ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 4, 2005

COHEN-MACAULAY DIMENSION OF MODULES OVER NOETHERIAN RINGS

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ABSTRACT. We extend a criterion of Gerko for a ring to be Cohen-Macaulay to arbitrary, not necessarily local, Noetherian rings. Our version reads as follows: The Noetherian ring R is Cohen-Macaulay if and only if, for all finitely generated R-modules M, $\text{CM-dim}_R M$ is finite.

1. Introduction. There are many important homological dimensions, defined for finitely generated module M over a commutative Noetherian ring R. The classic one is projective dimension P-dim, which characterizes regular rings by a famous result of Auslander, Buchbaum and Serre. Another dimension corresponding to the complete intersection property of ring is defined by Avramov, Gasharov and Peeva [**4**] and is denoted by CI-dim. Gerko also defined a dimension which reflects the complete intersection property of the ring called polynomial complete intersection dimension and denoted PCI-dim [**7**]. Oana Veliche [**9**] called it lower complete intersection dimension and used notion CI_*-dim to denote it. The notion of G-dimension was introduced by Auslander and Bridge, denoted G-dim, and has some relation to the Gorenstein property of R [**1**]. There is another dimension, defined by Veliche, called upper Gorenstein dimension or G^*-dimension, denoted G^*-dim that characterizes Gorenstein local rings. Dimension which reflects Cohen-Macaulay property of rings is defined also by Gerko, called Cohen-Macaulay dimension and denoted CM-dim [**7**].

Putting them together and using the same terminology as in [**9**], we have notions of homological dimensions of finitely generated module M, denoted $\text{H-dim}_R M$ for H=P, CI, CI_*, G, G^* or CM. We say that, not necessary local, ring R has property (H) with H=P, (respectively,

2000 AMS *Mathematics Subject Classification.* Primary 13C15, 13C14, Secondary 13H05, 13D22.
Key words and phrases. Cohen-Macaulay ring, Cohen-Macaulay dimension.
This research was in part supported by a grant from IPM (No. 81130030 and No. 81130117).
Received by the editors on July 15 2002, and in revised form on March 17, 2003.

CI, CI_*, G, G^* or CM), where R is regular (respectively, complete intersection, complete intersection, Gorenstein, Gorenstein or Cohen-Macaulay).

In case R is local, a feature common to all these dimensions states that ring R has property H if and only if $\text{H-dim}_R M$ is finite for every finitely generated R-module M. Case H=P is classical. For a proof of the result, see [**4**, 1.3] (respectively, [**7**, 2.5], [**1**, 4.20], [**9**, 2.7] or [**7**, 3.9]) when H=CI (respectively, CI_*, G, G^* or CM).

When H=G, Goto has generalized the above result to the nonlocal case [**8**]. An extended version, when H=CI has been proved by Sega [**3**, 6.2]. The main aim of this note is to extend the above result to an assertion about arbitrary Noetherian ring, when H=CM. The main theorem can be stated as follows:

Theorem 1.1. *Let R be a commutative Noetherian ring. The following are equivalent*:

i) *The ring R has property* H;

ii) $\text{H-dim}_R M$ *is finite for every finitely generated R-module M*;

iii) $\text{H-dim}_R R/I$ *is finite for every ideal I of R*;

iv) $\text{H-dim}_R R/\mathfrak{m}$ *is finite for every maximal ideal* $\mathfrak{m} \in \operatorname{Max}(R)$.

Our extension of H-dimension is as follows:

Definition 1.2. Let M be a finitely generated R-module. We define the H-dimension of M by

$$\text{H-dim}_R M = \sup\{\text{H-dim}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in \operatorname{Supp}_R(M)\}.$$

Note that, by [**7**], in case R is local this definition is compatible with the original one. We prepare the ground for the proof of Theorem 1.1 by introducing a new invariant related to any finitely generated R-module M, called restricted dimension of M. Section 2 is devoted to the study of this dimension. We show that it is well behaved on short exact sequences, in the sense that if two terms of a short exact sequence have finite restricted dimension then so does the third.

