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CLASSIFICATION OF 3-DIMENSIONAL ISOLATED RATIONAL HYPERSURFACE SINGULARITIES WITH $\mathbf{C}^*$-ACTION

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1. Introduction. In [**2**] Artin first introduced the definition of rational surface singularity. He classified all rational surface singularities embeddable in $\mathbf{C}^3$. These are precisely those Du Val singularities in $\mathbf{C}^3$ defined by one of the following polynomial equations:

$$\begin{aligned} A_n:&\quad x^2+y^2+z^{n+1}, &&\text{for } n\geq 1\\ D_n:&\quad x^2+y^2z+z^{n-1}, &&\text{for } n\geq 4\\ E_6:&\quad x^2+y^3+z^4\\ E_7:&\quad x^2+y^3+yz^3\\ E_8:&\quad x^2+y^3+z^5. \end{aligned}$$

It is well known that any canonical singularity, i.e., singularity that occurs in a canonical model of a surface of general type, is analytically isomorphic to one of the rational double points listed above.

In [**3**] Burns defined higher dimensional rational singularity as follows. Let (V,p) be an n-dimensional isolated singularity. Let $\pi: M\to V$ be a resolution of singularity. And p is said to be a rational singularity if $R^i\pi_*\mathcal{O}_M=0$ for $1\leq i\leq n-1$. In [**14**], Yau shows for Gorenstein singularity that it is sufficient to require $R^{n-1}\pi_*\mathcal{O}_M=0$. He [**14**] proves that $R^{n-1}\pi_*\mathcal{O}_M\cong H^0(V-\{p\},\Omega^n)/L^2(V-\{p\},\Omega^n)$ where Ω^n is the sheaf of germs of holomorphic n-forms and $L^2(V-\{p\},\Omega^n)$ is the space of holomorphic n forms on $V-\{p\}$ which are L^2-integrable. The geometric genus p_g of the singularity (V,p) is defined to be

$$p_g:=\dim R^{n-1}\pi_*\mathcal{O}_M=\dim H^0(V-\{p\},\Omega^n)/L^2(V-\{p\},\Omega^n).$$

It turns out that p_g is an important invariant of (V,p).

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In [**16**], we give algebraic classification of rational CR structures on the topological 5-sphere with transversal holomorphic S^1-action in $\mathbf{C}^4$. Here, algebraic classification of compact strongly pseudoconvex CR manifolds X means classification up to algebraic equivalence, i.e., roughly up to isomorphism of the normalization of the complex analytic variety V which has X as boundary. The problem is intimately related to the study of three dimensional isolated rational weighted homogeneous hypersurface singularities with link homeomorphic to S^5. For this, we need the classification of three-dimensional isolated rational hypersurface singularities with a $\mathbf{C}^*$-action. This list is only available at the home page of one of us. Since there is a desire for a complete list of this classification, cf. Theorem 3.3, we decided to publish it for the convenience of readers.

The idea of our proof is very easy. If $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial in $\mathbf{C}^4$ and $V = \{z \in \mathbf{C}^4 : h(z) = 0\}$ has an isolated singularity at the origin, then Kouchnirenko [**6**] and Orlik-Randell [**9**] observed that V can be deformed into one of the 19 classes of weighted homogeneous singularities listed in Section 2 while keeping the differential structure of the link $K_V := S^7 \cap V$ constant. We prove that the above deformation is actually a deformation that preserves weights and embedded topological type. By a theorem of Merle-Teissier [**8**], the geometric genus p_g of the singularity can be expressed in terms of its weights. The MAPLE program [**4**] helps us to finish the classification. In fact, if we use the similar method as above, we can also classify the weighted homogenous rational surface singularities embeddable in $\mathbf{C}^3$, which are exactly the A_n, D_n, E_6, E_7, and E_8 singularities described above.

In Section 2, we shall give a classification (up to deformation which preserves weights) of weighted homogeneous polynomials of four variables with isolated singularity at the origin. This list was obtained first by Kouchnirenko [**6**] and Orlik-Randell [**9**], see also [**5**], independently. In Section 3, we classify all three-dimensional isolated rational hypersurface singularities with $\mathbf{C}^*$-action.

