

SPACES OF GENERALIZED TYPE  $\mathcal{H}_\mu$ ,  
SPACES OF TYPE  $S$  AND THE  
HANKEL TRANSFORMATION

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ABSTRACT. In this paper, membership in the spaces of type  $\mathcal{H}_\mu$ , as introduced by Betancor and the author, is characterized by symmetric decay conditions on a function and on its Hankel transform, and on a function and its derivatives. This is applied to obtain intrinsic descriptions of the even functions in the Gelfand-Shilov spaces of type  $S$ .

**1. Introduction.** The space  $\mathcal{H}_\mu$ ,  $\mu \in \mathbf{R}$ , introduced by Zemanian, consists of all those smooth, complex-valued functions  $\varphi = \varphi(x)$ ,  $x \in I = ]0, \infty[$ , such that

$$(1.1) \quad \gamma_{k,m}^\mu(\varphi) = \sup_{x \in I} |x^k (x^{-1}D)^m x^{-\mu-1/2} \varphi(x)| < +\infty, \quad k, m \in \mathbf{N}_0.$$

When endowed with the topology generated by the system of seminorms  $\{\gamma_{k,m}^\mu\}_{k,m \in \mathbf{N}_0}$ ,  $\mathcal{H}_\mu$  becomes a Fréchet space where the Hankel transformation

$$(h_\mu \varphi)(x) = \int_0^\infty (xt)^{1/2} J_\mu(xt) \varphi(t) dt, \quad x \in I$$

is an automorphism, provided that  $\mu \geq -1/2$ ; here,  $J_\mu$  denotes the Bessel function of the first kind and order  $\mu$  [17, Chapter 5]. Thus the space  $\mathcal{H}_\mu$  behaves with respect to the Hankel transformation as the Schwartz space  $S$  with respect to the Fourier transformation.

Inspired by the work of Gelfand and Shilov [10] on the Fourier transformation and the spaces of type  $S$  ( $S_\alpha$ ,  $S^\beta$  and  $S_\alpha^\beta$ ) various

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2000 AMS *Mathematics Subject Classification*. Primary 46F05, 44A15.

*Key words and phrases*. Spaces of type  $S$ , spaces of type  $\mathcal{H}_\mu$ , Hankel transformation.

Partially supported by Universidad de La Laguna (Proyecto 1802010205) and Gobierno de Canarias (PI2001/039).

Received by the editors on October 26, 2002, and in revised form on December 16, 2002.

authors, through different approaches, have tried to design specific analogues in the realm of the Hankel transformation [8, 11–13]. Betancor and the author [4] introduced new spaces of type  $\mathcal{H}_\mu$  which overcame a deficiency of previous constructions. Namely, we proved that the Hankel transformation is an isomorphism between the spaces introduced, thus yielding a complete analogy to the spaces of type  $S$ , which bear the property that the Fourier transformation is an isomorphism from  $S_\alpha$  onto  $S^\alpha$  and from  $S_\alpha^\beta$  onto  $S_\beta^\alpha$ . Van Eijndhoven and van Berkel [16] had established, by entirely different means, that the Hankel-type transformation

$$(H_\mu\varphi)(x) = \int_0^\infty (xt)^{-\mu} J_\mu(xt)\varphi(t)x^{2\mu+1} dt, \quad x \in I$$

is an isomorphism from the spaces  $Se_\alpha$ , respectively  $Se_\alpha^\beta$ , onto the spaces  $Se^\alpha$ , respectively  $Se_\beta^\alpha$ , consisting of all the even functions in the corresponding spaces of type  $S$ . However, these spaces lacked a simple intrinsic description.

Van Eijndhoven [14] and Chung, Chung and Kim [5, 6] have given symmetric characterizations for  $S$  and the spaces of type  $S$  in terms of the decay at infinity of a function and of its Fourier transform.

Closely connected with these, characterizations for the same spaces involving growth conditions on a function and boundedness of its derivatives separately have been found by Chung, Chung and Kim [5, 6]; an alternative proof of such a characterization for  $S$  is due to Chung, Kim and Lee [7]. The Zemanian space  $\mathcal{H}_\mu$  has been similarly characterized by Betancor [1].

