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ON THE VANISHING OF THE ETA INVARIANT OF DIRAC OPERATORS ON LOCALLY SYMMETRIC MANIFOLDS

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ABSTRACT. In this note we prove a vanishing theorem for the Eta invariant of the spin Dirac operator on a locally symmetric space.

1. Introduction. Atiyah, Patodi and Singer [**2**] first defined the η-invariant of any self-adjoint elliptic operator A on a compact manifold as a measure of the asymmetry of $\operatorname{Spec}(A)$. If X is a compact oriented odd-dimensional locally symmetric manifold, then the generalized Dirac operator $\mathbf{D}$ (after choosing the essentially unique G-invariant connection) associated to a locally homogeneous Clifford module bundle over X is such an operator. Relying on Selberg trace formula analysis, Moscovici and Stanton [**7**] prove

Theorem 1.1. *Let G be a semi-simple Lie group with a maximal compact subgroup K, and let $\dim(G/K)$ be odd. Suppose that Γ is a cocompact discrete torsion free subgroup and suppose G has no factors locally isomorphic to $SL(3,\mathbf{R})$ or $SO(p,q)$, for p,q odd. Then for the generalized Dirac operator $\mathbf{D}$ on $\Gamma\backslash G/K$*

$$\eta(\mathbf{D}) = 0. \tag{1}$$

In this note we present another proof of this theorem which is not based on an evaluation of the trace of the odd heat kernel operator $\mathbf{D}e^{-t\mathbf{D}^2}$ by means of orbital integrals. Our proof is modeled after the proof of the vanishing theorems of cohomology of the locally symmetric space $\Gamma\backslash G/K$ and in particular after the algebraic proof of the triviality of the analytic torsion $\tau_1(\Gamma\backslash G/K)$ for the trivial representation of Γ in Speh [**8**]. In 3.1 we expand $Tr(\mathbf{D}e^{-t\mathbf{D}^2})$ using representation-theoretic data involving certain unitary representations of G. Then in

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4.1 we relate $\mathrm{Tr}\,(\widetilde{\mathbf{D}}_\pi)$ to the trace of the principal series representations that appear in the Grothendieck group decomposition of the unitary representation π of G. Finally, we use the fundamental result of [**7**] that $\mathrm{Tr}\,(\widetilde{\mathbf{D}}_{I(Q,\xi,\nu)}) = 0$ for the principal series representation $I(Q,\xi,\nu)$ if G does not have a cuspidal parabolic subgroup of split rank 1 to complete the argument.

Remark. The vanishing of the η-invariant is equivalent to the vanishing of the secondary characteristic classes of the bundle over G/K associated to the trivial representation of Γ.

2. Preliminaries. In this section we first recall the definition of the η-invariant and then discuss the generalized Dirac operator on the C^∞-sections of a homogeneous Clifford bundle over a locally symmetric space.

Let A be a self-adjoint elliptic operator on a compact manifold X. We define for $\mathrm{Re}\,(s) \gg 0$

$$\eta(s,A) = \sum_{\lambda\in \mathrm{Spec}\,(A)-\{0\}} \frac{\mathrm{sgn}\,\lambda}{|\lambda^s|} = \mathrm{Tr}\,(A(A^2)^{-(s+1)/2}). \tag{2}$$

It turns out that this is a holomorphic function which can be analytically continued to a meromorphic function of $\mathbf{C}$. Moreover, we have the identity

$$\eta(s,A) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2}\,\mathrm{Tr}\,(Ae^{-tA^2})\,dt \tag{3}$$

which allows us to work with the Mellin transform integrand $\mathrm{Tr}\,(Ae^{-tA^2})$. It can be shown that $s = 0$ is not a pole, so one can define

$$\eta(A) \,=\, \eta(0,A). \tag{4}$$

Thus we can associate the η-invariant to any Dirac-type operator on a compact Riemannian manifold of odd dimension (on the even-dimensional ones, Dirac operators have symmetric spectra).

Let G be a semi-simple connected Lie group with maximal compact subgroup K such that $\dim(G/K) = \dim \widetilde{X} = 2n+1$. We may assume that G is simple because $\widetilde{X}$, being simply connected, is a product of symmetric spaces which are quotients of simple Lie groups. The Lie algebra $\mathbf{g}$ of G has Cartan decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ where $\mathbf{k}$ is the Lie algebra of K. Thus we can identify $\mathbf{p}$ with the tangent space to $\widetilde{X}$ at eK. We denote by $\mathrm{Spin}\,(\mathbf{p})$ the $\mathbf{Z}_2$-covering group of $SO(\mathbf{p})$. Since $\dim \mathbf{p} = 2n+1$, the Clifford algebra $Cl(\mathbf{p})$ possesses exactly two distinct simple modules which collapse into one when restricted to $\mathrm{Spin}\,(\mathbf{p})$. We may assume that K maps into $\mathrm{Spin}\,(\mathbf{p})$, by passing to a covering group if necessary, and we refer to this homomorphism as the spin representation (σ, s) of K.