2. Classification of weighted homogeneous polynomials in four variables with isolated singularity at the origin. Orlik and Wagreich [**10**] and Arnold [**1**] showed that if $h(z_0, z_1, z_2)$ is a weighted homogeneous polynomial in $\mathbf{C}^3$ and $V = \{z \in \mathbf{C}^3 : h(z) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the seven classes of weighted homogeneous singularities (Table 1) while keeping the differential structure of the link $K_V := S^5 \cap V$ constant. Let $(w_0, w_1, w_2) = (wt(z_0), wt(z_1), wt(z_2))$ be the weight type and μ the Milnor number.

Recall that two isolated hypersurface singularities $(V, 0), (W, 0)$ in $\mathbf{C}^{n+1}$ are said to have the same topological type if $(\mathbf{C}^{n+1}, V, 0)$ is homeomorphic to $(\mathbf{C}^{n+1}, W, 0)$, cf., [**13**].

In [**11**], we prove that the above deformation is actually a deformation that preserves weights and embedded topological type. Therefore any weighted homogeneous singularity has the same topological type of one of the seven classes above.

If $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial in $\mathbf{C}^4$ and $V = \{z \in \mathbf{C}^4 : h(z) = 0\}$ has an isolated singularity at the origin, then Kouchnirenko [**6**] and Orlik and Randell [**9**] observed that V can be deformed into one of the following 19 classes of weighted homogeneous singularities (Table 2) while keeping the differential structure of the link $K_V := S^7 \cap V$ constant. Let $w_i = wt(z_i)$ and μ be the Milnor number. The meaning of the linear forms $\alpha(x, y, z, w)$ in the list will be explained later, cf., the proof of Theorem 3.3.

Theorem 2.1. *Suppose $h(z_0, z_1, z_2, z_3)$ is a polynomial and $V_k = \{(z_0, z_1, z_2, z_3) \in \mathbf{C}^4 : h(z_0, z_1, z_2, z_3) = 0\}$ has an isolated singularity at* 0*. Then $h(z_0, z_1, z_2, z_3) = f(z_0, z_1, z_2, z_3) + g(z_0, z_1, z_2, z_3)$ where f is one of the* 19 *classes above with only an isolated singularity at* 0 *and f and g have no monomial in common. If h is weighted homogeneous of type (w_0, w_1, w_2, w_3), then so are f and g. Let $V_f = \{(z_0, z_1, z_2, z_3) \in \mathbf{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$, and let*

$$K_f = V_f \cap S^7, \quad K_h = V_h \cap S^7.$$

Then K_f is equivariantly diffeomorphic to K_h.

Proof. If none of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ appears in $h(z_0, z_1, z_2, z_3)$, then $\frac{\partial h}{\partial z_j}(z_0, 0, 0, 0) = 0$, $0 \leq j \leq 3$. This contradicts the fact that h has an isolated singularity at 0. Therefore, one of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ appears in h. Similarly one of the monomials in each of the following

TABLE 1.

Class I $\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$, $(w_0, w_1, w_2) = (a_0, a_1, a_2)$, $\mu = (a_0 - 1)(a_1 - 1)(a_2 - 1)$.

Class II $\{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$, $(w_0, w_1, w_2) = \left(a_0, a_1, \dfrac{a_1 a_2}{a_1 - 1}\right)$, $\mu = (a_0 - 1)(a_1 a_2 - a_1 + 1)$.

Class III $\{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}$, $(w_0, w_1, w_2) = \left(a_0, \dfrac{a_1 a_2 - 1}{a_2 - 1}, \dfrac{a_1 a_2 - 1}{a_1 - 1}\right)$, $\mu = (a_0 - 1) a_1 a_2$.

