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THE GLOBAL STRUCTURE OF UNIFORMLY ASYMPTOTICALLY ZHUKOVSKIJ STABLE SYSTEMS

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ABSTRACT. In this paper, we prove that the omega limit set of a uniformly asymptotically Zhukovskij stable orbit of a flow defined on a locally compact metric space is a closed orbit or a fixed point, and also it is a uniform attractor. If each orbit of the flow is uniformly asymptotically Zhukovskij stable, we obtain the global structure of the system, and further, if the space is compact, then the sum of fixed points and closed orbits is finite.

1. Introduction. There are many types of stabilities in the mathematical and physical literature, see [**13**]. Among the most important ones, Lyapunov stability is rather restrictive since it is an isochronous correspondence of orbits, for example, even an anharmonic oscillator is unstable in this sense [**6**, p. 41]. In this paper we shall consider a relaxed concept of stability, i.e., Zhukovskij stability [**7**]. It implies that orbits should be close to each other in the phase space and also repeat the 'tracery' of each other with a certain time lag; obviously such a stability is a kind of phase stability. The problem of studying periodicity for a limit orbit is old, and the literature on the subject is enormous, see [**2, 3, 8, 10**–**12**] and references therein. In 1966, Sell [**12**] proves that a bounded phase asymptotically stable solution of an autonomous system approaches an asymptotically stable periodic solution. In [**3**] Cronin gives conditions to guarantee that Lagrange stable solutions of a differential system in R^n are phase asymptotically stable, in the sense of [**12**], and therefore limit to a phase asymptotically stable periodic solution. Later, Li and Muldowney [**8**] obtain further results and simpler criteria to ensure that Lagrange stable solutions have periodic orbits as their limit sets, especially they show in [**8**, Theorem 2.1] that the omega limit set of a Lagrange stable orbit $\Gamma^+(x)$

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is a periodic orbit that attracts its neighbors with bounded time phase if and only if $\Gamma^+(x)$ itself attracts its neighbors with bounded time phase. We refer to [**8**] for an excellent brief summary on this subject. We deal with the problem in a locally compact metric space and generalize the above results with unbounded time phase.

Consider a system in R^2 defined by differential equations in polar coordinates:

$$\frac{dr}{dt} = r(1-r)^3, \qquad \frac{d\theta}{dt} = r. \tag{2}$$

Obviously the origin is a fixed point and the unit circle is an asymptotically stable closed orbit. The other solutions of (2) have the relation: $2\theta + \alpha = 1/(1-r)^2$, where $\alpha = \alpha(r_0, \theta_0)$ is a constant dependent on the initial value $r_0 = r(\theta_0)$. We choose $r_0 = r(0) > 1$ to fix a solution $r = r(\theta)$ outside of the unit circle $r = 1$; by some computation it is easy from $d\theta/r(\theta) = dt$ to obtain that $T = 2k\pi - \int_0^{2k\pi} d\theta/\sqrt{2(\theta+\alpha)} + 1$ is the time that $r = r(\theta)$ surrounds the unit circle k times. Let $\beta(k) = \int_0^{2k\pi} d\theta/\sqrt{2(\theta+\alpha)} + 1$, so $\beta(k) \to +\infty$ as $k \to +\infty$. Thus $r = r(\theta)$ is not asymptotically stable in the sense of Sell, see [**12**], since the closed orbit $r = 1$ needs time $2k\pi$ for circling itself k times and the difference of time between solutions $r = r(\theta)$ and $r = 1$ tends to infinity. This example shows that a periodic orbit may attract its neighbors with unbounded time phase. On the other hand, it is not difficult to see that the orbit $r = r(\theta)$ is uniformly asymptotically Zhukovskij stable, see the following Definition 1. This example also shows that the conclusion of our Theorem 2.3 really generalizes the Sell theorem [**12**, Theorem 1].