Moscovici and Stanton [**7**] show that if $\widetilde{\mathbf{E}}$ is a G-homogenous Clifford module bundle over $\widetilde{X}$, it is associated to a finite-dimensional representation of K of the form $(\sigma \otimes \tau,\, S \otimes V)$. Hence we can characterize the space $\Gamma(\widetilde{\mathbf{E}})$ of smooth sections of $\mathbf{E}$ as the K-invariants $[C^\infty(G) \otimes S \otimes V]^K$ where K acts on $C^\infty(G)$ via the right regular representation $R(G)$.

An essentially unique Dirac operator exists which is G-homogeneous and anti-commutes with the Cartan involution

$$\widetilde{\mathbf{D}} = \sum_i R(X_i) \otimes c(X_i) c(\omega^{\mathcal{C}}) \tag{5}$$

where $\{X_i\}$ is an oriented orthonormal basis of $\mathbf{p}$, $c(\cdot)$ denotes Clifford multiplication on the fiber E over eK, and $\omega^{\mathcal{C}}$ is the complex volume element in $Cl(\mathbf{p})$ [**7**]. This invariant operator is elliptic and formally self-adjoint.

We define

$$\widetilde{\mathbf{D}}_\pi = \sum_i \pi(X_i) \otimes c(X_i) c(\omega^{\mathcal{C}}) : [H_\pi^\infty \otimes S \otimes V]^K \to [H_\pi^\infty \otimes S \otimes V]^K \tag{6}$$

associated to a unitary representation π of G with smooth vectors H_π^∞. Then

$$\widetilde{\mathbf{D}}_\pi^2 = -\pi(\Omega) \otimes I \otimes I - I \otimes \sigma(\Omega_K) \otimes I + I \otimes I \otimes \tau(\Omega_K) \tag{7}$$

where Ω is the Casimir operator of G and Ω_K is the Casimir operator of K with respect to the Killing form on $\mathbf{g}$. See Borel-Wallach [**3**] and Atiyah-Schmid [**1**].

Let $X = \Gamma\backslash\widetilde{X}$ for a discrete co-compact torsion-free subgroup Γ of G. By homogeneity of $\widetilde{\mathbf{E}}$ we can form the bundle $\mathbf{E} = \Gamma\backslash\widetilde{\mathbf{E}}$. Then smooth sections on $\mathbf{E}$ can be identified with $[C^\infty(\Gamma\backslash G \otimes S \otimes V)]^K$. $\widetilde{\mathbf{D}}$ induces the generalized Dirac operator $\mathbf{D} : [C^\infty(\Gamma\backslash G \otimes S \otimes V)]^K \to [C^\infty(\Gamma\backslash G \otimes S \otimes V)]^K$ which is also elliptic and self-adjoint.

3. The trace of the odd heat kernel. In this section we give a representation-theoretic interpretation of $\mathrm{Tr}\,(\mathbf{D}e^{-t\mathbf{D}^2})$. Let dx denote both the Haar measure on G and the associated measure on $\Gamma\backslash G$. The Hilbert space $L^2(\Gamma\backslash G)$ of square-integrable functions with respect to dx is the completion of $C^\infty(\Gamma\backslash G)$. By a theorem of Gel'fand and Piateskii-Shapiro [**4**] we can write

$$L^2(\Gamma\backslash G) \cong \bigoplus m(\pi,\Gamma) H_\pi \tag{8}$$

where we sum over all irreducible representations $\pi : G \to U(H_\pi)$ in the unitary dual $\widehat{G_u}$ and $m(\pi,\Gamma) = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$. Hence

$$[L^2(\Gamma\backslash G) \otimes S \otimes V]^K \cong \bigoplus m(\pi,\Gamma)[H_\pi \otimes S \otimes V]^K. \tag{9}$$

Lemma 3.1. *Suppose $\Gamma\backslash G$ is compact, and let Ω be the Casimir operator of G. For $\lambda \in \mathbf{R}$,*

(10)

$$\dim \ker(\mathbf{D}^2 - \lambda) = \sum_{\substack{\pi \in \widehat{G_u} \\ \pi(\Omega) = -\lambda - \sigma(\Omega_K) + \tau(\Omega_K)}} m(\pi,\Gamma) \dim[H_\pi^\infty \otimes S \otimes V]^K.$$

Proof. Since the operator $\mathbf{D}^2$ is elliptic we can write $[C^\infty(\Gamma\backslash G) \otimes S \otimes V]^K$ in a unique way as a sum of its eigenspaces. By (7), the action of $\mathbf{D}^2$ corresponds to the action of the Casimir element on $C^\infty(\Gamma\backslash G)$, so the decomposition claimed in the lemma is the eigenspace decomposition.