Class IV $\{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$, $(w_0, w_1, w_2) = \left(a_0, \dfrac{a_0 a_1 a_2}{a_0 a_2 - a_0 + 1}, \dfrac{a_0 a_2}{a_0 - 1}\right)$,
$\mu = a_0 a_2 (a_1 - 1) + a_0 - 1$.

Class V $\{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$, $(w_0, w_1, w_2) = \left(\dfrac{a_0 a_1 a_2 + 1}{a_1 a_2 - a_2 + 1}, \dfrac{a_0 a_1 a_2 + 1}{a_0 a_2 - a_0 + 1}, \dfrac{a_0 a_1 a_2 + 1}{a_0 a_1 - a_1 + 1}\right)$,
$\mu = a_0 a_1 a_2$.

Class VI $\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, where $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$,
$(w_0, w_1, w_2) = \left(a_0, \dfrac{a_0 a_1}{a_0 - 1}, \dfrac{a_0 a_2}{a_0 - 1}\right)$, $\mu = \dfrac{(a_0 a_1 - a_0 + 1)(a_0 a_2 - a_0 + 1)}{a_0 - 1}$.

Class VII $\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, where $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2 (a_0 a_1 - 1)$,
$(w_0, w_1, w_2) = \left(\dfrac{a_0 a_1 - 1}{a_1 - 1}, \dfrac{a_0 a_1 - 1}{a_0 - 1}, \dfrac{a_2 (a_0 a_1 - 1)}{a_1 (a_0 - 1)}\right)$, $\mu = \dfrac{a_0 (a_0 a_1 a_2 - a_0 a_1 + a_1 - a_2)}{a_0 - 1}$.

sets appears in h : $\{z_0z_1^{a_1}, z_1^{a_1}, z_1^{a_1}z_2, z_1^{a_1}z_3\}$, $\{z_0z_2^{a_2}, z_1z_2^{a_2}, z_2^{a_2}, z_2^{a_2}z_3\}$, $\{z_0z_3^{a_3}, z_1z_3^{a_3}, z_2z_3^{a_3}, z_3^{a_3}\}$. Taking a monomial from each of the 4 sets above, we get 256 polynomials. One can check that these 256 polynomials are equivalent to one of the 19 classes above up to permutation of coordinates. Notice that in Type VIII, for example, the monomial $z_2^pz_3^q$ is added to make sure that f has an isolated singularity at 0. Obviously if h is weighted homogeneous of type (w_0, w_1, w_2, w_3), then so are f and g.

The proof of Theorem 3.1.4 in [**10**] shows that K_f is equivariantly diffeomorphic to K_h. $\square$

We shall use the theory developed in [**11**] and [**12**] to show that $(V_f, 0)$ and $(V_h, 0)$ have the same embedded topological type.

Definition. Given a real manifold B of dimension m and a family $\{(M_t, N_t) : t \in B,\ N_t$ is a closed submanifold of a compact differentiable manifold $M_t\}$, we say that (M_t, N_t) depends C^∞ on t and that $\{(M_t, N_t) : t \in B\}$ is a C^∞ family of compact manifolds with submanifolds, if there is a C^∞ manifold $\mathcal{M}$, a closed submanifold $\mathcal{N}$ and a C^∞ map w from $\mathcal{M}$ onto B such that $\bar{w} := w|\mathcal{N}$ is also a C^∞ map from $\mathcal{N}$ onto B satisfying the following conditions

(i) $M_t = w^{-1}(t) \supseteq N_t = \bar{w}^{-1}(t)$.

(ii) The rank of the Jacobian of w, respectively $\bar{w}$, is equal to m at every point of $\mathcal{M}$, respectively $\mathcal{N}$.

Theorem 2.2 (e.g. [**11**]). *Let $((\mathcal{M}, \mathcal{N}), (w, \bar{w}))$ be a C^∞ family of compact manifolds with submanifolds, with B connected. Then $(M_t, N_t) = (w^{-1}(t), \bar{w}^{-1}(t))$ is diffeomorphic to (M_{t_0}, N_{t_0}) for any $t, t_0 \in B$.*

Now we fix weights $(w_0, \dots, w_n)$ with $w_i \geq 2$. Suppose that there is a weighted homogeneous polynomial $f(z_0, \dots, z_n)$ with the weights $(w_0, \dots, w_n)$ such that f has an isolated singularity at the origin. Let Δ be the intersection of the plane

$$\sum_{i=0}^{n} \frac{x_i}{w_i} = 1$$

TABLE 2.