Moreover it will be shown that it is a refitment of any of the above-mentioned homological dimensions. As a corollary of this, we get, for any ideal I of R and any finitely generated R-module M, an inequality $\operatorname{grade}(I,R) \leq \text{H-dim}_R M + \operatorname{grade}(I,M)$, which will be used in proving Theorem 1.1. Throughout the paper R is a commutative and Noetherian ring and M is a finitely generated R-module.

2. Restricted dimension. In this section we introduce and study a new homological dimension assigning to any finitely generated R-module.

Definition 2.1. Let M be a finitely generated R-module. We define the restricted dimension $\text{r-dim}_R M$ of M by

$$\text{r-dim}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

The restricted dimension is often finite. Moreover, it follows from [**2**, Theorem 2.4] that $\text{r-dim}_R M \leq \dim R$ for all R-module M. In [**6**], this invariant is introduced and called (large) restricted flat dimension. In the following we summarize some basic properties of the restricted dimension.

Proposition 2.2. *Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of R-modules. If two of them have finite restricted dimension, then so does the third.*

Proof. Suppose, for instance, that $\text{r-dim}_R M$ and $\text{r-dim}_R M'$ are finite. So there exists an integer t such that for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq t \quad \text{and} \quad \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M'_{\mathfrak{p}} \leq t.$$

On the other hand, it is easy, using the long exact sequence of 'Ext' modules for instance, to see that, for any $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$\operatorname{depth}_{R_{\mathfrak{p}}} M''_{\mathfrak{p}} \geq \operatorname{Max}\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, \operatorname{depth}_{R_{\mathfrak{p}}} M'_{\mathfrak{p}}\} - 1.$$

Therefore

$$\begin{aligned}\operatorname{depth} R_{\mathfrak{p}} &- \operatorname{depth}_{R_{\mathfrak{p}}} M''_{\mathfrak{p}} \\ &\leq \operatorname{depth} R_{\mathfrak{p}} - \operatorname{Max}\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, \operatorname{depth}_{R_{\mathfrak{p}}} M^{\omega}_{\mathfrak{p}}\} + 1\end{aligned}$$

The result now follows, in this case by taking supremum on both end. The other cases can be proved by using a similar argument. □

Proposition 2.3. *Let M be a finitely generated R-module. Let $\underline{x} = x_1, \dots, x_n$ be an R-regular sequence such that $\underline{x}M = 0$. Then*

$$\text{r-dim}_{\bar{R}}M = \text{r-dim}_R M - n,$$

where $\bar{R} = R/(\underline{x})$.

Proof. Let $\mathfrak{p}$ be a prime ideal in $\mathrm{Supp}_R(M)$. It follows from the isomorphism

$$\mathrm{Ext}^n_{R_\mathfrak{p}}\left(\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}, M_\mathfrak{p}\right) \cong \mathrm{Ext}^n_{\overline{R}_{\overline{\mathfrak{p}}}}\left(\frac{\overline{R}_{\overline{\mathfrak{p}}}}{\overline{\mathfrak{p}}\overline{R}_{\overline{\mathfrak{p}}}}, M_{\overline{\mathfrak{p}}}\right)$$

that $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \mathrm{depth}_{\overline{R}_{\overline{\mathfrak{p}}}} M_{\overline{\mathfrak{p}}}$, where $\overline{\mathfrak{p}}$ denotes $\mathfrak{p}/(\underline{x})$. Also we have

$$\mathrm{depth}\, R_\mathfrak{p} = \mathrm{depth}\, \overline{R}_{\overline{\mathfrak{p}}} + r.$$

Hence

$$\begin{aligned}\sup\{&\mathrm{depth}\, \overline{R}_{\overline{\mathfrak{p}}} - \mathrm{depth}_{\overline{R}} M_{\overline{\mathfrak{p}}} : \mathfrak{p} \in \mathrm{Supp}_{\bar{R}}(M)\}\\ &= \sup\{\mathrm{depth} R_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in \mathrm{Supp}_R(M)\} - r. \ \square\end{aligned}$$

Proposition 2.4. *Let I be an ideal of R and x an R/I-regular element. Then*

$$\text{r-dim}_R R/I = \mathrm{Max}\,\{\text{r-dim}_R(R/(I + xR)) + 1, \text{r-dim}_R(R_x/IR_x)\}.$$

Proof. Set $A := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I,\ x \in \mathfrak{p}\}$ and $B := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I,\ x \notin \mathfrak{p}\}$. So $\sup\{\mathrm{depth} R_\mathfrak{p} - \mathrm{depth}(R/I)_\mathfrak{p} : \mathfrak{p} \supseteq I\}$ is equal to the maximum of the following two numbers $\sup\{\mathrm{depth} R_\mathfrak{p} - \mathrm{depth}(R/I)_\mathfrak{p} : \mathfrak{p} \in A\}$ and $\sup\{\mathrm{depth} R_\mathfrak{p} - \mathrm{depth}(R/I)_\mathfrak{p} : \mathfrak{p} \in B\}$.

