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DISCRETE COCOMPACT SUBGROUPS OF $\mathrm{G}_{5,3}$ AND RELATED C^*-ALGEBRAS

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ABSTRACT. The discrete cocompact subgroups of the five-dimensional Lie group $\mathrm{G}_{5,3}$ are determined up to isomorphism. Each of their group C^*-algebras is studied by determining all of its simple infinite dimensional quotient C^*-algebras. The K-groups and trace invariants of the latter are also obtained.

1. Introduction. Consider the Lie group $\mathrm{G}_{5,3}$ equal to $\mathbf{R}^5$ as a set with multiplication given by

$$\begin{aligned}&(h,j,k,m,n)(h',j',k',m',n')\\&=(h+h'+nj'+m'n(n-1)/2+mk',j+j'+nm',k+k',m+m',n+n'),\end{aligned}$$

and inverse

$$(h,j,k,m,n)^{-1}=(-h+nj+mk-mn(n+1)/2,-j+nm,-k,-m,-n).$$

The group $\mathrm{G}_{5,3}$ is one of only six nilpotent, connected, simply connected, five-dimensional Lie groups; it seemed the most tractable of them for our present purposes. (Our notation is as in Nielsen [**8**], where a detailed catalogue of Lie groups like this one is given.) In [**6**, Section 3] the authors have studied a natural discrete cocompact subgroup $\mathrm{H}_{5,3}$, the lattice subgroup $\mathrm{H}_{5,3}=\mathbf{Z}^5\subset\mathrm{G}_{5,3}$. In Section 2 of this paper we study the group $\mathrm{G}_{5,3}$ more closely, determining the isomorphism classes of all its discrete cocompact subgroups, Theorem 1. These are given by five integer parameters α, β, γ, δ, ε that satisfy certain conditions, see $(*)$ and $(**)$ of Theorem 1, and are denoted by $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\varepsilon)$. It is shown that each such subgroup is isomorphic to a cofinite subgroup of $\mathrm{H}_{5,3}=\mathrm{H}_{5,3}(1,0,1,1,0)$. Conversely, each cofinite subgroup

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of $\mathrm{H}_{5,3} \subset \mathrm{G}_{5,3}$ is a discrete cocompact subgroup of $\mathrm{G}_{5,3}$. In Sections 3 and 4, the group C^*-algebras of the $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$'s are examined by obtaining their simple infinite dimensional quotients. Some of these are shown to be crossed products of certain types of Heisenberg C^*-algebras (in Packer's terminology [**11**]) and the rest are matrix algebras over irrational rotation algebras, Theorem 5. In Section 5 the K-groups of the simple quotients are calculated, Theorem 6, as are their trace invariants, Theorem 8. The paper ends with a discussion of the classification of the simple quotients.

We use one of the conventional notations for crossed products as in, for example, [**12**] or [**19**]. Hence, if a discrete group G acts on a C^*-algebra A, we write $C^*(A, G)$ to denote the associated C^*-crossed product algebra. We use a similar notation for twisted crossed products, i.e., when there is a cocycle instead of an action, as in Theorem 2. (See the preliminaries of [**6**] for more details.)

2. Determination of the discrete cocompact subgroups.

Theorem 1. *Every discrete cocompact subgroup* H *of* $\mathrm{G}_{5,3}$ *has the following form: there are integers* α, β, γ, δ *and* ε *satisfying* $\alpha, \gamma, \delta > 0$, *and*

$$(*) \qquad 0 \le \varepsilon \le \gcd\{\gamma, \delta\}/2$$

and

$$(**) \qquad 0 \le \beta \le \gcd\{\alpha, \gamma, \delta, \varepsilon\}/2,$$

yielding $\mathrm{H} \cong \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ $(= \mathbf{Z}^5$ *as a set*) *with multiplication* (m)

$$\begin{cases} (h,j,k,m,n)(h',j',k',m',n') \\ = (h + h' + \gamma n j' + \alpha\gamma m' n(n-1)/2 + \beta n m' + \delta m k' + \varepsilon n k', \\ \qquad\qquad j + j' + \alpha n m', k + k', m + m', n + n'). \end{cases}$$

Different choices for α, β, γ, δ *and* ε *give non-isomorphic groups. Each such group is, in fact, isomorphic to a cofinite subgroup of* $\mathrm{H}_{5,3}$ (*the lattice subgroup of* $\mathrm{G}_{5,3}$), *and each cofinite subgroup of* $\mathrm{H}_{5,3}$ *is isomorphic to some* $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$.

Proof. Using the discreteness and cocompactness as in [**7**], the second commutator subgroup of H tells us that there is a member (with entries that don't need to be identified indicated by $*$)

$$e_5 = (*,\ *,\ *,\ \mathfrak{a}, z)$$

of H, where $z > 0$ is the smallest positive number that can appear as the last coordinate of a member of H. Continuing in this vein, we get

$$\begin{aligned} e_4 &= (*,\ *,\ *,\ y,\ 0),\\ e_3 &= (*,\ \mathfrak{b},\ x,\ 0,\ 0),\\ e_2 &= (*,\ w,\ 0,\ 0,\ 0) \quad \text{and}\\ e_1 &= (v,\ 0,\ 0,\ 0,\ 0), \end{aligned}$$

where $x > 0$ is the smallest positive number that can appear as the third coordinate of a member of H whose last two coordinates are 0, and similarly for v, w and y. Also, all other coordinates are ≥ 0, and the bottom non-zero coordinate in each column is greater than the coordinates above it, e.g., $w > \mathfrak{b} \geq 0$ and w is also greater than the second coordinate of e_5 or of e_4. These considerations show that the map

$$\pi : (h, j, k, m, n) \longmapsto e_1^h\, e_2^j\, e_3^k\, e_4^m\, e_5^n, \quad \mathbf{Z}^5 \to \text{H},$$

is one-to-one and onto. We want the multiplication (m) for $\mathbf{Z}^5$ that makes π a homomorphism, hence an isomorphism; (m) is determined using the commutators,

(C)

$$\begin{cases} [e_5, e_4] = (*,\ zy,\ 0,\ 0,\ 0) = e_1^{\beta} e_2^{\alpha}, & [e_5, e_3] = (z\mathfrak{b} + x\,\mathfrak{a},\ 0,\ 0,\ 0,\ 0) = e_1^{\varepsilon},\\ [e_5, e_2] = (zw,\ 0,\ 0,\ 0,\ 0) = e_1^{\gamma}, & [e_4, e_3] = (xy,\ 0,\ 0,\ 0,\ 0) = e_1^{\delta},\\ [e_5, e_1] = 0, \quad [e_4, e_1] = 0, & [e_3, e_1] = 0, \quad [e_2, e_1] = 0,\\ [e_4, e_2] = 0, \quad [e_3, e_2] = 0, & \end{cases}$$

for some integers α, β, γ, δ, ε. Using the commutators to collect terms in

$$(e_1^h\, e_2^j\, e_3^k\, e_4^m\, e_5^n)(e_1^{h'}\, e_2^{j'}\, e_3^{k'}\, e_4^{m'}\, e_5^{n'})$$

gives the multiplication formula (m) for $\mathbf{Z}^5$, and also the equation

$$e_5^n\, e_4^{m'} = e_1^{\alpha\gamma m' n(n-1)/2 + \beta n m'}\, e_2^{\alpha m' n}\, e_4^{m'}\, e_5^n,$$

which the reader may find helpful in checking computations later.