Type I $\{z_0^{a_0}+z_1^{a_1}+z_2^{a_2}+z_3^{a_3}=0\}$, $(w_0,w_1,w_2,w_3)=(a_0,a_1,a_2,a_3)$,

$$\mu=(a_0-1)(a_1-1)(a_2-1)(a_3-1),\ \ \alpha(x,y,z,w)=\frac{x}{a_0}+\frac{y}{a_1}+\frac{z}{a_2}+\frac{w}{a_3}.$$

Type II $\{z_0^{a_0}+z_1^{a_1}+z_2^{a_2}+z_2z_3^{a_3}=0\}$, $(w_0,w_1,w_2,w_3)=\left(a_0,a_1,a_2,\frac{a_2a_3}{a_2-1}\right)$,

$$\mu=(a_0-1)(a_1-1)[a_2(a_3-1)+1],\ \ \alpha(x,y,z,w)=\frac{x}{a_0}+\frac{y}{a_1}+\frac{z}{a_2}+\frac{(a_2-1)w}{a_2a_3}.$$

Type III $\{z_0^{a_0}+z_1^{a_1}+z_2^{a_2}z_3+z_2z_3^{a_3}=0\}$, $(w_0,w_1,w_2,w_3)=\left(a_0,a_1,\frac{a_2a_3-1}{a_3-1},\frac{a_2a_3-1}{a_2-1}\right)$,

$$\mu=(a_0-1)(a_1-1)a_2a_3,\ \ \alpha(x,y,z,w)=\frac{x}{a_0}+\frac{y}{a_1}+\frac{(a_3-1)z}{a_2a_3-1}+\frac{(a_2-1)w}{a_2a_3-1}.$$

Type IV $\{z_0^{a_0}+z_0z_1^{a_1}+z_2^{a_2}+z_2z_3^{a_3}=0\}$, $(w_0,w_1,w_2,w_3)=\left(a_0,\frac{a_0a_1}{a_0-1},a_2,\frac{a_2a_3}{a_2-1}\right)$,

$$\mu=[a_0(a_1-1)+1][a_2(a_3-1)+1],\ \ \alpha(x,y,z,w)=\frac{x}{a_0}+\frac{(a_0-1)y}{a_0a_1}+\frac{z}{a_2}+\frac{(a_2-1)w}{a_2a_3}.$$

Type V $\{z_0^{a_0}z_1+z_0z_1^{a_1}+z_2^{a_2}+z_2z_3^{a_3}=0\}$, $(w_0,w_1,w_2,w_3)=\left(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},a_2,\frac{a_2a_3}{a_2-1}\right)$,

$$\mu=a_0a_1[a_2(a_3-1)+1],\ \ \alpha(x,y,z,w)=\frac{(a_1-1)x}{a_0a_1-1}+\frac{(a_0-1)y}{a_0a_1-1}+\frac{z}{a_2}+\frac{(a_2-1)w}{a_2a_3}.$$

TABLE 2. (CONT'D.)

Type VI $\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_2^{a_2}z_3 + z_2z_3^{a_3} = 0\}$,

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2a_3-1}{a_3-1}, \frac{a_2a_3-1}{a_2-1}\right),$$

$$\mu = a_0a_1a_2a_3, \quad \alpha(x,y,z,w) = \frac{(a_1-1)x}{a_0a_1-1} + \frac{(a_0-1)y}{a_0a_1-1} + \frac{(a_3-1)z}{a_2a_3-1} + \frac{(a_2-1)w}{a_3a_2-1}.$$