Motivated by [1] and [6], in Section 2 below we give symmetric characterizations for the spaces of generalized type  $\mathcal{H}_\mu$ , Definition 2.1, in terms of the decay at infinity of a function and of its  $h_\mu$ -transform. In [16], from a completely different approach, analogous descriptions were obtained for the even functions in the spaces of type  $S$  with respect to the  $H_\mu$ -transformation. A comparison of our results with those in [16] reveals that, when  $\mu \geq -1/2$ ,

$$\mathcal{H}_\mu = x^{\mu+1/2}Se$$

( $Se$  the space of all even functions in  $S$ ) in the following sense: if  $\psi \in Se$ , then the function  $\varphi(x) = \psi(x)$  ( $x \in I$ ) is such that

$x^{\mu+1/2}\varphi(x) \in \mathcal{H}_\mu$  and, conversely, for every  $\varphi \in \mathcal{H}_\mu$  the even extension of  $\psi(x) = x^{-\mu-1/2}\varphi(x)$  ( $x \in I$ ) up to  $\mathbf{R}$  lies in  $S$ . Although this particular result is not new, see [15, Corollary 4.8] and the remarks following it, our approach allows us to show that, interestingly enough, the same relationship holds between the spaces of type  $\mathcal{H}_\mu$  introduced in [4] and the spaces of type  $Se$  discussed in [16], thus providing an intrinsic description of the latter. This is done in Section 3.

Throughout this paper, unless otherwise stated, we shall assume  $\mu \geq -1/2$ . The set of positive integers will be denoted by  $\mathbf{N}$ , while  $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ . For  $k \in \mathbf{N}$  we shall write  $T_{\mu,k} = N_{\mu+k-1} \cdots N_\mu$ , where  $N_\mu = x^{\mu+1/2}Dx^{-\mu-1/2}$ ; for  $k = 0$ ,  $T_{\mu,k}$  will be the identity operator. Also,  $I$  will stand for the positive real axis, and  $C$  will denote a positive constant, depending only on the opportune subscripts, if any, not necessarily the same at each occurrence. We shall represent by  $\|\cdot\|_\iota$  the usual norm of the Lebesgue space  $L^\iota(I)$ , where  $\iota = 2, \infty$ .

**2. Characterization of the spaces of generalized type  $\mathcal{H}_\mu$ .**

First of all, following the spirit of the spaces of type  $\mathcal{H}_\mu$  introduced by Betancor and the author [4], we define the spaces of generalized type  $\mathcal{H}_\mu$ .

**Definition 2.1.** Given sequences  $\{M_p\}_{p \in \mathbf{N}_0}$  and  $\{N_p\}_{p \in \mathbf{N}_0}$  of positive numbers, the spaces  $\mathcal{H}_{\mu, M_p}$ ,  $\mathcal{H}_\mu^{N_p}$ , and  $\mathcal{H}_{\mu, M_p}^{N_p}$ , consist of all those smooth, complex-valued functions  $\varphi = \varphi(x)$ ,  $x \in I$ , such that, respectively,

$$\begin{aligned} \left\| x^{k-1/2}T_{\mu,m}\varphi(x) \right\|_\infty &\leq C_m A^k M_k, \\ \left\| x^{k-1/2}T_{\mu,m}\varphi(x) \right\|_\infty &\leq C_k B^m N_m, \end{aligned}$$

and

$$\left\| x^{k-1/2}T_{\mu,m}\varphi(x) \right\|_\infty \leq C A^k B^m M_k N_m$$

for some  $A, B > 0$  and all  $k, m \in \mathbf{N}_0$ .

Some notation and terminology are in order. For each sequence  $\{M_p\}_{p \in \mathbf{N}_0}$  of positive numbers, its *associated function*  $M = M(\rho)$  is

defined as

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \log \left( \frac{\rho^p}{M_p} \right), \quad \rho \in I.$$

The set  $\mathfrak{S}$  consists of all those sequences  $\{M_p\}_{p \in \mathbf{N}_0}$  of positive numbers such that  $M_0 = 1$ , satisfying the following two properties:

(M.1) (*logarithmic convexity*).  $M_p^2 \leq CM_{p-1}M_{p+1}$ ,  $p \in \mathbf{N}$ .

(M.2) (*stability under ultradifferential operators*). There exist  $R, H > 0$  such that

$$M_{p+q} \leq RH^{p+q} M_p M_q, \quad p, q \in \mathbf{N}_0.$$

Also, given two sequences  $\{M_p\}_{p \in \mathbf{N}_0}$  and  $\{N_p\}_{p \in \mathbf{N}_0}$  of positive numbers satisfying (M.1), we write  $M_p \subset N_p$  if there are constants  $L, C > 0$  such that  $M_p \leq CL^p N_p$ ,  $p \in \mathbf{N}_0$ .