For a start in putting the restrictions on α, β, γ, δ, ε, (C) tells us that $\alpha, \gamma, \delta > 0$ (since v, w, x, y and $z > 0$). Let Z denote the center of $\mathrm{H} = \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$, $Z = (\mathbf{Z}, 0, 0, 0, 0)$. Then, as for G_4, with quotients and subgroups it is shown that different (positive) α, γ, δ give non-isomorphic groups, e.g., H/Z gives α and Z modulo the subgroup $[\mathrm{H}, [\mathrm{H}, \mathrm{H}]]$ gives γ; also, if $K_3 \subset \mathrm{H}$ is the largest subset for which all commutators are central, i.e., $xyx^{-1}y^{-1} \in Z$ for all $x \in K_3$ and $y \in \mathrm{H}$, and K_4 is the centralizer of the commutator subgroup, then

$$Z \supset (\delta\,\mathbf{Z}, 0, 0, 0, 0) = \{xyx^{-1}y^{-1} \mid x \in K_3,\ y \in K_4\}$$

and $Z/(\delta\,\mathbf{Z}, 0, 0, 0, 0) = \mathbf{Z}_\delta$, the cyclic group of order δ.

Then we have an isomorphism of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ onto $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon + d\gamma + e\delta)$, which is simpler to give in terms of generators,

$$(\circledast) \qquad \begin{gathered} e_3 \longmapsto e_3' = e_2^d e_3, \quad e_5 \longmapsto e_5' = e_4^e e_5, \\ \text{and} \quad e_i \longmapsto e_i' = e_i \quad \text{otherwise.} \end{gathered}$$

Here we are merely changing the basis for $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$, and the only commutator (using (m) and (C)) that changes is $[e_5', e_3'] = e_1^{\varepsilon + e\delta + d\gamma}$, so the resulting isomorphism is of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ onto $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon + d\gamma + e\delta)$, which shows we can require

$$0 \le \varepsilon < \gcd\{\gamma, \delta\}.$$

This, accompanied by another isomorphism,

$$(\circledast') \qquad \begin{gathered} (h, j, k, m, n) \longmapsto (-h, -j, k, -m, n), \\ \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon) \longrightarrow \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, -\varepsilon), \end{gathered}$$

assures that we can have

$$(*) \qquad 0 \le \varepsilon \le \gcd\{\gamma, \delta\}/2,$$

the required range for ε.

Now, to control β,

$$(\dagger) \qquad \begin{cases} e_1 \mapsto e_1 = e_1', \quad e_2 \mapsto e_1^{-q} e_2 = e_2', \quad e_3 \to e_3 = e_3', \\ e_4 \mapsto e_2^r e_3^g e_4 \quad \text{and} \quad e_5 \mapsto e_3^{-f} e_5 = e_5' \end{cases}$$

is an isomorphism of $\mathrm{H}_{5,3}(\alpha,\,\beta,\,\gamma,\,\delta,\,\varepsilon)$ onto $\mathrm{H}_{5,3}(\alpha,\,\beta+q\alpha+r\gamma+f\delta+g\varepsilon,\,\gamma,\,\delta,\,\varepsilon)$, which yields

$$0 \le \beta < \gcd\{\alpha,\,\gamma,\,\delta,\,\varepsilon\}.$$

Then the isomorphism

$$(\dagger') \qquad (h,j,k,m,n) \longmapsto (-h,j,k,-m,-n)$$

of $\mathrm{H}_{5,3}(\alpha,\,\beta,\,\gamma,\,\delta,\,\varepsilon)$ onto $\mathrm{H}_{5,3}(\alpha,\,-\beta+\alpha\gamma,\,\gamma,\,\delta,\,\varepsilon)$ leads to the conclusion

$$(**) \qquad 0 \le \beta \le \gcd\{\alpha,\,\gamma,\,\delta,\,\varepsilon\}/2.$$

It must still be shown that changing ε or β within the allowed limits (namely, ε and β must satisfy $(*)$ and $(**)$, respectively) gives a non-isomorphic group.

So, suppose that $\varphi : \mathrm{H} = \mathrm{H}_{5,3}(\alpha,\,\beta,\,\gamma,\,\delta,\,\varepsilon) \to \mathrm{H}_{5,3}(\alpha,\,\beta',\,\gamma,\,\delta,\,\varepsilon') = \mathrm{H}'$ is an isomorphism. Then

$$\begin{aligned}
\varphi : Z = K_1 = (\mathbf{Z},0,0,0,0) &\longrightarrow (\mathbf{Z},0,0,0,0) = K_1' = Z',\\
K_2 = (\mathbf{Z},\mathbf{Z},0,0,0) &\longrightarrow (\mathbf{Z},\mathbf{Z},0,0,0) = K_2',\\
K_3 = (\mathbf{Z},\mathbf{Z},\mathbf{Z},0,0) &\longrightarrow (\mathbf{Z},\mathbf{Z},\mathbf{Z},0,0) = K_3', \quad \text{and}\\
K_4 = (\mathbf{Z},\mathbf{Z},\mathbf{Z},\mathbf{Z},0) &\longrightarrow (\mathbf{Z},\mathbf{Z},\mathbf{Z},\mathbf{Z},0) = K_4',
\end{aligned}$$

since the Z's are the centers, the K_2's consist of those $s \in \mathrm{H}$ for which s^r is in the commutator subgroup of H for some $r \in \mathbf{Z}$, and the K_3's and K_4's are as above. So we must have

$$\begin{aligned}
&\varphi(0,0,0,0,1) = (*,*,-f,e,a) = S_5 \quad \text{with } a = \pm 1,\\
&\varphi(0,0,0,1,0) = (*,r,g,b,0) = S_4 \quad \text{with } b = \pm 1, \quad \text{and}\\
&\varphi(0,0,1,0,0) = (*,d,c,0,0) = S_3 \quad \text{with } c = \pm 1;
\end{aligned}$$

furthermore, commutators give

$$\varphi(\beta,\alpha,0,0,0) = [S_5,S_4] = S_5S_4S_5^{-1}S_4^{-1} = (*,\alpha ab,0,0,0),$$

hence $\varphi(0,1,0,0,0) = (q,ab,0,0,0) = S_2$, and

$$\varphi(\gamma,0,0,0,0) = [S_5,S_2] = (\gamma a^2 b,0,0,0,0),$$

so $\varphi(1,0,0,0,0) = (b,0,0,0,0) = S_1$, but also

$$\varphi(\delta,0,0,0,0) = [S_4, S_3] = (\delta bc, 0,0,0,0),$$

so $c = 1$. Furthermore, $\varphi(\varepsilon,0,0,0,0) = [S_5,S_3] = (a\varepsilon' + e\delta + ad\gamma, 0,0,0,0)$, which shows that the manipulations at ($\circledast$) and ($\circledast'$) above give the only way of changing ε in $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$; that is, if

$$(*) \qquad 0 \le \varepsilon,\, \varepsilon' \le \gcd\{\gamma,\, \delta\}/2$$

and $\varepsilon = \pm\varepsilon' + a_1\delta + a_2\gamma$ with $a_1, a_2 \in \mathbf{Z}$, then $\varepsilon = \varepsilon'$. Now consider

$$\begin{aligned}\varphi(h,j,k,m,n) &= \varphi((h,0,0,0,0)(0,j,0,0,0)\\ &\qquad \times (0,0,k,0,0)(0,0,0,m,0)(0,0,0,0,n))\\ &= (hS_1)\cdot(jS_2)\cdot(kS_3)\cdot S_4^m \cdot S_5^n\\ &= hS_1 + jS_2 + kS_3 + S_4^m\cdot S_5^n \in \mathrm{H}'.\end{aligned}$$

Note that $S_5^n \ne nS_5$, but $S_5^n = (*,*,-nf,ne,na)$, and also $S_4^m = (*,mr,mg,mb,0)$; further, the (jS_2) term puts a jq in the first entry of $\varphi(h,j,k,m,n)$, so also $(j+j'+\alpha nm')q$ in the first entry of $\varphi(h,j,k,m,n)\cdot\varphi(h',j',k',m',n')$ (product in $\mathrm{H}_{5,3}(\alpha,\beta',\gamma,\delta,\varepsilon)$). Then, equating the coefficients of the nm' terms in the first entry of

$$\varphi(e_5^n e_4^{m'}) \quad \text{and} \quad \varphi(e_5^n)\varphi(e_4^{m'}) = S_5^n S_4^{m'}$$

gives

$$b(-\alpha\gamma/2 + \beta) + q\alpha = ab\beta' - ab\alpha\gamma/2 + ag\varepsilon + ar\gamma + (eg+bf)\delta,$$

or

$$\beta = \pm\beta' + a_1\alpha + a_2\gamma + a_3\delta + a_4\varepsilon \quad \text{for some} \quad a_i \in \mathbf{Z}, \quad 1 \le i \le 4,$$

which shows that the manipulations at (†) and (†′) above give the only way of changing just β in $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$.