Type VII $\{z_0^{a_0} + z_1^{a_1} + z_1z_2^{a_2} + z_2z_3^{a_3} = 0\}$, $\quad (w_0,w_1,w_2,w_3) = \left(a_0, a_1, \frac{a_1a_2}{a_1-1}, \frac{a_1a_2a_3}{a_1(a_2-1)+1}\right)$,

$$\mu = (a_0-1)[a_1a_2(a_3-1) + a_1 - 1], \quad \alpha(x,y,z,w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{(a_1-1)z}{a_1a_2} + \frac{[a_1(a_2-1)+1]w}{a_1a_2a_3}.$$

Type VIII $\left\{z_0^{a_0} + z_1^{a_1} + z_1z_2^{a_2} + z_1z_3^{a_3} + z_2^pz_3^q = 0,\ \frac{p(a_1-1)}{a_1a_2} + \frac{q(a_1-1)}{a_1a_3} = 1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, a_1, \frac{a_1a_2}{a_1-1}, \frac{a_1a_3}{a_1-1}\right),$$

$$\mu = \frac{(a_0-1)[a_1(a_2-1)+1][a_1(a_3-1)+1]}{a_1-1}, \quad \alpha(x,y,z,w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{(a_1-1)z}{a_1a_2} + \frac{(a_1-1)w}{a_1a_3}.$$

Type IX $\left\{z_0^{a_0} + z_1^{a_1}z_3 + z_2^{a_2}z_3 + z_1z_3^{a_3} + z_1^pz_2^q = 0,\ \frac{p(a_3-1)}{a_1a_3-1} + \frac{qa_1(a_3-1)}{a_2(a_1a_3-1)} = 1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_1a_3-1}{a_3-1}, \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}, \frac{a_1a_3-1}{a_1-1}\right),$$

$$\mu = \frac{(a_0-1)a_3[a_2(a_1a_3-1) - a_1(a_3-1)]}{a_3-1},$$

$$\alpha(x,y,z,w) = \frac{x}{a_0} + \frac{(a_3-1)y}{a_1a_3-1} + \frac{a_1(a_3-1)z}{a_2(a_1a_3-1)} + \frac{(a_1-1)w}{a_1a_3-1}.$$

TABLE 2. (CONT'D.)

Type X $\{z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_1 z_3^{a_3} = 0\}$,

$$(w_0, w_1, w_2, w_3) = \left(a_0, \frac{a_1a_2a_3+1}{a_3(a_2-1)+1}, \frac{a_1a_2a_3+1}{a_1(a_3-1)+1}, \frac{a_1a_2a_3+1}{a_2(a_1-1)+1}\right),$$

$\mu = (a_0 - 1)a_1a_2a_3$,

$$\alpha(x,y,z,w) = \frac{x}{a_0} + \frac{[a_3(a_2-1)+1]y}{a_1a_2a_3+1} + \frac{[a_1(a_3-1)+1]z}{a_1a_2a_3+1} + \frac{[a_2(a_1-1)+1]w}{a_1a_2a_3+1}.$$

Type XI $\{z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} = 0\}$,

$$(w_0, w_1, w_2, w_3) = \left(a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)}\right),$$

$\mu = a_0a_1a_2(a_3 - 1) + a_0(a_1 - 1) + 1$,

$$\alpha(x,y,z,w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0a_1} + \frac{[a_0(a_1-1)+1]z}{a_0a_1a_2} + \frac{[a_0a_1(a_2-1)+(a_0-1)]w}{a_0a_1a_2a_3}.$$

Type XII $\left\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1 z_3^{a_3} + z_1^p z_2^q = 0,\ \frac{p(a_0-1)}{a_0a_1} + \frac{q(a_0-1)}{a_0a_2} = 1\right\}$,

$$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_2}{a_0-1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1}),$$

$$\mu = \frac{(a_0(a_2-1)+1)(a_0a_1(a_3-1)+a_0-1)}{a_0-1},$$

$$\alpha(x,y,z,w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0a_1} + \frac{(a_0-1)z}{a_0a_2} + \frac{[a_0(a_1-1)+1]w}{a_0a_1a_3}.$$

TABLE 2. CONT'D.

