishes the proof of part 1 of Lemma 2.

2) If $\xi \in \Upsilon^0$ and $C > 1$, (18) is reduced to

$$\begin{gathered} Cy^{\alpha} - 1 = 0 \\ y + \frac{1}{C}x - \frac{1}{C} = 0 \\ 1-\alpha-BC\alpha-(2-\alpha)x = 0 \end{gathered}$$

and the (only) singularity $\Psi^{-1}(p_2)$, is

$$\Psi^{-1}(p_2) = \Big(1 - C^{(\alpha-1)/\alpha},\ \frac{1}{C^{1/\alpha}}\Big).$$

Since $0 < \alpha < 1$, it follows that $\operatorname{tr} DY^{\alpha,\beta}\big|_{P_1^{-1}(0)}(u_2) = -1 - BC\alpha + (2-\alpha)C^{1-(1/\alpha)} < 0$. This proves part 2 of Lemma 2.

3) If $\xi \in \Upsilon^+$, we first analyze the condition of loss hyperbolicity ($\operatorname{tr} DY_\eta^{\alpha,\beta}(p_2) = 0$), in terms of the parameters of vector field (4), that is,

$$\begin{gathered} 1 - u - u^{\alpha-1}v^\beta = 0 \\ u^\alpha v^{\beta-1} - C = 0 \\ 1 - 2u - \alpha u^{\alpha-1}v^\beta + B\beta u^\alpha v^{\beta-1} - BC = 0. \end{gathered}$$

This system defines implicitly the condition of nonhyperbolicity

$$[1-\alpha-BC(1-\beta)]^{\alpha+\beta-1} - C^\beta(2-\alpha)^{\alpha+2\beta-2}[1+BC(1-\beta)]^{1-\beta} = 0. \tag{19}$$

To study (19) in the $\alpha\beta$-plane, let $C > 0$ and $B > 0$ be arbitrary but fixed values, and we consider the functions

$$\begin{aligned} F(\alpha,\beta) &= [1-\alpha-BC(1-\beta)]^{\alpha+\beta-1} - C^\beta(2-\alpha)^{\alpha+2\beta-2}[1+BC(1-\beta)]^{1-\beta} \\ l(\alpha,\beta) &= 1 - \alpha - BC(1-\beta). \end{aligned}$$

Then

$$F(\alpha,\beta)|_{l^{-1}(0)} = -C^\beta(2-\alpha)^{\alpha+2\beta-2}[1+BC(1-\beta)]^{1-\beta} < 0. \tag{20}$$

Moreover,

$$\begin{cases} F(\alpha,1) < 0 & \text{if } C > ((1-\alpha)/(2-\alpha))^\alpha \\ F(\alpha,1) = 0 & \text{if } C = ((1-\alpha)/(2-\alpha))^\alpha \\ F(\alpha,1) > 0 & \text{if } C < ((1-\alpha)/(2-\alpha))^\alpha. \end{cases}$$

Therefore, given $C < 1$, there exists $0 < \alpha^* < 1$ such that

$$\begin{aligned} \alpha < \alpha^* &\implies F(\alpha,1) > 0 \\ \alpha > \alpha^* &\implies F(\alpha,1) < 0. \end{aligned}$$

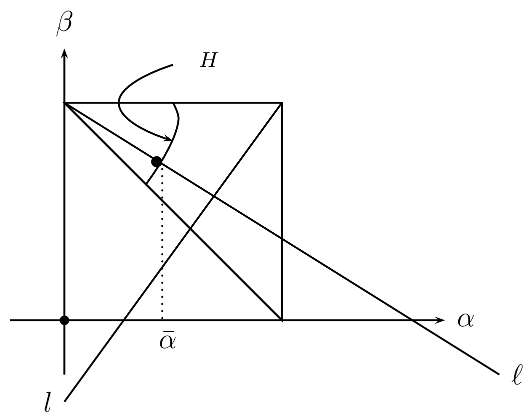


FIGURE 6.

This change of signs proves that $(\alpha^*, 1) \in F^{-1}(0)$ and now, from (20), the surface of the zeros of $F$ and $l^{-1}(0)$ does not have intersections, and this proves that the graphics $F^{-1}(0)$ in $l^{-1}(0, \infty)$ intersects to the straight line $\alpha + \beta = 1$, see Figure 6.

Then the arc of $F^{-1}(0)$ given by

$$H = F^{-1}(0) \cap l^{-1}(0, \infty) \cap \{(\alpha, \beta) \mid \alpha + \beta > 1,\ 0 < \alpha,\ \beta < 1\}$$

is a curve of Hopf bifurcation and (4) has, at least, a weak focus of order one at the singularity $p_2 = (u_2, v_2)$.