Here is an isomorphism φ of $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ onto a subgroup of the lattice subgroup $\mathrm{H}_{5,3} = \mathbf{Z}^5 \subset \mathrm{G}_{5,3}$ in terms of generators; $\mathrm{H}_{5,3}$ has multiplication

$$(\mathrm{m}') \qquad \begin{cases} (h,j,k,m,n)(h',j',k',m',n') \\ \qquad = (h+h'+nj'+m'n(n-1)/2+mk', \\ \qquad\qquad j+j'+nm', k+k', m+m', n+n'), \end{cases}$$

i.e., $\alpha = \gamma = \delta = 1$ and $\beta = \varepsilon = 0$. First suppose $\varepsilon > 0$. Then, with $\mathfrak{d} = \alpha\gamma\varepsilon$ and generators

$$e_1 = (1,0,0,0,0),\ e_2 = (0,1,0,0,0),\ \ldots,\ e_5 = (0,0,0,0,1)$$

for $\mathrm{H}_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ satisfying
(C)

$$\begin{cases} [e_5,e_4] = e_1^{\beta} e_2^{\alpha},\ [e_5,e_3] = e_1^{\varepsilon},\ [e_5,e_2] = e_1^{\gamma},\ [e_4,e_3] = e_1^{\delta}, \\ [e_5,e_1] = 0,\ [e_4,e_1] = 0,\ [e_3,e_1] = 0,\ [e_2,e_1] = 0,\ [e_4,e_2] = 0, \\ [e_3,e_2] = 0, \end{cases}$$

φ is given by

$$\begin{aligned} \varphi : e_1 &\longmapsto e_1' = (\delta\mathfrak{d}^2, 0, 0, 0, 0), \\ e_2 &\longmapsto e_2' = (\gamma\delta\mathfrak{d}(\mathfrak{d}-1)/2, \gamma\delta\mathfrak{d}, 0, 0, 0), \\ e_3 &\longmapsto e_3' = (0, \delta\varepsilon\mathfrak{d}, \delta\varepsilon\mathfrak{d}, 0, 0), \\ e_4 &\longmapsto e_4' = (0, \beta\delta\mathfrak{d}, 0, \alpha\gamma\delta, 0), \end{aligned}$$

and

$$e_5 \longmapsto e_5' = (0,0,0,0,\mathfrak{d}).$$

That φ is an isomorphism is verified by showing that $\{e_1', e_2', e_3', e_4', e_5'\} \subset \mathrm{H}_{5,3}$ satisfies (C). (Here φ is given by

$$\begin{aligned} (h,j,k,m,n) \longmapsto (&\delta\mathfrak{d}^2 h + (\gamma\delta\mathfrak{d}(\mathfrak{d}-1)/2)j, \\ &\gamma\delta\mathfrak{d}j + \delta\varepsilon\mathfrak{d}k + \beta\delta\mathfrak{d}m, \delta\varepsilon\mathfrak{d}k, \alpha\gamma\delta m, \mathfrak{d}n).\,) \end{aligned}$$

When $\varepsilon = 0$, use $\mathfrak{d} = \alpha\gamma$ and $e_3' = (0,0,\delta\mathfrak{d},0,0)$.

It is easy to see that the image $\mathrm{H}_1 = \varphi(\mathrm{H}_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon))$ is cofinite in $\mathrm{H}_{5,3}$. Consider the coset $s\mathrm{H}_1$ for $s = (h,j,k,m,n) \in \mathrm{H}_{5,3}$; since $e_5' = (0,0,0,0,\mathfrak{d})$, we can choose $r_5 \in \mathbf{Z}$ so that $s{e_5'}^{r_5}$ has its last coordinate in $[0,\mathfrak{d})$. Then choose $r_4 \in \mathbf{Z}$ so that $s{e_5'}^{r_5}{e_4'}^{r_4}$ has its second last coordinate in $[0,\alpha\gamma\delta)$. Continuing like this, we arrive at

$$s{e_5'}^{r_5}\,{e_4'}^{r_4}\,{e_3'}^{r_3}\,{e_2'}^{r_2}\,{e_1'}^{r_1} \in K$$

where

$$K = \big([0,\delta\,\mathfrak{d}^2) \times [0,\gamma\delta\,\mathfrak{d}) \times [0,\delta\,\varepsilon\,\mathfrak{d}) \times [0,\alpha\gamma\,\delta) \times [0,\mathfrak{d})\big) \cap \mathbf{Z}^5 \subset \mathrm{H}_{5,3},$$

so every coset $s\mathrm{H}_1$ for $s \in \mathrm{H}_{5,3}$ has a representative in K, which is a finite set. It follows that the quotient map $\mathrm{H}_{5,3} \to \mathrm{H}_{5,3}/\mathrm{H}_1$ maps K onto $\mathrm{H}_{5,3}/\mathrm{H}_1$, which is therefore finite. (A similar argument shows that $\mathrm{G}_{5,3}/\mathrm{H}_1$ is cocompact.)

Finally, note that since any cofinite subgroup of $\mathrm{H}_{5,3}$ is also a discrete cocompact subgroup of $\mathrm{G}_{5,3}$, it must therefore be isomorphic to some $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$. This completes the proof. $\square$

Remarks. 1. The image $\mathrm{H}_1 = \varphi(\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon))$ above is not a normal subgroup of $\mathrm{H}_{5,3}$, e.g.,

$$(0,0,1,0,0)e_5'(0,0,-1,0,0) = (\mathfrak{d},0,0,0,0) \notin \mathrm{H}_1.$$

This makes it seem unlikely that $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ can be embedded in $\mathrm{H}_{5,3}$ as a normal subgroup; however, the existence of such an embedding is still a possibility.

2. The theorem gives an isomorphism φ of $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ onto a subgroup of $\mathrm{H}_{5,3}$; conversely, there is always an isomorphism φ' of $\mathrm{H}_{5,3}$ onto a subgroup of $\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$, and as for φ, it is easier to give φ' in terms of the generators $\{e_i \mid 1 \le i \le 5\}$ of $\mathrm{H}_{5,3}$, which satisfy

$$(\mathrm{C}') \quad \begin{cases} [e_5,e_4] = e_2, \quad [e_5,e_3] = 0, \quad [e_5,e_2] = e_1, \quad [e_4,e_3] = e_1, \\ [e_5,e_1] = 0, \quad [e_4,e_1] = 0, \quad [e_3,e_1] = 0, \quad [e_2,e_1] = 0, \\ [e_4,e_2] = 0, \quad [e_3,e_2] = 0. \end{cases}$$

Then

$$\begin{aligned} \varphi' : e_1 &\longmapsto e_1' = (\alpha\gamma^2\delta^2, 0, 0, 0, 0), \\ e_2 &\longmapsto e_2' = (\alpha\gamma^2\delta(\delta-1)/2, \alpha\gamma\,\delta, 0, 0, 0), \\ e_3 &\longmapsto e_3' = (0, -\alpha\,\delta\varepsilon, \alpha\delta\,\gamma, 0, 0), \\ e_4 &\longmapsto e_4' = (0, -\beta, 0, \gamma, 0), \end{aligned}$$

and

$$e_5 \longmapsto e_5' = (0,0,0,0,\delta).$$

That φ' is an isomorphism is verified by showing that $e_1', e_2', e_3', e_4', e_5' \in \mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ satisfy (C$'$). (Here φ' is given by

$$\begin{aligned} (h,j,k,m,n) \longmapsto (&\alpha\,\gamma^2\,\delta^2 h + j\alpha\,\gamma^2\,\delta(\delta-1)/2, \\ &\alpha\,\gamma\,\delta j - \alpha\,\delta\,\varepsilon k - \beta m, \alpha\,\gamma\,\delta k, \gamma m, \delta n).) \end{aligned}$$

So, as for the three-dimensional groups $H_3(p)$ and the four-dimensional groups $H_4(p_1, p_2, p_3)$, here we have an infinite family of non-isomorphic groups, each of which is isomorphic to a subgroup of any other one.