Type XIII $\left\{z_0^{a_0}+z_0z_1^{a_1}+z_1z_2^{a_2}+z_1z_3^{a_3}+z_2^pz_3^q=0,\ \frac{p[a_0(a_1-1)+1]}{a_0a_1a_2}+\frac{q[a_0(a_1-1)+1]}{a_0a_1a_3}=1\right\}$,

$$(w_0,w_1,w_2,w_3)=(a_0,\frac{a_0a_1}{a_0-1},\frac{a_0a_1a_1}{a_0(a_1-1)+1},\frac{a_0a_1a_3}{a_0(a_1-1)+1}),$$

$$\mu=\frac{[a_0a_1(a_2-1)+a_0-1][a_0a_1(a_3-1)+a_0-1]}{a_0(a_1-1)+1},$$

$$\alpha(x,y,z,w)=\frac{x}{a_0}+\frac{(a_0-1)y}{a_0a_1}+\frac{[a_0(a_1-1)+1]z}{a_0a_1a_2}+\frac{[a_0(a_1-1)+1]w}{a_0a_1a_3}.$$

Type XIV $\left\{z_0^{a_0}+z_0z_1^{a_1}+z_0z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_2^q+z_2^rz_3^s=0,\ \frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=1 \text{ and } \frac{r(a_0-1)}{a_0a_2}+\frac{s(a_0-1)}{a_0a_3}=1\right\}$,

$$(w_0,w_1,w_2,w_3)=\left(a_0,\frac{a_0a_1}{a_0-1},\frac{a_0a_2}{a_0-1},\frac{a_0a_3}{a_0-1}\right),$$

$$\mu=\frac{[a_0(a_1-1)+1][a_0(a_2-1)+1][a_0(a_3-1)+1]}{(a_0-1)^2},$$

$$\alpha(x,y,z,w)=\frac{x}{a_0}+\frac{(a_0-1)y}{a_0a_1}+\frac{(a_0-1)z}{a_0a_2}+\frac{(a_0-1)w}{a_0a_3}.$$

Type XV $\left\{z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_2z_3^{a_3}+z_1^pz_2^q=0,\ \frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)}=1\right\}$,

$$(w_0,w_1,w_2,w_3)=\left(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_2(a_0a_1-1)}{a_1(a_0-1)},\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}\right),$$

TABLE 2. CONT'D.

$$\mu = \frac{a_0[a_2(a_3-1)(a_0a_1-1)+a_1(a_0-1)]}{a_0-1},$$

$$\alpha(x,y,z,w) = \frac{(a_1-1)x}{a_0a_1-1}+\frac{(a_0-1)y}{a_0a_1-1}+\frac{a_1(a_0-1)z}{a_2(a_0a_1-1)}+\frac{[a_2(a_0a_1-1)-a_1(a_0-1)]w}{a_2a_3(a_0a_1-1)}.$$

Type XVI $$\left\{z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_2^q+z_2^rz_3^s=0,\ \frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)}=1 \right.$$

$$\left. \text{and } \frac{ra_1(a_0-1)}{a_2(a_0a_1-1)}+\frac{sa_1(a_0-1)}{a_3(a_0a_1-1)}=1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}\right),$$

$$\mu = \frac{a_0[a_2(a_0a_1-1)-a_1(a_0-1)][a_3(a_0a_1-1)-a_1(a_0-1)]}{a_1(a_0-1)^2},$$

$$\alpha(x,y,z,w) = \frac{(a_1-1)x}{a_0a_1-1}+\frac{(a_0-1)y}{a_0a_1-1}+\frac{a_1(a_0-1)z}{a_2(a_0a_1-1)}+\frac{a_1(a_0-1)w}{a_3(a_0a_1-1)}.$$