In order to determine the topological type of the singularity $p_2$, we consider the translation of $p_2$ to the origin and the respective Jordan canonical form of the system (4). For that, first we use the following $C^\infty$-conjugation

(21)

$$\varphi : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \quad \text{such that} \quad \varphi(x, y) = (a_{11}x - \gamma y + u_2,\ a_{21}x + v_2) = (u, v)$$

where

$$\gamma = \sqrt{\det DY_\eta^{\alpha,\beta}(\bar{u_2}, \bar{v_2})}\ y \qquad \det D\varphi(x, y) = \gamma a_{21} > 0.$$

The qualitatively equivalent vector field in the new coordinates is $Z_\eta^{\alpha,\beta} = \varphi_* Y_\eta^{\alpha,\beta}$. As the vector field $Z_\eta^{\alpha,\beta}$ is analytic in a neighborhood of the origin, we have that

$$\frac{1}{\gamma}Z_\eta^\alpha(x,y)=\Big(-y+\sum_{i,j=2}^{5}A_{i,j}x^iy^j+\text{H.O.T.}\Big)\frac{\partial}{\partial x}$$
$$+\Big(x+\sum_{i,j=2}^{5}B_{i,j}x^iy^j+\text{H.O.T.}\Big)\frac{\partial}{\partial y},$$

where H.O.T. denotes the higher order term and $A_{i,j}=A_{i,j}(\alpha,\eta)$, $B_{i,j}=B_{i,j}(\alpha,\eta)$.

For $k=0,1$, let $L_k$ be the first two Liapunov quantities at the origin [**5, 14**] of the vector field $(1/\gamma)Z_\eta^\alpha$. Since the trace of linear part of the vector field at the origin is zero, we have that $L_0=0$. Now, as $L_1$ depends on the 3-jet of $(1/\gamma)Z_\eta^\alpha$, see [**7**], then

(22)
$$L_1=(A_{02}A_{11}+A_{12}+A_{11}A_{20}+3A_{30}+2A_{02}B_{02}+3B_{03}-B_{02}B_{11}$$
$$-2A_{20}B_{20}-B_{11}B_{20}+B_{21})/8.$$

Using the Mathematica software [**19**], we have

$$L_1=\frac{\alpha\beta(-2+\alpha+2\beta)B^2C^2(-1-BC+\beta BC)(1-\alpha-BC+\beta BC)^2}{16(-1+\alpha+BC-\beta BC)^2\gamma^3}.$$

As $u_2>0$, then $1-\alpha-BC(1-\beta)\neq 0$, and consequently

$$L_1=\frac{\alpha\beta(-2+\alpha+2\beta)B^2C^2(-1-BC+\beta BC)}{16\gamma^3}.$$

Therefore, if we define $\ell(\alpha,\beta)=\alpha+2\beta-2$, then $p_2$ is:

$$\begin{cases}\text{an attracting weak focus of order one} & \text{if } (\alpha,\beta)\in H\cap\ell^{-1}(0,\infty)\\ \text{a repelling weak focus of order one} & \text{if } (\alpha,\beta)\in H\cap\ell^{-1}(-\infty,0)\\ \text{a weak focus, at least, of order two} & \text{if } (\alpha,\beta)\in H\cap\ell^{-1}(0)\end{cases}$$

Now, if $\alpha+2\beta-2=0$, the weakness of $p_2$ depends only of $L_2$. On the other hand, it is known that $L_2$ depends on the 5-jet of $(1/\gamma)Z_\eta^\alpha$. Then, again using the Mathematica software, we have