3. Infinite dimensional simple quotients of $C^*(H_{5,3}(\alpha,\beta,\gamma,\delta,\varepsilon))$**.** We begin by obtaining concrete representations on $L^2(\mathbf{T}^2)$ of the faithful simple quotients, i.e., those arising from a faithful representation of $H_{5,3}(\alpha,\beta,\gamma,\delta,\varepsilon)$, and consider first the case $\varepsilon = 0$. In this case $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ has an abelian normal subgroup $N = (\mathbf{Z},\mathbf{Z},0,\mathbf{Z},0)$, with quotient

$$H_{5,3}(\alpha,\beta,\gamma,\delta,0)/N \cong (0,0,\mathbf{Z},0,\mathbf{Z}) = \mathbf{Z}^2,$$

also abelian and embedded in $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ as a subgroup, so that $H_{5,3}(\alpha,\beta,\gamma,\delta,0)$ is isomorphic to a semi-direct product $N \times \mathbf{Z}^2$; in this situation, the simple quotients of $C^*(H_{5,3}(\alpha,\beta,\gamma,\delta,0))$ can be presented as C^*-crossed products using flows from commuting homeomorphisms, as follows.

Note. Here, and below, the term *flow* designates a pair (G, X) consisting of a compact Hausdorff space X with a group G acting continuously on it. Some authors refer to such a pair as a dynamical system.

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, and consider the flow $\mathcal{F}' = (\mathbf{Z}^2, \mathbf{T}^2)$ generated by the commuting homeomorphisms

$$\psi_1' : (w,v) \longmapsto (\lambda^\gamma w, \lambda^\beta w^\alpha v) \quad \text{and} \quad \psi_2' : (w,v) \longmapsto (w, \lambda^{-\delta} v).$$

The flow $\mathcal{F}'$ is minimal, so the C^*-crossed product $\mathcal{C}' = C^*(\mathcal{C}(\mathbf{T}^2), \mathbf{Z}^2)$ is simple [**1**, Corollary 5.16].

Let v and w denote, as well as members of $\mathbf{T}$, the functions in $\mathcal{C}(\mathbf{T}^2)$ defined by

$$(w,v) \longmapsto v \quad \text{and} \quad w,$$

respectively. Define unitaries U, V, W and X on $L^2(\mathbf{T}^2)$ by

$$(\mathcal{U}') \qquad \begin{aligned} U &: f \longmapsto f \circ \psi_1', & V &: f \longmapsto vf, \\ W &: f \longmapsto f \circ \psi_2' \quad \text{and} & X &: f \longmapsto wf. \end{aligned}$$

These unitaries satisfy

$$
\text{(CR}'\text{)}\quad \begin{array}{lll} UV = \lambda^{\beta} X^{\alpha} VU, & UX = \lambda^{\gamma} XU, & VW = \lambda^{\delta} WV, \\ UW = WU, & VX = XV, & WX = XW, \end{array}
$$

equations which ensure that

$$
\pi : (h, j, k, m, n) \longmapsto \lambda^h X^j W^k V^m U^n
$$

is a representation of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, 0)$. Denote by $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, 0)$ the C^*-subalgebra of $B(L^2(\mathbf{T}^2))$ generated by π, i.e., by U, V, W and X. Since $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, 0)$ is generated by a representation of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, 0)$, it is a quotient of the group C^*-algebra $C^*(\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, 0))$. It follows readily that $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, 0)$ is isomorphic to the simple C^*-crossed product $\mathcal{C}'$ above, and hence is simple.

However, when $0 < \varepsilon \le \gcd\{\gamma, \delta\}/2$ (which implies $\gamma > 1$, by $(*)$), $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ is only an extension $(\mathbf{Z}, \mathbf{Z}, 0, \mathbf{Z}, 0) \times (0, 0, \mathbf{Z}, 0, \mathbf{Z}) = N \times \mathbf{Z}^2$, and not a semi-direct product. Nonetheless, we can modify the flow $\mathcal{F}'$ representing $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, 0)$ above to ge t a concrete representation of $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$. Consider the flow $\mathcal{F} = (\mathbf{Z}^2, \mathbf{T}^2)$ generated by the commuting homeomorphisms

$$
\psi_1 : (w, v) \longmapsto (\lambda w, \lambda^{\beta} w^{\alpha\gamma} v) \quad \text{and} \quad \psi_2 : (w, v) \longmapsto (w, \lambda^{-\delta} v).
$$

The flow $\mathcal{F}$ is minimal, so the C^*-crossed product $\mathcal{C} = C^*(\mathcal{C}(\mathbf{T}^2), \mathbf{Z}^2)$ is simple. Define unitaries on $L^2(\mathbf{T}^2)$ by

$$
(\mathcal{U}) \qquad \begin{array}{ll} U : f \longmapsto f \circ \psi_1, & V : f \longmapsto vf, \\ W : f \longmapsto w^{\varepsilon} f \circ \psi_2 \quad \text{and} & X : f \longmapsto w^{\gamma} f. \end{array}
$$

These unitaries satisfy

$$
\text{(CR)}\quad \begin{array}{lll} UV = \lambda^{\beta} X^{\alpha} VU, & UX = \lambda^{\gamma} XU, & VW = \lambda^{\delta} WV, \\ UW = \lambda^{\varepsilon} WU, & VX = XV, & WX = XW, \end{array}
$$

equations which ensure that

$$
\pi : (h, j, k, m, n) \longmapsto \lambda^h X^j W^k V^m U^n
$$

is a representation of $H_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$. Denote by $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ the C^*-subalgebra of $B(L^2(\mathbf{T}^2))$ generated by π. Now $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ is isomorphic only to a subalgebra of $\mathcal{C}$ (as may be shown using conditional expectations); a unitary that is missing is $X' : f \mapsto wf$, since $\gamma > 1$.

Note. The reason we did not use $\mathcal{F}$ when $\varepsilon = 0$, and $\gamma > 1$, is that $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ 0)$ seems to be isomorphic only to a subalgebra of $\mathcal{C}$ in that case too, whereas with $\mathcal{F}'$, $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ 0) \cong \mathcal{C}'$.

Since the flow method can no longer be used to prove the simplicity of the algebra $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ (when $0 < \varepsilon \leq \gcd\{\gamma, \delta\}/2$), we use the strong result of Packer [**10**].

Theorem 2. *Let* $\lambda = e^{2\pi i\theta}$ *for an irrational* θ.