Proposition 3.1. *Suppose* $\Gamma\backslash G/K$ *is a compact locally symmetric space. Then, for the generalized Dirac operator* $\mathbf{D}$*, we have*

$$\mathrm{Tr}\,(\mathbf{D}e^{-t\mathbf{D}^2}) = \sum_{\lambda} \sum_{\substack{\pi\in\widehat{G_u} \\ \pi(\Omega)=-\lambda-\sigma(\Omega_K)+\tau(\Omega_K)}} \mathrm{Tr}\,(\widetilde{\mathbf{D}}_\pi)e^{-t\lambda}. \tag{11}$$

Proof. By (9) and the lemma we have

$$\begin{aligned} \mathrm{Tr}\,(\mathbf{D}e^{-t\mathbf{D}^2}) &= \sum_{\substack{\pi\in\widehat{G_u} \\ \pi(\Omega)=-\lambda-\sigma(\Omega_K)+\tau(\Omega_K)}} m(\pi,\lambda)\mathrm{Tr}\,(\widetilde{\mathbf{D}}e^{-t\tilde{\mathbf{D}}^2}([H_\pi^\infty\otimes S\otimes V]^K)) \quad (12) \\ &= \sum_{\lambda} \sum_{\substack{\pi\in\widehat{G_u} \\ \pi(\Omega)=-\lambda-\sigma(\Omega_K)+\tau(\Omega_K)}} m(\pi,\lambda)\mathrm{Tr}\,(\widetilde{\mathbf{D}}([H_\pi^\infty\otimes S\otimes V]^K))e^{-t\lambda}. \quad (13) \end{aligned}$$

□

4. Conclusion. In this section we finish the proof of the main theorem. Let $Q = MAN$ be a cuspidal parabolic subgroup of G, ξ an irreducible unitary representation of M and ν a character of A. Let $I(Q,\xi,\nu) = \mathrm{ind}_Q^G\ \xi\otimes\nu\otimes 1$ be the induced principal series representation. By an explicit calculation, Moscovici and Stanton [**7**] prove the following

Proposition 4.1. $\mathrm{Tr}\,(\widetilde{\mathbf{D}}_{I(Q,\xi,\nu)}) = 0$ *if* G *does not have a cuspidal parabolic subgroup of split rank* 1 *and* $\dim(G/K)$ *is odd.*

The simple Lie groups that have cuspidal parabolic subgroups of real rank 1 and for which $\dim G/K$ is odd are locally isomorphic to $SL(3,\mathbf{R})$ or $SO(p,q)$ with p,q both odd. Hence to complete the proof of the main theorem we only need the following

Lemma 4.1. *If* $\mathrm{Tr}\,(\widetilde{\mathbf{D}}_{I(Q,\xi,\nu)}) = 0$ *for all principal series representations of* G*, then* $\mathrm{Tr}\,(\widetilde{\mathbf{D}}_{\pi}) = 0$ *for any unitary representation* $U(H_\pi)$ *of* G*.*

Proof. In the Grothendieck group, every unitary representation π of G can be represented uniquely as a sum of principal series representations with coefficients $m(U(H_\pi), \xi \otimes \nu)$.

Since

$$[H_\pi \otimes S \otimes V]^K = \Big[\sum m(H_\pi^\infty, \xi \otimes \nu) I^\infty(Q,\xi,\nu) \otimes S \otimes V\Big]^K \tag{14}$$

$$= \sum m(H_\pi^\infty, \xi \otimes \nu)[I^\infty(Q,\xi,\nu) \otimes S \otimes V]^K \tag{15}$$

and both $\widetilde{\mathbf{D}}$ and the trace are linear,

$$\mathrm{Tr}\,(\widetilde{\mathbf{D}}_\pi) = \sum m(H_\pi^\infty, \xi \otimes \nu) Tr(\widetilde{\mathbf{D}}_{I(Q,\xi,\nu)}). \qquad \square \tag{16}$$

In conclusion we would like to note that the same proof extends without any difficulty to the case of the twisted η-invariants.

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