Type XVII $$\left\{z_0^{a_0}z_1+z_0z_1^{a_1}+z_1z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_3^q+z_0^rz_2^s=0,\ \frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_3(a_0a_1-1)}=1 \right.$$

$$\left. \text{and } \frac{r(a_1-1)}{a_0a_1-1}+\frac{sa_0(a_1-1)}{a_2(a_0a_1-1)}=1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_0(a_1-1)}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}\right),$$

$$\mu = \frac{[a_2(a_0a_1-1)-a_0(a_1-1)][a_3(a_0a_1-1)-a_1(a_0-1)]}{(a_0-1)(a_1-1)},$$

TABLE 2. (CONT'D.)

$$\alpha(x,y,z,w) = \frac{(a_1-1)x}{a_0a_1-1} + \frac{(a_0-1)y}{a_0a_1-1} + \frac{a_0(a_1-1)z}{a_2(a_0a_1-1)} + \frac{a_1(a_0-1)w}{a_3(a_0a_1-1)}.$$

Type XVIII $\left\{z_0^{a_0}z_2+z_0z_1^{a_1}+z_1z_2^{a_2}+z_1z_3^{a_3}+z_2^pz_3^q=0,\ \frac{p[a_0(a_1-1)+1]}{a_0a_1a_2+1}+\frac{qa_2[a_0(a_1-1)+1]}{a_3(a_0a_1a_2+1)}=1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1a_2+1}{a_1(a_2-1)+1}, \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}, \frac{a_0a_1a_2+1}{a_0(a_1-1)+1}, \frac{a_3(a_0a_1a_2+1)}{a_2[a_0(a_1-1)+1]}\right),$$

$$\mu = \frac{a_0a_1[a_0a_1a_2(a_3-1)+a_2(a_0-1)+a_3]}{a_0(a_1-1)+1},$$

$$\alpha(x,y,z,w) = \frac{[a_1(a_2-1)+1]x}{a_0a_1a_2+1} + \frac{[a_2(a_0-1)+1]y}{a_0a_1a_2+1} + \frac{[a_0(a_1-1)+1]z}{a_0a_1a_2+1} + \frac{a_2[a_0(a_1-1)+1]w}{a_3(a_0a_1a_2+1)}.$$

Type XIX $\{z_0^{a_0}z_2+z_0z_1^{a_1}+z_2^{a_2}z_3+z_1z_3^{a_3}=0\}$,

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1a_2a_3-1}{a_1[a_3(a_2-1)+1]-1}, \frac{a_0a_1a_2a_3-1}{a_3[a_2(a_0-1)+1]-1},\right.$$
$$\left.\frac{a_0a_1a_2a_3-1}{a_0[a_1(a_3-1)+1]-1}, \frac{a_0a_1a_2a_3-1}{a_2[a_0(a_1-1)+1]-1}\right),$$

$$\mu = a_0a_1a_2a_3, \alpha(x,y,z,w) = \frac{[a_1(a_3(a_2-1)+1)-1]x}{a_0a_1a_2a_3-1} + \frac{[a_3(a_2(a_0-1)+1)-1]y}{a_0a_1a_2a_3-1}$$
$$+\frac{[a_0(a_1(a_3-1)+1)-1]z}{a_0a_1a_2a_3-1} + \frac{[a_2(a_0(a_1-1)+1)-1]w}{a_0a_1a_2a_3-1}.$$

with the first quadrant of $\mathbf{R}^{n+1}$. Let $\mathbf{C}[\Delta] = \{f \in \mathbf{C}[z_0, \ldots, z_n] : \operatorname{supp} f \subset \Delta\}$ where $\operatorname{supp} f = \{(d_0, \ldots, d_n) \in \mathbf{R}^{n+1} : z_0^{d_0} z_1^{d_1} \cdots z_n^{d_n}$ occurs in $f\}$. Let N be the number of the integer points which are in Δ. There is a canonical correspondence between $\mathbf{C}[\Delta]$ and $\mathbf{C}^N$. Thus we may introduce a Zariski topology on $\mathbf{C}[\Delta]$.

Theorem 2.3. *Notation as above. Let*

$$U = \{f \in \mathbf{C}[\Delta] : f \textit{ has an isolated singularity at the origin}\}$$

Then U is a nonempty Zariski open set of $\mathbf{C}[\Delta]$.