(a) *There is a unique* (*up to isomorphism*) *simple* C^**-algebra* $A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ *generated by unitaries* U, V, W *and* X *satisfying*

$$\text{(CR)}\qquad \begin{array}{lll} UV = \lambda^\beta X^\alpha VU, & UX = \lambda^\gamma XU, & VW = \lambda^\delta WV, \\ UW = \lambda^\varepsilon WU, & VX = XV, & WX = XW, \end{array}$$

Furthermore, for a suitable $\mathbf{C}$*-valued cocycle on* $H_3(\alpha) \times \mathbf{Z}$,

$$A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon) \cong C^*(\mathbf{C}, H_3(\alpha) \times \mathbf{Z}).$$

(b) *Let* π' *be a representation of* $H'_{5,3} = H_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ *such that* $\pi = \pi'$, *as scalars, on the center* $(\mathbf{Z}, 0, 0, 0, 0)$ *of* $H'_{5,3}$, *and let* A *be the* C^**-algebra generated by* π'. *Then* $A \cong A_\theta^{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon) = A_\theta'^{5,3}$ (*say*) *via a unique isomorphism* ω *such that the following diagram commutes.*

$$\begin{tikzcd} H'_{5,3} \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & A_\theta'^{5,3} \arrow[dl, "\omega"] \\ & A & \end{tikzcd}$$

Proof. To use Packer's result, we regard $H_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon)$ as an extension

$$H_{5,3}(\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon) \cong \mathbf{Z} \times (0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z} \times (H_3(\alpha) \times \mathbf{Z})$$

(with $\mathrm{H}_3(\alpha) \cong (0, \mathbf{Z}, 0, \mathbf{Z}, \mathbf{Z}) \subset \mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$); this extension has cocycle

$$\begin{aligned}[s, s'] &= [(j,k,m,n),(j',k',m',n')] \\ &= \lambda^{\gamma n j' + \alpha\gamma m' n(n-1)/2 + \beta n m' + \delta m k' + \varepsilon n k'}, \end{aligned}$$

$$(\mathrm{H}_3(\alpha) \times \mathbf{Z}, \mathrm{H}_3(\alpha) \times \mathbf{Z}) \longrightarrow \mathbf{T}.$$

The application of Packer's result requires the consideration of the related function

$$\chi^{s'}(s) = [s', s]\overline{[s, s^{-1}s's]} \quad \text{for} \quad s, s' \in (0, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}) \cong \mathrm{H}_3(\alpha) \times \mathbf{Z}.$$

It must be shown that $\chi^{s'}$ is non-trivial on the centralizer of s' in $\mathrm{H}_3(\alpha) \times \mathbf{Z}$ if s' has finite conjugacy class in $\mathrm{H}_3(\alpha) \times \mathbf{Z}$; this is easy because the only elements of $\mathrm{H}_3(\alpha) \times \mathbf{Z}$ that have finite conjugacy class are in the center $Z_1 = (\mathbf{Z}, \mathbf{Z}, 0, 0)$ of $\mathrm{H}_3(\alpha) \times \mathbf{Z}$, so their centralizer is all of $\mathrm{H}_3(\alpha) \times \mathbf{Z}$. Thus the C^*-crossed product $C^*(\mathbf{C}, \mathrm{H}_3(\alpha) \times \mathbf{Z})$ is simple; it is isomorphic to $\mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ because, with basis members

$$\begin{aligned} e_1 &= (1,0,0,0), & e_2 &= (0,1,0,0), \\ e_3 &= (0,0,1,0) \quad \text{and} & e_4 &= (0,0,0,1) \end{aligned}$$

for $\mathrm{H}_3(\alpha) \times \mathbf{Z}$, the unitaries

$$U' = \delta_{e_4}, \quad V' = \delta_{e_3}, \quad W' = \delta_{e_2} \quad \text{and} \quad X' = \delta_{e_1}$$

in $l_1(\mathrm{H}_3(\alpha) \times \mathbf{Z}) \subset C^*(\mathbf{C}, \mathrm{H}_3(\alpha) \times \mathbf{Z})$ satisfy (CR). □

4. Other simple quotients of $C^*(\mathbf{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon))$. Now assume that λ is a primitive qth root of unity and that U, V, W and X are unitaries generating a simple quotient A of $C^*(\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon))$, i.e., they satisfy

$$\text{(CR)} \quad \begin{aligned} UV &= \lambda^\beta X^\alpha VU, & UX &= \lambda^\gamma XU, & VW &= \lambda^\delta WV, \\ UW &= \lambda^\varepsilon WU, & VX &= XV, & WX &= XW, \end{aligned}$$

We may assume that A is irreducibly represented. Then, if

$$(c') \qquad \begin{cases} q_1 \text{ is the order of } \lambda^\gamma \text{ and} \\ q_2 \text{ is the lcm of the orders of } \lambda^\delta \text{ and } \lambda^\varepsilon, \end{cases}$$

W^{q_2} and X^{q_1} are scalar multiples of the identity, by irreducibility. Since W can be multiplied by a scalar without changing (CR), we may assume $W^{q_2} = 1$. However, $X^{q_1} = \mu'$, a multiple of the identity. Put $X = \mu X_1$ for $\mu^{q_1} = \mu'$, so that $X_1^{q_1} = 1$, and substitute $X = \mu X_1$ in (CR) to get

$$(\mathrm{CR}_1) \quad \begin{cases} UV = \lambda^{\beta}\mu^{\alpha} X_1^{\alpha} VU, \quad UX_1 = \lambda^{\gamma} X_1 U, \quad VX_1 = X_1 V, \\ WX_1 = X_1 W, \quad VW = \lambda^{\delta} WV, \quad UW = \lambda^{\varepsilon} WU \\ \text{and} \quad W^{q_2} = 1 = X_1^{q_1}. \end{cases}$$

1. If μ is also a root of unity, then (CR_1), along with irreducibility, shows that U and V, as well as W and X, are (multiples of) finite order unitaries, so A is finite dimensional.

2. If μ is not a root of unity, the dynamical system $\mathcal{F} = (\mathbf{Z}^2, \mathbf{T}^2)$ used above to get a concrete representation of $\mathrm{A}_{\theta}^{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ can be modified to get a concrete representation of A on $L^2(\mathbf{Z}_{q_1} \times \mathbf{T})$, where $\mathbf{Z}_{q_1}$ is the subgroup of $\mathbf{T}$ with q_1 elements. We shall now show that A is isomorphic to $M_{q_2} \otimes C^*(C(\mathbf{Z}_{q_1} \times \mathbf{T}), \mathbf{Z})$, where q_2 is as in (c') and the action of $\mathbf{Z}$ on $\mathbf{Z}_{q_1} \times \mathbf{T}$ is generated by a minimal transformation $\phi(w, v) = (\lambda' w, \xi_1 \lambda^{\gamma\alpha q_2} v)$ for suitable λ' of order q_1 and ξ_1, see Theorem 3 below.

First consider the universal C^*-algebra $\mathfrak{A}$ generated by unitaries satisfying

$$(\mathrm{CR}_1) \quad \begin{cases} UV = \lambda^{\beta}\mu^{\alpha} X_1^{\alpha} VU, \quad UX_1 = \lambda^{\gamma} X_1 U, \quad VX_1 = X_1 V, \\ WX_1 = X_1 W, \quad VW = \lambda^{\delta} WV, \quad UW = \lambda^{\varepsilon} WU \\ \text{and} \quad W^{q_2} = 1 = X_1^{q_1}. \end{cases}$$

A change of variables is useful. Pick relatively prime integers c, d such that $d\delta + c\varepsilon = 0$, and let a, b be integers such that $ad - bc = 1$. Put

$$U' = U^a V^b \quad \text{and} \quad V' = U^c V^d.$$

Then keeping X and W the same, (CR_1) becomes

$$(\mathrm{CR}_2) \quad \begin{cases} U'V' = \xi X_1^{\alpha} V'U', \quad U'X_1 = \lambda^{a\gamma} X_1 U', \quad WX_1 = X_1 W, \\ V'W = WV', \quad U'W = \lambda^{\delta'} WU', \quad V'X = \lambda^{c\gamma} XV' \\ \text{and} \quad W^{q_2} = 1 = X_1^{q_1} \end{cases}$$

where $\xi = \lambda^\beta \mu^\alpha \lambda^s$ for some integer s, and $\delta' = b\delta + a\varepsilon$. It is clear that $\lambda^{\delta'}$ is a primitive q_2th root of unity and that the algebra $\mathfrak{A}$ is generated by U', V', W and X_1, since $ad - bc = 1$.