Examples of members of  $\mathfrak{S}$  are the Gevrey sequences  $\{(p!)^\alpha\}_{p \in \mathbf{N}_0}$ ,  $\{p^{p\alpha}\}_{p \in \mathbf{N}_0}$ , and  $\{\Gamma(1 + p\alpha)\}_{p \in \mathbf{N}_0}$ , with  $\alpha > 0$ . Here the convention is made that  $p^{p\alpha} = 1$  when  $p = 0$ . The choices  $M_p = p^{p\alpha}$ ,  $N_p = p^{p\beta}$  ( $\alpha, \beta > 0$ ,  $p \in \mathbf{N}_0$ ) result in the spaces  $\mathcal{H}_{\mu, \alpha}$ ,  $\mathcal{H}_\mu^\beta$  and  $\mathcal{H}_{\mu, \alpha}^\beta$  of type  $\mathcal{H}_\mu$  discussed in [4]. All the theory developed in [4] can be seen to hold, *mutatis mutandis*, for the spaces of generalized type  $\mathcal{H}_\mu$  whose defining sequences belong to  $\mathfrak{S}$ .

Let us recall other descriptions of the Zemanian space  $\mathcal{H}_\mu$ , different from that given in the introduction.

**Proposition 2.2.** *For  $\varphi \in C^\infty(I)$ , the following are equivalent:*

- (i)  $\varphi \in \mathcal{H}_\mu$ .
- (ii) For every  $k, m \in \mathbf{N}_0$ ,  $\|x^k T_{\mu, m} \varphi(x)\|_2 < +\infty$ .
- (iii) For each  $k, m \in \mathbf{N}_0$ ,  $\|x^k T_{\mu, m} \varphi(x)\|_\infty < +\infty$ .

*Proof.* This can be inferred from [2, Theorem 3.3] and [3, Proposition 2.15].  $\square$

An alternative  $L^\infty$ -norm description as well as an  $L^2$ -norm description of  $\mathcal{H}_{\mu, M_p}^{N_p}$  is given by the following analogue of Proposition 2.2.

**Proposition 2.3.** *Let  $\{M_p\}_{p \in \mathbf{N}_0}$ ,  $\{N_p\}_{p \in \mathbf{N}_0}$  be two sequences of positive numbers with the property (M.2). The space  $\mathcal{H}_{\mu, M_p}^{N_p}$  consists of all those  $\varphi \in \mathcal{H}_{\mu}$  satisfying*

$$\|x^k T_{\mu, m} \varphi(x)\|_{\iota} \leq C A^k B^m M_k N_m, \quad k, m \in \mathbf{N}_0$$

for some  $A, B > 0$ , where either  $\iota = 2$  or  $\iota = \infty$ .

*Proof.* Let  $\varphi \in \mathcal{H}_{\mu}$ , and let  $k, m \in \mathbf{N}_0$ . The argument in the proof of [4, Proposition 2.1.6] shows that

$$\begin{aligned} \|x^k T_{\mu, m} \varphi(x)\|_2 &\leq C \left\{ \|x^{k-1/2} T_{\mu, m} \varphi(x)\|_{\infty} \right. \\ &\quad \left. + \|x^{(k+2)-1/2} T_{\mu, m} \varphi(x)\|_{\infty} \right\}, \\ \|x^{k-1/2} T_{\mu, m} \varphi(x)\|_{\infty} &\leq C \left\{ \|x^k T_{\mu, m} \varphi(x)\|_2 + \|x^k T_{\mu, m+2} \varphi(x)\|_2 \right\}. \end{aligned}$$

A slight modification of this argument combined with that in the proof of [4, Proposition 2.1.1] gives

$$\begin{aligned} \|x^k T_{\mu, m} \varphi(x)\|_2 &\leq C \left\{ \|x^k T_{\mu, m} \varphi(x)\|_{\infty} + \|x^{k+2} T_{\mu, m} \varphi(x)\|_{\infty} \right\}, \\ \|x^k T_{\mu, m} \varphi(x)\|_{\infty} &\leq C \left\{ \|x^{k-1/2} T_{\mu, m} \varphi(x)\|_{\infty} \right. \\ &\quad \left. + \|x^{(k+1)-1/2} T_{\mu, m} \varphi(x)\|_{\infty} \right\}. \end{aligned}$$

Under (M.2), this yields easily the desired result.  $\square$

Condition (iii) in Proposition 2.2 can be relaxed in the terms stated in part (ii) of Proposition 2.4 below, as shown by Betancor [1, Proposition 1]. Furthermore, a characterization for  $\mathcal{H}_{\mu}$  through symmetric decay conditions on a function and on its Hankel transform can be given.