Let (X, d) be a locally compact metric space with metric d, on which there is a flow $f : X \times R \to X$. Write $x \cdot t = f(x, t)$ and let $A \cdot J = \{x \cdot t \,|\, x \in A,\ t \in J\}$ for $A \subset X$ and $J \subset R$. Then $x \cdot R$ and $x \cdot R^+$ are the orbit and the positive semi-orbit, respectively, of a point $x \in X$. The omega limit set of x is the set $\omega(x) = \{y \in X \mid$ there is a sequence $t_n \in R^+$ such that $t_n \to +\infty$ and $x \cdot t_n \to y\}$. A set Y is invariant under f if Y is a subset in X with $Y \cdot R = Y$, and an invariant set Y is a minimal set provided (i) Y is a closed, nonempty set and (ii) if Z is a closed, nonempty, invariant subset of Y, then $Z = Y$. In addition, throughout this paper we let $B_\delta(x) = \{y \in X \,|\, d(x, y) < \delta\}$ and $S_\delta(x) = \{y \in X \,|\, d(x, y) \leq \delta\}$ be the open ball and the closed ball, respectively, with center x and radius $\delta > 0$. For $p \in X$ and $A \subset X$, let $d(p, A) = \inf\{d(p, z) \,|\, z \in A\}$, and then we define $N_r(A) = \{z \in X \,|\, d(z, A) < r\}$ for $r > 0$; it is called the generalized open r-ball about A of radius r.

Zhukovskij stability [**7**]. The orbit $x \cdot R$ of a point x in X is Zhukovskij stable provided that, given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that, for any $y \in B_\delta(x)$, one can find a time parameterization τ_y such that $d(x \cdot t, y \cdot \tau_y(t)) < \varepsilon$ holds for $t \geq 0$, where τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$. Moreover, if $d(x \cdot t, y \cdot \tau_y(t)) \to 0$ as $t \to +\infty$ also holds, the orbit $x \cdot R$ is said to be asymptotically Zhukovskij stable.

Definition 1. The orbit $x \cdot R$ of a point x is uniformly asymptotically Zhukovskij stable provided that, given any $\varepsilon > 0$, there is a $\delta > 0$ such that for each $t' \geq 0$ and $y \in B_\delta(x \cdot t')$, one can find a time parameterization τ_y such that $d(x \cdot (t + t'), y \cdot \tau_y(t)) < \varepsilon$ holds for $t \geq 0$, and also

$$(1) \qquad d(x \cdot (t + t'), y \cdot \tau_y(t)) \to 0 \quad \text{as} \quad t \to +\infty,$$

where τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$.

We shall prove in this paper that the omega limit set of a uniformly asymptotically Zhukovskij stable orbit of a flow f on a locally compact metric space is minimal; more precisely, it is a closed orbit or a fixed point, and also it is a uniform attractor. Further, if each orbit of the flow is uniformly asymptotically Zhukovskij stable, we show that in every component of the space X either all the orbits have empty omega limit sets or all the orbits are uniformly attracted to a fixed point or a closed orbit. Also, if X is compact and each orbit is uniformly asymptotically Zhukovskij stable, then the sum of fixed points and closed orbits is finite. The concrete conditions for a dynamical system to have the global Zhukovskij stability will be the subject of a subsequent paper.

2. The omega limit set. In this section, we always consider the case $\omega(x) \neq \varnothing$ for a point $x \in X$.

Lemma 2.1. *If the orbit $x \cdot R$ of a point x is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is minimal.*

Proof. Otherwise, $\omega(x)$ has a proper closed invariant subset $A \subset \omega(x)$ with $A \neq \varnothing$. Choose a point $p \in \omega(x) \setminus A$. Then $\lambda = d(p, A) > 0$. Now, for a point $q \in A$, we can find a sufficiently large t' satisfying $d(x \cdot t', q) < \delta$ (with δ the number defined as in Definition 1 for $\varepsilon = \lambda/2$). Also there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \geq t'$, such that $t_i \to +\infty$ and $x \cdot t_i \to p$. Since A is invariant, it follows that $q \cdot R \subset A$. However, for large t_i, we have $d(x \cdot t_i, p) < \lambda/2$. It follows that $d(x \cdot t_i, q \cdot R) \geq d(p, A) - d(x \cdot t_i, p) \geq \lambda/2$ for large t_i. It is contradictory to (1) in Definition 1, since $d(x \cdot t', q) < \delta$. Thus $\omega(x)$ is minimal. □

Corollary 1. *If $\omega(x)$ contains at least two points, since $\omega(x)$ is minimal, there are no fixed points in $\omega(x)$. Further, if there is a closed orbit γ in $\omega(x)$, then $\omega(x) = \gamma$.*

From the proof of Lemma 2.1, it is also easy to conclude:

Corollary 2. *Any closed nonempty invariant set A must be δ-apart from a uniformly asymptotically Zhukovskij stable semi-orbit $x \cdot R^+$ if $A \cap \omega(x) = \varnothing$.*