Let $B = C^*(X_1, V')$ and let $C(\mathbf{Z}_{q_2}) = C^*(W)$ be the C^*-algebra generated by W. Since W commutes with X_1 and V', we can form the tensor product algebra $B \otimes C(\mathbf{Z}_{q_2}) = C^*(X_1, V', W)$. The automorphism $\mathrm{Ad}_{U'}$ acts on this tensor product as $\sigma \otimes \tau$, where σ and τ are automorphisms of B and $C(\mathbf{Z}_{q_2})$, respectively, given by

$$\sigma(X_1) = \lambda^{a\gamma} X_1, \quad \sigma(V') = \xi_1 X_1^\alpha V' \quad \text{and} \quad \tau(W) = \zeta W.$$

Therefore, by the universality of $\mathfrak{A}$ and of the C^*-crossed product $C^*(B \otimes C(\mathbf{Z}_{q_2}), \mathbf{Z})$, these algebras are isomorphic. By Rieffel's Proposition 1.2 [**17**], the latter of these is isomorphic to $M_{q_2}(D)$, where $D = C^*(B, \mathbf{Z}) = C^*(X_1, V', U'^{q_2})$, and the action of $\mathbf{Z}$ on B is generated by σ^{q_2}.

Now, the unitaries X_1, V' and U'^{q_2} generating D satisfy

$$(\star) \qquad \begin{cases} U'^{q_2} V' = \xi^{q_2} \lambda^{s'} X_1^{\alpha q_2} V' U'^{q_2}, & V' X_1 = \lambda^{c\gamma} X_1 V', \\ U'^{q_2} X_1 = \lambda^{a\gamma q_2} X_1 U'^{q_2} \quad \text{and} & X_1^{q_1} = 1, \end{cases}$$

for some $s' \in \mathbf{Z}$.

Now we apply another change of variables. Choose relatively prime integers c', d' such that $cd' + aq_2 c' = 0$, then pick integers a', b' with $a'd' - b'c' = 1$, and put

$$U'' = U'^{q_2 a'} V'^{b'} \quad \text{and} \quad V'' = U'^{q_2 c'} V^{d'}.$$

Then $(\star)$ becomes (keeping X_1 the same)

$$(\star\star) \qquad \begin{cases} U'' V'' = \xi_1 X_1^{\alpha q_2} V'' U'', & V'' X_1 = X_1 V'', \\ U'' X_1 = \lambda' X_1 U'' \quad \text{and} & X_1^{q_1} = 1, \end{cases}$$

where $\xi_1 = \xi^{q_2} \lambda^{s'}$ for some integer s', $\lambda' = \lambda^{\gamma(aq_2 a' + cb')}$ has order q_3 dividing q_1 (the order of λ^γ), and perhaps $q_3 \neq q_1$.

Now, with $\mathbf{Z}_{q_1} \subset \mathbf{T}$ representing the subgroup with q_1 members, one observes that D is isomorphic to the crossed product of $C^*(C(\mathbf{Z}_{q_1} \times \mathbf{T}), \mathbf{Z})$ from the flow generated by $\phi(w, v) = (\lambda' w, \xi_1 \lambda^{\gamma\alpha q_2} v)$. (Note that the flow is not minimal unless the order of λ' is exactly q_1.) This proves the following.

Theorem 3. *The universal C^*-algebra $\mathfrak{A}$ generated by unitaries U, V, W and X_1 satisfying* (CR_1) *as for* 2 *near the beginning of this section,* (*see also* (c')) *is isomorphic to $M_{q_2}(D)$, where $D = C^*(C(\mathbf{Z}_{q_1} \times \mathbf{T}), \mathbf{Z})$, as above.*

Therefore, we now obtain all simple algebras satisfying (CR_1).

Corollary 4. *Every simple C^*-algebra generated by unitaries satisfying* (CR_1)*, with μ not a root of unity, is isomorphic to a matrix algebra over an irrational rotation algebra.*

Proof. By Theorem 3, any such simple algebra Q is a quotient of $M_{q_2}(D)$. Hence $Q = M_{q_2}(Q')$ where Q' is a simple quotient of D. But such a Q' is generated by unitaries satisfying $(\star\star)$, but with X_1, of order q_1, replaced by another unitary X_2, which after suitable rescaling, has order equal to the order of the λ' appearing in $(\star\star)$. But this algebra is known to be a matrix algebra over an irrational rotation algebra, see for example Theorem 3 of [**5**]. $\square$

We state

Theorem 5. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of* $C^*(\mathrm{H}_{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon))$ *if and only if* A *is isomorphic to* $\mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ *for an irrational θ, or to an algebra as in Corollary* 4.

5. ***K*-theory and the trace invariant.** In this section we shall calculate the K-groups of the C^*-algebra $A := \mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ by means of the Pimsner-Voiculescu six term exact sequence [**16**]. Since one of the groups in the sequence turns out to have torsion elements, the application of this result requires careful examination.

Theorem 6. *For the C^*-algebra* $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$, *one has* $K_0 = K_1 = \mathbf{Z}^6 \oplus \mathbf{Z}_\alpha$.

Proof. To prove this theorem, we combine two applications of the PV sequence corresponding to two presentations P1 and P2 of A as follows.

P1. In view of (CR), let $B_1 = C^*(X, V, U)$ and let Ad_W, with

$$\mathrm{Ad}_W(X) = X, \quad \mathrm{Ad}_W(V) = \lambda^{-\delta} V, \quad \mathrm{Ad}_W(U) = \lambda^{-\varepsilon} U,$$

generate an action of $\mathbf{Z}$ on B_1, so that $A = C^*(B_1, \mathbf{Z})$. Applying the PV sequence to B_1, viewed as the crossed product of $C(\mathbf{T}^2) = C^*(X, V)$ by the automorphism Ad_U, it is not hard to see that $K_0(B_1) = \mathbf{Z}^3$ and $K_1(B_1) = \mathbf{Z}^3 \oplus \mathbf{Z}_\alpha$. Since Ad_W is homotopic to the identity, the PV sequence immediately gives

$$K_1(A) = \mathbf{Z}^6 \oplus \mathbf{Z}_\alpha.$$

However, since in the short exact sequence

$$0 \longrightarrow K_0(B_1) \xrightarrow{i_*} K_0(A) \xrightarrow{\delta} K_1(B_1) \longrightarrow 0$$

$K_1(B_1)$ has torsion, we cannot readily obtain $K_0(A)$. For this, the next presentation will help.

P2. In view of (CR), we can also let $B_2 = C^*(X, V, W) = C(\mathbf{T}) \otimes A_{\delta\theta}$, where $C(\mathbf{T}) = C^*(X)$ and $A_{\delta\theta} = C^*(V, W)$. Let $\sigma = \mathrm{Ad}_U$, with

$$\sigma(X) = \lambda^\gamma X, \quad \sigma(V) = \lambda^\beta X^\alpha V, \quad \sigma(W) = \lambda^\varepsilon W,$$

generate an action of $\mathbf{Z}$ on B_2, so that $A = C^*(B_2, \mathbf{Z})$. In this case the PV sequence becomes

$$(*) \qquad \begin{array}{ccccc} K_0(B_2) & \xrightarrow{id_* - \sigma_*} & K_0(B_2) & \xrightarrow{i_*} & K_0(A) \\ \uparrow{\scriptstyle \delta_1} & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(A) & \xleftarrow{i_*} & K_1(B_2) & \xleftarrow{id_* - \sigma_*} & K_1(B_2) \end{array}$$