**Proposition 2.4.** *For  $\varphi \in C^{\infty}(I)$ , the following are equivalent:*

- (i)  $\varphi \in \mathcal{H}_{\mu}$ .
- (ii) For every  $k \in \mathbf{N}_0$ ,

$$\|x^k \varphi(x)\|_{\infty} < +\infty \quad \text{and} \quad \|T_{\mu, k} \varphi\|_{\infty} < +\infty.$$

(iii) For each  $k \in \mathbf{N}_0$ , there holds

$$\|x^k \varphi(x)\|_\infty < +\infty \quad \text{and} \quad \|x^k (h_\mu \varphi)(x)\|_\infty < +\infty.$$

*Proof.* We only need to show the equivalence between (i) and (iii).

That (i) implies (iii) is straightforward from Proposition 2.2 and the fact that  $h_\mu$  is an automorphism of  $\mathcal{H}_\mu$  [17, Theorem 5.4-1]. Conversely, assume  $\varphi \in C^\infty(I)$  satisfies (iii). If  $k \in \mathbf{N}_0$ , then  $g_k(x) = (-x)^k (h_\mu \varphi)(x)$ ,  $x \in I$ , is integrable on  $I$  with respect to the weight  $x^{\mu+1/2}$ . Moreover, (iii) implies that  $g_k$  is integrable on  $I$  as well. An induction procedure on the basis of [17, Equation 5.4-(2)] then shows that  $(h_{\mu+k} g_k)(x) = T_{\mu,k} \varphi(x)$ , with  $\|T_{\mu,k} \varphi\|_\infty < +\infty$ ,  $k \in \mathbf{N}_0$ . This completes the proof.  $\square$

Next we establish an analogue of Proposition 2.4 for the spaces of generalized type  $\mathcal{H}_\mu$ . In the proof of Theorem 2.5,  $A_i, B_j$ ,  $i, j \in \mathbf{N}$ , will denote suitable positive constants.

**Theorem 2.5.** *Let  $\{M_p\}_{p \in \mathbf{N}_0}$ ,  $\{N_p\}_{p \in \mathbf{N}_0}$  be sequences in  $\mathfrak{S}$  such that  $M_p N_p \supset p!$ , and let  $\varphi \in C^\infty(I)$ . The following are equivalent:*

(i)  $\varphi \in \mathcal{H}_{\mu, M_p}^{N_p}$ .

(ii) There exist  $A, B > 0$  such that

$$\|x^k \varphi(x)\|_\infty \leq CA^k M_k, \quad \|T_{\mu, m} \varphi\|_\infty \leq CB^m N_m$$

for all  $k, m \in \mathbf{N}_0$ .

(iii) There exist  $A, B > 0$  such that

$$\|x^k \varphi(x)\|_\infty \leq CA^k M_k, \quad \|x^m (h_\mu \varphi)(x)\|_\infty \leq CB^m N_m$$

whenever  $k, m \in \mathbf{N}_0$ .

(iv) There exist  $a, b > 0$  such that

$$\|\exp\{M(ax)\} \varphi(x)\|_\infty < +\infty, \quad \|\exp\{N(bx)\} (h_\mu \varphi)(x)\|_\infty < +\infty,$$

where  $M = M(\rho)$  and  $N = N(\rho)$  are the associated functions of the sequences  $\{M_p\}_{p \in \mathbf{N}_0}$  and  $\{N_p\}_{p \in \mathbf{N}_0}$ , respectively.

*Proof.* From Proposition 2.3 and the identity  $h_\mu(\mathcal{H}_{\mu, M_p}^{N_p}) = \mathcal{H}_{\mu, N_p}^{M_p}$ , cf. [4, Proposition 3.2.2], it is apparent that (i) implies both (ii) and (iii). The equivalence between (iii) and (iv) is also clear.