For $\mathfrak{p} \in A$, since x is a nonzero divisor over $R_\mathfrak{p}/IR_\mathfrak{p}$,

$$\operatorname{depth} \frac{R_\mathfrak{p}}{(I+Rx)R_\mathfrak{p}} = \operatorname{depth} \frac{R_\mathfrak{p}}{IR_\mathfrak{p}} - 1.$$

For $\mathfrak{p} \in B$, $R_\mathfrak{p} \cong (R_x)_{\mathfrak{p}R_x}$, so we have $\operatorname{depth} R_\mathfrak{p} = \operatorname{depth} (R_x)_{\mathfrak{p}R_x}$ and $\operatorname{depth} (R/I)_\mathfrak{p} = \operatorname{depth} (R_x/IR_x)_{\mathfrak{p}R_x}$. The result now is clear. □

Proposition 2.5. *For any R-module M, there exists an inequality*

$$\text{r-dim}_R M \leq \text{H-dim}_R M$$

with equality when $\text{H-dim}_R M$ *is finite.*

Proof. It suffices to assume that $\text{H-dim}_R M$ is finite. Since, for any $\mathfrak{q} \in \operatorname{Supp}_R(M)$, $\text{H-dim}_{R_\mathfrak{q}} M_\mathfrak{q}$ is less than or equal to $\text{H-dim}_R M$, it is also finite and hence by [**9**, 1.5] is equal to $\operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q}$. So the result follows. □

Proposition 2.6. *If I is an ideal of R and M a finitely generated R-module, then*

$$\operatorname{grade}(I,R) - \operatorname{grade}(I,M) \leq \text{r-dim}_R M.$$

In particular,

$$\operatorname{grade}(I,R) - \operatorname{grade}(I,M) \leq \text{H-dim}_R M.$$

Proof. By [**5**, 1.2.10(i)], there is the equality

$$\operatorname{grade}(I,M) = \inf\{\operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} \mid I \subseteq \mathfrak{q} \text{ with } \mathfrak{q} \in \operatorname{Spec}(R)\}.$$

Let $\mathfrak{p}$ be a prime containing I such that $\operatorname{grade}(I,M) = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$. If $\mathfrak{p} \notin \operatorname{Supp}_R(M)$, then $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \infty$ and the inequality holds. Suppose that $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Since, by [**5**, 1.2.10(i)],

$$\operatorname{grade}(I,R) = \inf\{\operatorname{depth} R_\mathfrak{q} \mid I \subseteq \mathfrak{q} \text{ with } \mathfrak{q} \in \operatorname{Spec}(R)\}$$

the inequality grade $(I, R) \leq \operatorname{depth} R_{\mathfrak{p}}$ holds for the chosen $\mathfrak{p}$. Therefore

$$\begin{aligned}\operatorname{grade}(I,R)-\operatorname{grade}(I,M) &\leq \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \sup\{\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \mid \ \mathfrak{q} \in \operatorname{Supp}_R(M)\}.\end{aligned}$$

The last assertion follows immediately from the previous proposition. □

3. Proof of theorem. We are now in a position to put all the various results of Section 2 together to produce a proof of the main theorem of this paper. The idea for the proof is motivated by the Goto's proof of [**8**, Theorem 1]. So the reader is referred to that paper for the proof of similar steps.