It is not hard to see that a basis for $K_1(B_2) = \mathbf{Z}^4$ is given by $\{[X],[V],[W],[\xi]\}$ where $\xi = X \otimes e + 1 \otimes (1-e)$ and $e = e(V,W)$ is a Rieffel projection in $A_{\delta\theta}$ of trace $\delta\theta \mod 1$. Also, a basis of $K_0(B_2) = \mathbf{Z}^4$ is given by $\{[1],[e],B_{XV},B_{XW}\}$ where $B_{XV} = [P_{XV}]-[1]$ is the Bott element in X,V and P_{XV} the usual Bott projection in the commuting unitaries X,V. The action of $id_* - \sigma_*$ on $K_1(B_2)$ is given by

$$id_* - \sigma_* : \quad [X] \mapsto 0, \quad [V] \mapsto -\alpha[X], \quad [W] \mapsto 0, \quad [\xi] \mapsto m\alpha[X]$$

for some integer m, as shown by the following lemma. The action of $id_* - \sigma_*$ on $K_0(B_2)$ is given by

$$id_* - \sigma_* : \quad [1] \mapsto 0, \quad [e] \mapsto \alpha B_{XW}, \quad B_{XW} \mapsto 0, \quad B_{XV} \mapsto 0.$$

Here, that $\sigma_*(B_{XV}) = B_{XV}$ is a well-known fact, see for example Lemma 3.2 of [**18**]. The action on $[e]$ is also shown in the following

Lemma 7. *We have $\sigma_*[e] = [e] - \alpha B_{XW}$ in $K_0(B_2)$ and $\sigma_*[\xi] = [\xi] + m\alpha[X]$ for some integer m.*

Proof. The proof of the first equality can be established using an argument quite similar to that of the proof of Lemma 4.2 of [**18**]. Hence the kernel of $id_* - \sigma_*$ on $K_0(B_2)$ is $\mathbf{Z}^3$. For the second equality, let $\eta = (id_* - \sigma_*)[\xi]$. From P1 and (*) we have

$$\begin{aligned} \mathbf{Z}^6 \oplus \mathbf{Z}_\alpha &\cong K_1(A) \cong \mathbf{Z}^3 \oplus \mathrm{Im}\,(i_*) \\ &\cong \mathbf{Z}^3 \oplus \frac{K_1(B_2)}{\mathrm{Im}\,(id_* - \sigma_*)} = \mathbf{Z}^3 \oplus \frac{K_1(B_2)}{\mathbf{Z}\alpha[X] + \mathbf{Z}\eta}. \end{aligned}$$

Thus

$$(**) \qquad \frac{K_1(B_2)}{\mathbf{Z}\alpha[X] + \mathbf{Z}\eta} \cong \mathbf{Z}^3 \oplus \mathbf{Z}_\alpha.$$

But since $K_1(B_2) \cong \mathbf{Z}^4$, it follows that the subgroup $\mathbf{Z}\alpha[X] + \mathbf{Z}\eta$ must have rank one.[1] Therefore, $\mathbf{Z}\alpha[X] + \mathbf{Z}\eta = \mathbf{Z}d[X]$ for some integer d. Substituting this into $(**)$, one gets $d = \alpha$ and so $\eta \in \mathbf{Z}\alpha[X]$. ◻

It now follows that in $K_1(B_2)$ one has $\mathrm{Im}\,(id_* - \sigma_*) = \mathbf{Z}\alpha[X]$ and that $\mathrm{Ker}\,(id_* - \sigma_*) = \mathbf{Z}^3$ whether m is zero or not. Therefore, from the exactness of $(*)$ we obtain $\mathrm{Im}\,(\delta_0) = \mathbf{Z}^3$ and hence, by Lemma 7,

$$K_0(A) = \mathbf{Z}^3 \oplus \mathrm{Im}\,(i_*) = \mathbf{Z}^3 \oplus \frac{K_0(B_2)}{\mathrm{Im}\,(id_* - \sigma_*)} = \mathbf{Z}^3 \oplus \frac{K_0(B_2)}{\mathbf{Z}\alpha B_{XW}} = \mathbf{Z}^6 \oplus \mathbf{Z}_\alpha,$$

which completes the proof of Theorem 6. $\square$

5.1 The trace invariant. Let us first note that when θ is irrational, the C^*-algebra $A_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ has a unique trace state τ. Such a trace clearly exists by defining $\tau(X^m W^n V^r U^s) = 0$ when $(m,n,r,s) \neq (0,0,0,0)$ and 1 otherwise. The uniqueness of a trace state follows from showing that, for any such trace τ, one has $\tau(X^m W^n V^r U^s) = 0$ when $(m,n,r,s) \neq (0,0,0,0)$. Indeed, using Ad_X in the trace, one gets $\tau(X^m W^n V^r U^s) = \tau(X^* X^m W^n V^r U^s X) = \lambda^{\gamma s} \tau(X^m W^n V^r U^s)$, which shows that $\tau(X^m W^n V^r U^s) = 0$ for $s \neq 0$, as $\gamma > 0$. One then looks at $\tau(X^m W^n V^r)$. Here one uses Ad_W to see that this trace is 0 for $r \neq 0$. For $\tau(X^m W^n)$ one uses Ad_V and for $\tau(X^m)$ one uses Ad_U. This proves uniqueness of the trace.

Theorem 8. *The range of the unique trace on* $K_0(\mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon))$ *is* $\mathbf{Z} + \mathbf{Z}\rho\theta + \mathbf{Z}\gamma\delta\theta^2$ *where* $\rho = \gcd\{\gamma, \delta, \varepsilon\}$.

Note that this agrees with the trace invariant $\mathbf{Z} + \mathbf{Z}\theta + \mathbf{Z}\theta^2$ of the algebra $\mathrm{A}_\theta^{5,3}$ as done in [**18**, Section 2], in the case $(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon) = (1,0,1,1,0)$.

Proof. First we make an appropriate change of variables for the unitary generators of the algebra $A = \mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$. Referring back to the defining relations (CR), pick integers a, b, c, d such that $b\delta + a\varepsilon = 0,\ ad - bc = 1$, and let

$$U' = U^a V^b, \qquad V' = U^c V^d.$$

Then the commutation relations (CR), with W remaining the same and X suitably scaled, become

$$\begin{aligned} &U'V' = X^\alpha V'U', \quad U'X = \lambda^{a\gamma} XU', \quad V'W = \lambda^{d\delta + c\varepsilon} WV', \\ &U'W = WU', \qquad V'X = \lambda^{c\gamma} XV', \quad WX = XW \end{aligned}$$

Let $B = C^*(X, U', V')$. It is isomorphic to the crossed product of $C^*(X, U') = A_{a\gamma\theta}$ by $\mathbf{Z}$ and automorphism $\mathrm{Ad}_{V'}$. An easy application of Pimsner's trace formula [**15**, Theorem 3] shows that

$$\tau_* K_0(B) = \mathbf{Z} + \mathbf{Z}a\gamma\theta + \mathbf{Z}c\gamma\theta = \mathbf{Z} + \mathbf{Z}\gamma\theta,$$

since $(a, c) = 1$. Next, it is not hard to see that an application of the Pimsner-Voiculescu sequence to the above crossed product presentation of B gives the basis $\{[X], [V'], [U'], [\xi]\}$ for $K_1(B)$, where $[X]$ has order α, $\xi = 1 - e + ew^*V'^*e$ is a unitary in B, e is a Rieffel projection in $A_{a\gamma\theta}$ of trace $(a\gamma\theta) \bmod 1$, and w is a unitary in $A_{a\gamma\theta}$ such that $V'^*eV' = wew^*$, which exists by Rieffel's cancellation theorem [**17**]. The underlying connecting homomorphism $\partial : K_1(B) \to K_0(A_{a\gamma\theta})$ gives $\partial[\xi] = [e]$ and $\partial[V'] = [1]$, the usual basis of $K_0(A_{a\gamma\theta})$.