Lemma 2.2 [**4**, p. 414]. *Let X be a Hausdorff topological space and $F : X \to X$ continuous. If for each open covering $\{W_\alpha\}$ of X there is at least one $x \in X$ such that both x and $F(x)$ belong to a common W_α, then F has a fixed point.*

Theorem 2.3. *If an orbit $x \cdot R$ is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is a fixed point or a closed orbit.*

Proof. Assume that $\omega(x)$ is not a singleton; we shall show that $\omega(x)$ is a closed orbit. Choose a point $p \in \omega(x)$, which is not a fixed point from Corollary 1. Now let a sequence $\{t_i\}_{i=1}^{\infty} \subset R^+$ be such that

$t_i \to +\infty$ and $x \cdot t_i \to p$. Thus there is a positive and small σ ($\sigma < \delta$ and δ defined as in Definition 1) such that the closed ball $S_\sigma(p)$ lies in the open ball $B_\delta(x \cdot t_k)$ for some $t_k \in \{t_i\}_{i=1}^{\infty}$ and so does the set $S_\sigma(p) \cdot [-\theta, \theta]$ for a sufficiently small $\theta > 0$. From the local compactness of X, we may also suppose that $S_\sigma(p)$ is compact. Since p is a regular point, by the tubular flow theorem [**9**, Chapter 5, Section 2], for a sufficiently small fixed σ there is a transversal $\Sigma \subset S_\sigma(p) \cdot [-\theta, \theta]$ such that for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$, the arc of $y \cdot R$ in $S_\sigma(p) \cdot [-\theta, \theta]$ crosses Σ at a unique time $t = \phi(y)$, where $\phi(y)$ is continuous on $y \in S_\sigma(p) \cdot [-\theta, \theta]$. Because of $S_\sigma(p) \cdot [-\theta, \theta] \subset B_\delta(x \cdot t_k)$, it follows from (1) in Definition 1 that for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$ there is a $T(y) > 0$ such that $d(x \cdot (t + t_k),\, y \cdot \tau_y(t)) < \sigma/2$ for $t \geq T(y)$. Thus, from the compactness of $S_\sigma(p) \cdot [-\theta, \theta]$ and the continuity of the flow f, one can find a positive $M = \sup\{T(y) \,|\, y \in S_\delta(p) \cdot [-\theta, \theta]\} < +\infty$ such that, for each $y \in S_\sigma(p) \cdot [-\theta, \theta]$, $d(x \cdot (t + t_k),\, y \cdot \tau_y(t)) < \sigma/2$ holds for $t \geq M$. Fix a $t_l > t_k$ such that $t_l - t_k \geq M$ and $d(x \cdot t_l, p) < \sigma/2$. Now we define a Poincaré map $F : \Sigma \to \Sigma$ as follows. If $y \in \Sigma$ ($\subset S_\sigma(p) \cdot [-\theta, \theta]$), then we have $d(x \cdot (t + t_k),\, y \cdot \tau_y(t)) < \sigma/2$ for $t \geq M$. This implies $d(p, y \cdot \tau_y(t_l - t_k) \leq d(p, x \cdot t_l) + d(x \cdot t_l, y \cdot \tau_y(t_l - t_k)) < \sigma/2 + \sigma/2 = \sigma$. So it follows $y' = y \cdot \tau_y(t_l - t_k) \in S_\sigma(p)$ and then $y \cdot (\tau_y(t_l - t_k) + \phi(y')) \in \Sigma$ for a $\phi(y') \in [-\theta, \theta]$. Thus we define $F(y) = y \cdot (\tau_y(t_l - t_k) + \phi(y'))$. The continuity of F comes from the continuities of the flow f, τ_y and $\phi(y)$. Note that F may not be the first return map. Next, if $\{W_\alpha\}$ is an open covering of Σ for its subspace topology from X, let $p \in W_\alpha = \Sigma \cap U$, where U is an open set in X. Choose an $r > 0$ with $B_r(p) \subset U \cap (S_\sigma(p) \cdot [-\theta, \theta])$, and let $x \cdot t_i \in B_{r/2}(p)$ for $t_i \geq T > t_l$. Thus, from (1) in Definition 1, we assert $d(F^n(p), x \cdot t_i) < r/2$ for $n \geq N$ and some $t_i \geq T' \geq T$, where $F^n(p)$ is the nth iterate of p. Hence, $d(F^N(p), p) \leq d(F^N(p),\, x \cdot t_m) + d(x \cdot t_m, p) < r/2 + r/2 = r$ holds for some $t_m \geq T'$ and similarly $d(F^{N+1}(p), p) \leq d(F^{N+1}(p),\, x \cdot t_n) + d(x \cdot t_n, p) < r/2 + r/2 = r$ for some $t_n \geq T'$. It follows that both $F^{N+1}(p)$ and $F^N(p)$ lie in $B_r(p)$. So we obtain that both $F(F^N(p))$ and $F^N(p)$ belong to W_α. By Lemma 2.2 we conclude that $F : \Sigma \to \Sigma$ has a fixed point q. Obviously, $q \cdot R$ is a closed orbit; from Corollaries 1 and 2, we immediately obtain $\omega(x) = q \cdot R$. This completes the proof. □