The proof of the previous theorem as well as the following theorem is the same as those of Theorem 3.4 and Theorem 3.5 in [**11**], respectively.

Theorem 2.4. *Suppose that $f(z_0, \ldots, z_n)$ and $g(z_0, \ldots, z_n)$ are weighted homogeneous polynomials with the same weights $(w_0, w_1, \ldots, w_n)$. If the variety V of f and the variety W of g have an isolated singularity at the origin, then $(\mathbf{C}^{n+1}, V, 0)$ is homeomorphically equivalent to $(\mathbf{C}^{n+1}, W, 0)$.*

Corollary 2.5. *Suppose that $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial with weights (w_0, w_1, w_2, w_3) and the variety $h^{-1}(0)$ has an isolated singularity at the origin. Then $h = f + g$ where f and g have no monomials in common, f is one of the 19 classes above and f and g are weighted homogeneous of type (w_0, w_1, w_2, w_3).*

Moreover $h^{-1}(0)$ and $f^{-1}(0)$ have the same embedded topological type.

3. Three-dimensional isolated rational hypersurface singularities with $\mathbf{C}^*$-action.

Definition 3.1. Let $(V, 0)$ be an n-dimensional variety with isolated singularity at 0. The geometric genus $p_g(V, 0)$ of the singularity is defined to be $\dim H^{n-1}(M, \mathcal{O})$ where M is a resolution of the singularity $(V, 0)$. $(V, 0)$ is called a rational singularity if $p_g(V, 0) = 0$.

Proposition 3.1 [**10**]. *Suppose $V \subseteq \mathbf{C}^{n+1}$ is an irreducible analytic variety, σ is a $\mathbf{C}^*$-action leaving V invariant,*

$$\sigma(t, (z_0, \dots, z_n)) = (t^{q_0} z_0, \dots, t^{q_n} z_n)$$

and $q_i > 0$ for all i. Then V is algebraic and the ideal of polynomials in $\mathbf{C}[z_0, \dots, z_n]$ vanishing on V is generated by weighted homogeneous polynomials.

Let $f(z_0, \dots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $\sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbf{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with vertex at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbf{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbf{C}^* = \mathbf{C} - \{0\}$. The following theorem was proved by Merle and Teissier.

Theorem 3.2 [**8**]. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. Then the geometric genus $p_g(V, 0) = \#\{p \in \mathbf{Z}^{n+1} \cap \Gamma_-(f) :$ p is positive$\}$.*

Now we are ready to give the classification of three-dimensional isolated rational hypersurface singularities with $\mathbf{C}^*$-action.

Theorem 3.3. *Let $(V, 0)$ be a three-dimensional isolated rational hypersurface singularity with $\mathbf{C}^*$-action. Then $(V, 0)$ is defined by a weighted homogeneous polynomial of one of the* 19 *cases of Table* 2 *such that the corresponding linear form α satisfies $\alpha(x, y, z, w) =$ 1. An explicit tabulation of the solutions of $\alpha(x, y, z, w) = 1$ for all cases is given in the online preprint at the Mathematics ArXiv, math.* AG/0303302.

Proof. In view of Corollary 2.5 and Theorem 3.2, it is clear that an isolated rational hypersurface singularity with $\mathbf{C}^*$-action is defined by one of the 19 types in Section 2 with $p_g = 0$. The equations of the Γ_- hyperplanes of these 19 types are respectively given by $\alpha(x, y, z, w) = 1$ in Table 2.

In order to find all hypersurfaces among these 19 types with $p_g = 0$, we only need to find all solutions of $\alpha(1, 1, 1, 1) > 1$ among these 19 types. We have used the MAPLE program [**4**] to perform the computations. The solutions are listed in the online preprint of the theorem.

Remark. The lists in (VIII), (XIII), (XIV), (XVI), (XVII) in Theorem 3.3 may be reduced slightly by change of coordinates.

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