To apply Pimsner's trace formula, one calculates the usual "determinant" on the aforementioned basis, since the kernel of $id_* - (\mathrm{Ad}_W)_*$ is all of $K_1(B)$, since Ad_W is homotopic to the identity. It is easy to see that this determinant, whose values are in $\mathbf{R}/\tau_* K_0(B)$, on the elements $[X], [V'], [U']$ gives the respective values $1, (d\delta + c\varepsilon)\theta, 1$. For ξ, since now Ad_W fixes $A_{a\gamma\theta}$, and in particular e and w, one obtains

$$\mathrm{Ad}_W(\xi)\xi^* = (1 - e + \lambda^{d\delta + c\varepsilon} ew^*V'^*e)(1 - e + eV'we) = 1 - e + \lambda^{d\delta + c\varepsilon} e.$$

Now a simple homotopy path connecting this element to 1 is just $t \mapsto 1 - e + e^{2\pi i\theta(d\delta + c\varepsilon)t}e$, and the corresponding determinant gives the value $(d\delta + c\varepsilon)\theta\tau(e)$. Since $\tau(e) = a\gamma\theta \mod 1$, the range of the trace is

$$\tau_* K_0(A) = \mathbf{Z} + \mathbf{Z}\gamma\theta + \mathbf{Z}(d\delta + c\varepsilon)\theta + \mathbf{Z}\gamma a(d\delta + c\varepsilon)\theta^2.$$

Now $a(d\delta + c\varepsilon) = ad\delta + ac\varepsilon - c(b\delta + a\varepsilon) = \delta$, and similarly $-b(d\delta + c\varepsilon) = \varepsilon$, thus showing that $d\delta + c\varepsilon = \gcd\{\delta, \varepsilon\}$. Therefore, one gets $\tau_* K_0(A) = \mathbf{Z} + \mathbf{Z}\gcd\{\gamma,\, \delta,\, \varepsilon\}\theta + \mathbf{Z}\gamma\delta\theta^2$. $\square$

5.2 Discussion of classification. Next, let us consider briefly the classification of the algebras $\mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$. First, it is easy to show that $\mathrm{A}_\theta^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon) \cong \mathrm{A}_{-\theta}^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$. Second, we note that the simple quotients $\mathrm{A}_\theta^{5,3} = \mathrm{A}_\theta^{5,3}(1, 0, 1, 1, 0)$ have been almost completely classified in [**18**]; specifically, they have been classified for all non-quartic irrationals, which are those that are not zeros of any polynomial of degree at most 4 with integer coefficients. But, generally, with $\lambda = e^{2\pi i\theta}$ for an irrational θ, the operator equations

$$\text{(CR)} \quad \begin{array}{lll} UV = \lambda^{\beta} X^{\alpha} VU, & UX = \lambda^{\gamma} XU, & VW = \lambda^{\delta} WV, \\ UW = \lambda^{\varepsilon} WU, & VX = XV, & WX = XW, \end{array}$$

for $\mathrm{A}_{\theta}^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon)$ can be modified by changing some of the variables, i.e., by substituting $X_0 = e^{2\pi i\theta\beta/\alpha} X$ and putting $\lambda_0 = \lambda^{\rho}$, where $\rho = \gcd\{\gamma,\, \delta,\, \varepsilon\}$, and then $\gamma_0 = \gamma/\rho$, $\delta_0 = \delta/\rho$ and $\varepsilon_0 = \varepsilon/\rho$ with $\gcd\{\gamma_0, \delta_0, \varepsilon_0\} = 1$. The equations (CR) become

(CR_0)

$$\begin{cases} UV = X_0^{\alpha} VU, \quad UX_0 = \lambda_0^{\gamma_0} X_0 U, \quad VW = \lambda_0^{\delta_0} WV, \quad VX_0 = X_0 V, \\ UW = \lambda_0^{\varepsilon_0} WU, \quad WX_0 = X_0 W, \quad \text{with} \quad \gcd\{\gamma_0, \delta_0, \varepsilon_0\} = 1, \end{cases}$$

which are the equations for $\mathrm{A}_{\rho\theta}^{5,3}(\alpha, 0, \gamma_0, \delta_0, \varepsilon_0)$, so

$$\mathrm{A}_{\theta}^{5,3}(\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon) \cong \mathrm{A}_{\rho\theta}^{5,3}(\alpha, 0, \gamma_0, \delta_0, \varepsilon_0)$$

where $\gcd\{\gamma_0, \delta_0, \varepsilon_0\} = 1$. This reduces the classification to the class of algebras $\mathrm{A}_{\theta}^{5,3}(\alpha, 0, \gamma,\, \delta,\, \varepsilon)$ where $\gcd\{\gamma,\, \delta,\, \varepsilon\} = 1$.

If two such C^*-algebras $A_j = \mathrm{A}_{\theta_j}^{5,3}(\alpha_j, 0, \gamma_j, \delta_j, \varepsilon_j)$, $j = 1, 2$, are isomorphic, where now $\rho_j = \gcd\{\gamma_j, \delta_j, \varepsilon_j\} = 1$, what constraints must hold between their respective parameters? As we observed in Theorem 6, one must have $\alpha_1 = \alpha_2$. By Theorem 8, one has

$$\mathbf{Z} + \mathbf{Z}\theta_1 + \mathbf{Z}\gamma_1\delta_1\theta_1^2 = \mathbf{Z} + \mathbf{Z}\theta_2 + \mathbf{Z}\gamma_2\delta_2\theta_2^2.$$

One can show that if one assumes that θ_j are non-quadratic irrationals, then these trace invariants are equal if, and only if, there is a matrix $S \in GL(2, \mathbf{Z})$ such that

$$\begin{pmatrix} \theta_2 \\ \gamma_2\delta_2\theta_2^2 \end{pmatrix} = S \begin{pmatrix} \theta_1 \\ \gamma_1\delta_1\theta_1^2 \end{pmatrix} \operatorname{mod} \begin{pmatrix} \mathbf{Z} \\ \mathbf{Z} \end{pmatrix}.$$

Further, one can more easily show that if θ_j are non-quartic irrationals, i.e., not roots of polynomials over $\mathbf{Z}$ of degree at most four, then the trace invariants are equal if, and only if,

$$\theta_2 = (\pm\theta_1) \operatorname{mod} 1, \quad \text{and} \quad \gamma_2\delta_2\theta_2^2 = (\pm\gamma_1\delta_1\theta_1^2 + m\theta_1) \operatorname{mod} 1,$$

for some integer m. If θ_j are in $(0,(1/2))$ for $j=1,2$, then this shows that $\theta_1=\theta_2$ and hence $\gamma_1\delta_1=\gamma_2\delta_2$. We therefore have one direction of what could be a classification theorem.

Theorem 9. *Let θ_1 and θ_2 be non-quartic irrationals in $(0,(1/2))$. If the C^*-algebras A_1 and A_2 are isomorphic, then $\theta_1=\theta_2$, $\alpha_1=\alpha_2$, and $\gamma_1\delta_1=\gamma_2\delta_2$.*

As to the converse, the necessary conditions by themselves seem to suggest that the Elliott invariant of both algebras are isomorphic. This will hold if it can be shown that the positive cone of $K_0(A_j)$ consists of those elements with positive trace. Further, if one can show that the algebras A_j fall into the classification class of Qing Lin and Chris Phillips, i.e., are direct limits of recursive subhomogeneous C^*-subalgebras, which is a highly nontrivial matter, then one will have obtained a complete classification theorem for these algebras. The difficulty in doing this is illustrated by their recent unpublished papers [**3, 13, 14**], in which [**3**] is a 200-page classification theorem. The authors are thankful to Chris Phillips for making these and related papers available to them.

ENDNOTES

1. If $0\to F_1\to G\to F_2\oplus H\to 0$ is a short exact sequence of finitely generated Abelian groups, where F_1,F_2 are free groups and H is torsion, then $\mathrm{rank}\,(G)=\mathrm{rank}\,(F_1)+\mathrm{rank}\,(F_2)$. This can be seen from the naturally obtained short exact sequence $0\to F_1\oplus F_2\to G\to H\to 0$, from which the result follows. (If G has rank greater than that of a subgroup K, then G/K contains a non-torsion element.)

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