$$L_2=L_2(\beta,B)=\frac{N_2(\beta,B)}{D_2(\beta,B)},$$

where

$$\begin{aligned}N_2 = &-(-1+\beta)^5\beta^4B^5C^5(1-5\beta+8\beta^2-4\beta^3+5BC-27\beta BC\\&+52\beta^2BC-42\beta^3BC+12\beta^4BC-8B^2C^2+113\beta B^2C^2\\&-431\beta^2B^2C^2+703\beta^3B^2C^2-553\beta^4B^2C^2+204\beta^5B^2C^2\\&-28\beta^6B^2C^2-62B^3C^3+646\beta B^3C^3-2392\beta^2B^3C^3\\&+4388\beta^3B^3C^3-4422\beta^4B^3C^3+2470\beta^5B^3C^3-708\beta^6B^3C^3\\&+80\beta^7B^3C^3+258\beta^4C^4-731\beta^5C^4+713\beta^6C^4-286\beta^7C^4\\&+40\beta^8C^4-103B^4C^4+993\beta B^4C^4-3830\beta^2B^4C^4\\&+7914\beta^3B^4C^4-9687\beta^4B^4C^4+7221\beta^5B^4C^4-3200\beta^6B^4C^4\\&+768\beta^7B^4C^4-76\beta^8B^4C^4-144\beta^5C^5+300\beta^6C^5-210\beta^7C^5\\&+60\beta^8C^5-6\beta^9C^5+228\beta^4BC^5-790\beta^5BC^5+1090\beta^6BC^5\\&-740\beta^7BC^5+242\beta^8BC^5-30\beta^9BC^5-71B^5C^5+605\beta B^5C^5\\&-2276\beta^2B^5C^5+4942\beta^3B^5C^5-6799\beta^4B^5C^5+6121\beta^5B^5C^5\\&-3590\beta^6B^5C^5+1316\beta^7B^5C^5-272\beta^8B^5C^5+24\beta^9B^5C^5\\&+18\beta^4B^2C^6-51\beta^5B^2C^6+51\beta^6B^2C^6-21\beta^7B^2C^6+3\beta^8B^2C^6\\&-18B^6C^6+123\beta B^6C^6-363\beta^2B^6C^6+603\beta^3B^6C^6\\&-615\beta^4B^6C^6+393\beta^5B^6C^6-153\beta^6B^6C^6+33\beta^7B^6C^6\\&-3\beta^8B^6C^6).\end{aligned}$$

and $D_2 = 9(-1-BC+\beta BC)^2(-1+2\beta-BC+\beta BC)^4\gamma^7$.

As $F(\alpha,((2-\alpha)/2)) = 0$, we obtain that $B = (2(1-\alpha-C^{(2-\alpha)/\alpha}))/(\alpha C(C^{(2-\alpha)/\alpha}+1))$. Consequently,

$$N_2 = -(-1+\beta)^5\beta^4B^5C^5N_{21}(\alpha, C).$$

In order to determine the sign of $N_{21}(\alpha, C)$, once more we use the Mathematica software. Then if $0<\alpha<1$ and $0<C<1$, it follows that $N_{21}(\alpha, C) < 0$. Therefore, $L_2 < 0$ and the singularity $p_2$ is an attracting weak focus of order two. The proof of statements iv) and v) of Lemma 2 are immediate, because $\operatorname{tr} DY_\eta^{\alpha,\beta}(p_2) < 0$, respectively $> 0$. This concludes the proof of Lemma 2. $\square$

*Proof of Theorem* 2. By Lemma 2 statement 3i, in the parameter space $\alpha\beta$ there is a neighborhood $\mho_c$ of the point $(\bar{\alpha}, \bar{\beta})$ such that the only singularity $p_2 = (\bar{\alpha}, \bar{\beta}) \in \Gamma$ of the vector field (4) is an attracting weak focus of order two. By the proof of Lemma 2 statement 3i we have that $L_0 = L_1 = 0$ and $L_2 < 0$. By the normal forms theory and by the versal unfolding of the weak focus [**7, 9**], the codimension of the singularity $p_2$ is two. Under the same hypothesis, Takens [**17**] (Case k=2, Figure 1.2, pp. 487, 488) and also Arrowsmith and Place [**2**] (Type(2,-) with reversal time, Figure 4.11, pp. 211–213) describe in detail the bifurcation diagram for the type of singularity of codimension two and has a diagram as in Figure 1, where $S_e$ is a bifurcation curve in which the unstable and stable limit cycles collapse (semi-stable limit cycles) and $H$ is a Hopf bifurcation curve. $\square$

*Proof of Theorem* 3. i) Let us assume that either $\xi \in \Lambda^-$ or $\xi \in \Lambda^+ \cup \Lambda^0 \cup \Upsilon^0$ and $C \leq 1$. In order to study the singularity of the vector field (4) at the origin , we consider the vector field (8) and the change of coordinates (7). Then (8) is a differentiable extension of (4) in $\varphi^{-1}(\bar{\Gamma})$ and

$$D\bar{Y}_\eta^{\alpha,\beta}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, is it easy to see that

$$\bar{Y}_\eta^{\alpha,\beta}(0,y) = -\frac{1-\alpha}{2}\, y^{(1+\beta)/(1-\beta)}\, \frac{\partial}{\partial x} \tag{23}$$

and

$$\bar{Y}_\eta^{\alpha,\beta}(x,0) = B\,\frac{1-\beta}{2}\, x^{(1+\alpha)/(1-\alpha)}\, \frac{\partial}{\partial y}.$$