Remark. Obviously, similar results about alpha limit sets also hold.

For the next result, we recall the definition of a uniform attractor. The first positive prolongational limit set of x is the set $J^+(x) = \{y \in X \mid \text{there are a sequence } x_n \in X \text{ and a sequence } t_n \in R^+ \text{ such that } x_n \to x, t_n \to +\infty \text{ and } x_n \cdot t_n \to y\}$. If K is a nonempty compact subset of X, the region of uniform attraction of K is the set $A_u(K) = \{x \in X \mid J^+(x) \neq \varnothing \text{ and } J^+(x) \subset K\}$. K is said to be a uniform attractor if $A_u(K)$ is a neighborhood of K.

Lemma 2.4 [**1**, Chapter 5, Proposition 1.2]. *For each neighborhood V of K, if there exists a neighborhood U of x and a $T \geq 0$ such that $U \cdot t \subset V$ holds for $t \geq T$, then $x \in A_u(K)$.*

Theorem 2.5. *If the orbit $x \cdot R$ of $x \in X$ is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then $\omega(x)$ is a uniform attractor.*

Proof. With a number δ defined in Definition 1, we only need to prove $N_{\delta/2}(\omega(x)) \subset A_u(\omega(x))$, i.e., for any $y \in N_{\delta/2}(\omega(x))$ we shall show $y \in A_u(\omega(x))$. Choose $\sigma > 0$ such that $S_\sigma(y) \subset N_{\delta/2}(\omega(x))$ and $S_\sigma(y)$ is compact from the local compactness of X. Given any $\lambda > 0$, $\lambda < \delta/4$, let $x \cdot t \in N_\lambda(\omega(x))$ for $t \geq T > 0$. Then it follows that, for every $z \in S_\sigma(y) \subset N_{\delta/2}(\omega(x))$, there exists a $p \in \omega(x)$ such that $d(z,p) < \delta/2$ and $d(x \cdot t_0, p) < \delta/2$ for some $t_0 \geq T$. Thus, $d(z, x \cdot t_0) \leq d(z,p) + d(p, x \cdot t_0) < \delta$ holds. By Definition 1 it follows that, for any $t \geq T_z \geq T$, $d(z \cdot \tau_z(t), x \cdot (t_0 + t)) < \lambda$. Now it is easy to see that $z \cdot \tau_z(t) \in N_{2\lambda}(\omega(x))$ for $t \geq T_z$, since $x \cdot t \in N_\lambda(\omega(x))$ for $t \geq T$. By the compactness of $S_\sigma(y)$, we take $T' = \sup\{\tau_z(T_z) \mid z \in S_\sigma(y)\}$. Then for all $z \in S_\sigma(y)$, $z \cdot t \in N_{2\lambda}(\omega(x))$ for $t \geq T'$, it implies $B_\sigma(y) \cdot t \subset N_{2\lambda}(\omega(x)) \subset N_{\delta/2}(\omega(x))$ for $t \geq T'$. So by Lemma 2.3 we have $y \in A_u(\omega(x))$. The proof is complete. □

3. The global structure. The goal of this section is to describe the global structure of the system if all the orbits are uniformly asymptotically Zhukovskij stable.