Let us prove that (ii) implies (i). If  $\varphi$  satisfies (ii) then  $\varphi \in \mathcal{H}_\mu$ , by Proposition 2.4. Hence, we only need to show that  $\varphi$  satisfies the  $L^2$ -norm estimate in Proposition 2.3. First of all, we observe that

$$\begin{aligned} \|x^k \varphi(x)\|_2 &= \left\{ \int_0^\infty |x^k(1+x^2)\varphi(x)|^2 \frac{dx}{(1+x^2)^2} \right\}^{1/2} \\ &\leq C \|x^k(1+x^2)\varphi(x)\|_\infty \\ &\leq C \{ \|x^k \varphi(x)\|_\infty + \|x^{k+2} \varphi(x)\|_\infty \} \\ &\leq C \{ A^k M_k + A^{k+2} M_{k+2} \} \\ &\leq CA_1^k M_k, \quad k \in \mathbf{N}_0. \end{aligned}$$

Now we proceed by induction. Let  $m \in \mathbf{N}$  and assume that  $\|x^k T_{\mu, n} \varphi(x)\|_2 \leq CA^k B^n M_k N_n$  whenever  $k, n \in \mathbf{N}_0, 0 \leq n \leq m-1$ , for some  $A, B > 0$ . We want to prove that  $\|x^k T_{\mu, m} \varphi(x)\|_2 \leq C\tilde{A}^k \tilde{B}^m M_k N_m$  for certain  $\tilde{A}, \tilde{B} > 0$  and all  $k \in \mathbf{N}_0$ .

To this end, let  $k \in \mathbf{N}$ , with  $2k \geq m$ . An integration by parts yields

$$\begin{aligned} (2.1) \quad &\|x^k T_{\mu, m} \varphi(x)\|_2^2 \\ &= \int_0^\infty x^{2k} T_{\mu, m} \varphi(x) T_{\mu, m} \overline{\varphi(x)} dx \\ &= \int_0^\infty \left[ (x^{-1}D)^m x^{-\mu-1/2} \overline{\varphi(x)} \right] \left[ x^{2k+m+\mu+1/2} T_{\mu, m} \varphi(x) \right] dx \\ &= \left| \int_0^\infty \left[ (Dx^{-1})^m x^{2k+m+\mu+1/2} T_{\mu, m} \varphi(x) \right] x^{-\mu-1/2} \overline{\varphi(x)} dx \right|. \end{aligned}$$

In this computation we have taken into account the fact that

$$\begin{aligned} (2.2) \quad &\left[ (x^{-1}D)^j x^{2k+m+\mu-1/2} T_{\mu, m} \varphi(x) \right] \left[ (x^{-1}D)^{m-j-1} x^{-\mu-1/2} \overline{\varphi(x)} \right] \Big|_{x=0+}^{x \rightarrow +\infty} \\ &= 0 \end{aligned}$$

when  $j \in \mathbf{N}_0$ ,  $0 \leq j \leq m - 1$ . To check this, let  $j \in \mathbf{N}_0$ ,  $0 \leq j \leq m - 1$ . By the Leibniz rule,

$$\begin{aligned}
 & \left[ (x^{-1}D)^j x^{2k+m+\mu-1/2} T_{\mu,m} \varphi(x) \right] \left[ (x^{-1}D)^{m-j-1} x^{-\mu-1/2} \overline{\varphi(x)} \right] \\
 &= \sum_{i=0}^j a_i x^{2(k+m+\mu-i)} \left[ (x^{-1}D)^{m+j-i} x^{-\mu-1/2} \varphi(x) \right] \\
 (2.3) \quad & \qquad \qquad \qquad \times \left[ (x^{-1}D)^{m-j-1} x^{-\mu-1/2} \overline{\varphi(x)} \right] \\
 &= \sum_{i=0}^j a_i x^{2k-i} [T_{\mu,m+j-i} \varphi(x)] [T_{\mu,m-j-1} \overline{\varphi(x)}]
 \end{aligned}$$

for some  $a_i \in \mathbf{R}$ , with  $i \in \mathbf{N}_0$ ,  $0 \leq i \leq j$ . Since  $\varphi \in \mathcal{H}_\mu$ , Proposition 2.2 yields (2.2) as a consequence of (2.3).