By Lemma 1 part 1, (4) has no singularities in $\Gamma$; then the vector field (8) neither has singularities in $\varphi^{-1}(\Gamma)$. For $0 < x < 1$, let $(x,0) \in \varphi^{-1}(\bar{\Gamma})$ be an initial condition and let $\gamma_x$ be the orbit of vector field (8) through this point. Then $\gamma_x$ is orthogonal to the coordinates axis and $\varphi(\gamma_x)$ is as we show in Figure 3, an orbit of (4) is tangent to both coordinate axes. Consequently, the families of orbits $\{\varphi(\gamma_x) \mid 0 < x < 1\}$ form an elliptical non Lipschitzian sector.

Now suppose that either $\xi \in \Lambda^0 \cup \Lambda^+$ or $C > 1$. By Lemma 1 part 2 and part 3 in $\Gamma$, there is a saddle-node or there are a hyperbolic saddle and a hyperbolic node point, respectively. In this last case, by Lemma 2 part 1 the abscissa of the node is greater than abscissa of the saddle. Let $W^{ss}$ and $W^{s}$ be the strong stable manifold of the saddle-node and the stable manifold of the hyperbolic saddle, respectively. Analogously to the above case i), and by (23), $\varphi^{-1}(W^{ss})$ and $\varphi^{-1}(W^{s})$ are orbits perpendicular to the $x$-axis. So, $W^{ss}$ and $W^{s}$ are tangent curves to the $u$-axis and consequently the $\alpha$-limit of both orbits is the origin of the coordinates system.

Similarly, let $W^{cu}$ and $W^{u}$, whose $\omega$-limit is not the node, be the unstable center manifold of the saddle-node and the unstable manifold of the hyperbolic saddle, respectively. These separatrices are tangent to the $v$-axis and their respective $\omega$-limits are the origin of the coordinate systems. Let $U$ be the open set enclosed by the closure of $W^{ss} \cup W^{cu}$, respectively, by the closure of $W^{s} \cup W^{u}$. Then, in $U$ there are no singularities, therefore using the same above argument, in $U$ there exists an elliptical sector.

ii) Let us suppose that either $\xi \in \Upsilon^{0}$ and $C > 1$ or $\xi \in \Upsilon^{+}$. We note that (23) does not depend on the conditions of case i). Then at the points of both coordinate axes, no uniqueness of solutions exists. So, we obtain the situation given in Figure 3. Now, from Lemma 1 part 4, there exists in $\Gamma$ only one singularity of (4) that is a center-focus. It is easy to see that this center-focus is extended to the origin. Indeed, if there exist separatrices limiting such extension, they should have an opposite orientation to the direction of flow, and this is a contradiction. Then there exists a hyperbolic sector of (4) at the origin. The proof of Theorem 3 is now complete. $\square$

**4. Discussion.** In this work we have studied a bioeconomics model of fishery at an open access that incorporated aspects of interference or congestion between units that carry out the effort of fishery and that consider the phenomenon of aggregation (or accumulation) of the fishing resource. We have obtained conditions to the existence of the equilibria points in the first quadrant and its stability has been analyzed.

We have proved the existence of one separatrix in the phase space that divides the behavior of the trajectories of the system where the solutions above the separatrix in finite time reach to the $y-axis$ for any value of the parameters. This situation implies that the point $(0,0)$ is an attractor for all trajectories above the separatrix (*threshold curve*), this meaning that the fishery collapses.

For certain values of the parameters we have verified that there are not equilibrium points in the first quadrant, Lemma 1.1. In this case, for any initial condition, the origin $(0,0)$ again is an attractor of the system, and this implies that the fishing resource goes to extinction and the fishery will be finished. Moreover, we have proved that (Theorem 2) there exists an open set in the parameter space in which the bistability phenomenon is possible, namely, the coexistence of two limit cycles around the unique locally asymptotically stable equilibrium point, one of them (outer) is stable and another one (inner) is unstable. This implies that for certain initial conditions the fishery will be able to self-regulate and for other initial conditions will tend to fixed values.

The dynamics of the system shows that every time that there exists congestion of ships, $\beta < 1$, dedicated to the capture of the resource and every time that there exists aggregation of the resource, $\alpha < 1$, it is highly probable that the fishery collapses. This supports the advice to carry out a permanent control on several fisheries commercially exploited. In short, our analyzed model approximately reflects what happens in several fisheries, although we have implicitly assumed some simplifications with respect to the population of species and the effort of the fishery.

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