Proof of Theorem 1.1. (i) $\Rightarrow$ (iii). Let I be an ideal of R. Since R has property H, for any prime $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ has the same property. So by [**9**, 1.7], $\text{H-dim}_{R_{\mathfrak{p}}}(R/I)_{\mathfrak{p}}$ is finite and hence, by [**9**, 1.5], it is equal to $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(R/I)_{\mathfrak{p}}$. So, in fact, we should show that $\text{r-dim}_R R/I = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(R/I)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ is finite. Suppose to the contrary that there exists an ideal I of R such that $\text{r-dim}_R R/I$ is not finite. Using the Noetherian property of R, choose I to be maximal among such counterexamples. By the same argument as in [**8**], one can prove that I has to be prime.

Let $n = \operatorname{ht}_R I$ be the height of I. So $\operatorname{grade}(I, R) = n$ as R is Cohen-Macaulay. Let $\underline{x} = x_1, \dots, x_n$ be a maximal R-sequence in I. By Proposition 2.3, we may pass through $\overline{R} = R/(x_1, \dots, x_n)$ and reduce the problem to the case that $\operatorname{ht}_R I = 0$. Let $\operatorname{Min} R$ denote the set of all minimal primes of R. Choose $x \in \cap_{\mathfrak{p} \in \operatorname{Min} R \setminus \{I\}} \mathfrak{p} \setminus I$. Using Proposition 2.4, after localizing at x and passing through R_x, we may assume that $\operatorname{Min} R = \{I\}$. Now, by the same argument as in [**8**, Theorem 1], we can see that R/I is Cohen-Macaulay.

Let $\mathfrak{p}$ be a prime ideal of R. So $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ is a Cohen-Macaulay $R_{\mathfrak{p}}$-module with $\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$, and hence $\operatorname{depth} R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$, which implies that

$$\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}) : \mathfrak{p} \in \operatorname{Spec}(R)\} = 0.$$

This is the desired contradiction, which completes the proof.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). Let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Since $\text{H-dim}_R R/\mathfrak{m}$ is finite, $\text{H-dim}_R R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ is finite, and so by [**9**, 1.7], $R_\mathfrak{m}$ has property H, which concludes the result, as $\mathfrak{m}$ was arbitrary.

(ii) $\Leftrightarrow$ (iii). Only the 'if' part needs proof. We prove it, using induction on the numbers of generators of M. Let M be cyclic. So it is isomorphic to R/I, for an ideal I of R, and hence the result is clear in this case. Suppose M is generated by n elements, where $n > 1$, and the result is true for all modules which can be generated by less than n elements. By the equivalence (i) $\Leftrightarrow$ (iii), it is enough to show that $\text{r-dim}_R M$ is finite. To this end, consider the exact sequence $0 \to L \to M \to N \to 0$, where L and N are generated by less than n elements and use induction assumption in conjunction with Proposition 2.2.

Corollary 3.1. *The Noetherian ring R is Cohen-Macaulay if and only if, for any finitely generated R-module M and any ideal I of R,*

$$\operatorname{ht} I \leq \operatorname{grade}(I, M) + \text{CM-dim}_R M.$$

Proof. For the 'if' part it is enough to put $M = R$ and use the fact that $\text{CM-dim}_R R = 0$. The 'only if' part is a consequence of Proposition 2.6 in view of the fact that, for any R-module M, $\text{CM-dim}_R M \leq \text{H-dim}_R M$. □

Acknowledgments. The authors are deeply grateful to the referee for several useful pointed comments on the paper.

REFERENCES

1. M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. **94** (1969).

2. M. Auslander and D. A. Buchsbaum, *Homological dimension in Noetherian rings*, II, Trans. Amer. Math. Soc. **88** (1958), 194–206.

3. L.L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318.

4. L.L. Avramov, V.N. Gasharov and I.V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. **86** (1997), 67–114.

5. W. Bruns and J. Herzog, *Cohen-Macaulay rings* (rev. ed.), Univ. Press Cambridge, Cambridge, 1996.

6. L.W. Christensen, H.-B. Foxby and A. Frankild, *Restricted homological dimensions and Cohen-Macaulayness*, J. Algebra **251** (2002), 479–502.

7. A.A. Gerko, *On homological dimensions*, Sb. Math. **192** (2001), 1165–1179.

8. Sh. Goto, *Vanishing of* $Ext^i_A(M,A)$, J. Math. Kyoto Univ. **22** (1982/83), 481–484.

9. O. Veliche, *Construction of modules with finite homological dimensions*, J. Algebra **250** (2002), 427–499.

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