Lemma 3.1. *If the orbit $x \cdot R$ is uniformly asymptotically Zhukovskij stable and $\omega(x) \neq \varnothing$, let $y \in B_\delta(x \cdot t')$ for $t' \geq 0$ with δ the number defined as in Definition* 1, *then $\omega(y) = \omega(x)$.*

Proof. At first, we show $\omega(x) \subset \omega(y)$. For any $p \in \omega(x)$, there exists a sequence $t_i \in R^+$ such that $t_i \to +\infty$ and $x \cdot t_i \to p$. Hence, given any $\delta > 0$, $x \cdot t_i \in B_{\delta/2}(p)$ for $i \geq N$. Now by (1) in Definition 1, we obtain $d(x \cdot (t+t'), y \cdot \tau_y(t)) < \delta/2$ for $t \geq T > t'$. Since τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$, $\tau_y(t_i - t') \to +\infty$, $i \to +\infty$, holds. Thus it follows $d(p, y \cdot \tau_y(t_i - t')) \leq d(p, x \cdot t_i) + d(x \cdot t_i, y \cdot \tau_y(t_i - t')) < \delta$ for large t_i. So $y \cdot \tau_y(t_i - t') \to p$, it implies $p \in \omega(y)$, i.e., $\omega(x) \subset \omega(y)$ since p is arbitrary. Now from Lemma 2.1 it follows that $\omega(x) = \omega(y)$.

For brevity, we shall call a flow on the locally compact metric space a UAZS flow if all the orbits are uniformly asymptotically Zhukovskij stable.

Lemma 3.2. *If f is a UAZS flow on the locally compact metric space X, then for each component C of X, one and only one of the following cases holds*:

(i) $\omega(x) = \varnothing$ *for all* $x \in C$;

(ii) $\omega(x) \neq \varnothing$ *for all* $x \in C$.

Further, if the second case happens, for $x \in C$ all the $\omega(x)$ are the same invariant subset in C.

Proof. Otherwise, there exist two points x_1 and x_2 in C such that $\omega(x_1) = \varnothing$ and $\omega(x_2) \neq \varnothing$. Let $C_1 = \{x \in C \mid \omega(x) \neq \varnothing\}$; then $C_1 \neq \varnothing$ and $C \setminus C_1 \neq \varnothing$. From Lemma 3.1 it is easy to see that C_1 is open in C. On the other hand, let $x \in C \setminus C_1$ and, for any $y \in B_\delta(x)$, by Definition 1 we have $d(x \cdot t, y \cdot \tau_y(t)) \to 0$ as $t \to +\infty$, which implies $\omega(y) = \varnothing$. In fact, if $y \cdot \tau_y(t_i)$ tends to a point p for a sequence $t_i \in R^+$, then $x \cdot t_i$ also tends to p. It is contradictory to $x \in C \setminus C_1$. Thus we conclude $B_\delta(x) \subset C \setminus C_1$, i.e., C_1 is closed in C. It is impossible, since C is connected. By using Lemma 3.1, we obtain the second part of this lemma similarly as the argument above. □

We conclude the global behavior of orbits for a UAZS flow on a locally compact metric space as follows:

Theorem 3.3. *If f is a UAZS flow on the locally compact metric space X, then either all the orbits in each component of X are uniformly attracted to a fixed point or a closed orbit, or all the orbits have empty omega limit set.*

Proof. It follows immediately from Lemma 3.2, Theorem 2.3 and Theorem 2.5. □

Corollary 3.4. *If f is a UAZS flow on R^n, then either all the orbits go to infinity, or all the orbits are uniformly attracted to a fixed point or a closed orbit. In particular, if $n = 2$, all the orbits tend to a fixed point.*

At last, we consider the case that (X, d) is a compact metric space. Let 2^X be the hyperspace consisting of all closed nonempty subsets of X; it is also a compact metric space under the Hausdorff metric H_d, see [**5**]. If f is a UAZS flow, by the compactness of X, one can take a commom $\delta > 0$ that is suitable for every orbit $x \cdot R$ as in Definition 1. Denote by $\mathcal{C}(X)$ the set of all minimal sets of the flow f. Thus, for A and B in $\mathcal{C}(X)$, if $A \neq B$, from Corollary 2 we have $A \cap B = \varnothing$ and also $d(A, B) = \inf\{d(a, b) \,|\, a \in A \text{ and } b \in B\} > \delta$. So $H_d(A, B) > \delta$; it follows that $\mathcal{C}(X)$ is a discrete subset of 2^X and each pair of its members is δ-apart. By the compactness of 2^X, we get that $\mathcal{C}(X)$ is finite and also conclude:

Theorem 3.5. *If (X, d) is a compact metric space and f is a UAZS flow on X, then the sum of fixed points and closed orbits is finite. Further, each component of X contains a unique fixed point or closed orbit.*

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