Again by the Leibniz rule,

$$\begin{aligned}
 & \left[ (Dx^{-1})^m x^{2k+m+\mu+1/2} T_{\mu,m} \varphi(x) \right] x^{-\mu-1/2} \overline{\varphi(x)} \\
 (2.4) \quad &= \sum_{j=0}^m \binom{m}{j} 2^j (k+m+\mu) \\
 & \qquad \qquad \qquad \cdots (k+m+\mu-j+1) x^{2k-j} \overline{\varphi(x)} T_{\mu,2m-j} \varphi(x).
 \end{aligned}$$

After plugging (2.4) into (2.1), bearing in mind that  $\{M_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$ ,  $\{N_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$ , and  $M_p N_p \supset p!$ , we may write

$$\begin{aligned}
 \|x^k T_{\mu,m} \varphi(x)\|_2^2 &\leq \sum_{j=0}^m \binom{m}{j} 2^j (k+m+\mu) \cdots (k+m+\mu-j+1) \\
 & \qquad \qquad \qquad \times \int_0^\infty |x^{2k-j} \varphi(x)| |T_{\mu,2m-j} \varphi(x)| dx \\
 &\leq C \sum_{j=0}^m \binom{m}{j} \binom{k+m+\mu}{j} j! 2^j \|T_{\mu,2m-j} \varphi\|_\infty \\
 (2.5) \quad & \qquad \qquad \times \{ \|x^{2k-j} \varphi(x)\|_\infty + \|x^{2k-j+2} \varphi(x)\|_\infty \}
 \end{aligned}$$



$$\begin{aligned}
 &\leq C \sum_{j=0}^m \binom{m}{j} \binom{k+m+\mu}{j} j! 2^j B^{2m-j} N_{2m-j} \\
 &\quad \times (A^{2k-j} M_{2k-j} + A^{2k-j+2} M_{2k-j+2}) \\
 &\leq C \sum_{j=0}^m \binom{m}{j} \binom{k+m+\mu}{j} j! 2^j A^{2k-j} B^{2m-j} \\
 &\quad \times M_{2k-j} N_{2m-j} (1 + RA^2 H^{2k-j+2} M_2) \\
 &\leq CA_2^{2k} B_2^{2m} M_{2k} N_{2m} \sum_{j=0}^m \binom{m}{j} \binom{k+m+\mu}{j} \frac{j!}{M_j N_j} \\
 &\leq CA_3^{2k} B_3^{2m} M_k^2 N_m^2, \quad k \in \mathbf{N}, 2k \geq m.
 \end{aligned}$$

On the other hand, if  $k \in \mathbf{N}$  and  $2k < m$ , then (2.2) holds for  $j \in \mathbf{N}_0$ ,  $0 \leq j \leq 2k - 1$ , because  $\varphi \in \mathcal{H}_\mu$ . Integrate by parts, apply the Leibniz rule, use the induction hypotheses and take into account that  $\{M_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$ ,  $\{N_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$ , and  $M_p N_p \supset p!$ , to obtain

(2.6)

$$\begin{aligned}
 \|x^k T_{\mu,m} \varphi(x)\|_2^2 &\leq \sum_{j=0}^{2k} \binom{2k}{j} \binom{k+m+\mu}{j} j! 2^j \\
 &\quad \times \int_0^\infty |x^{2k-j} T_{\mu,m-2k} \varphi(x)| |T_{\mu,m+2k-j} \varphi(x)| dx \\
 &\leq C \sum_{j=0}^{2k} \binom{2k}{j} \binom{k+m+\mu}{j} j! 2^j \|T_{\mu,m+2k-j} \varphi\|_\infty \\
 &\quad \times \{ \|x^{2k-j} T_{\mu,m-2k} \varphi(x)\|_2 + \|x^{2k-j+2} T_{\mu,m-2k} \varphi(x)\|_2 \} \\
 &\leq C \sum_{j=0}^{2k} \binom{2k}{j} \binom{k+m+\mu}{j} j! 2^j A^{2k-j} B^{m+2k-j} B^{m-2k} \\
 &\quad \times M_{2k-j} N_{m+2k-j} N_{m-2k} (1 + RA^2 H^{2k-j+2} M_2) \\
 &\leq CA_4^{2k} B_4^{2m} M_{2k} N_m^2 \sum_{j=0}^{2k} \binom{2k}{j} \binom{k+m+\mu}{j} \frac{j!}{M_j N_j} \\
 &\leq CA_5^{2k} B_5^{2m} M_k^2 N_m^2, \quad k \in \mathbf{N}, 2k < m.
 \end{aligned}$$

The case  $k = 0$  remains to be discussed. Given  $r \in \mathbf{N}$ ,  $n \in \mathbf{N}_0$  and  $x \in I$ , one has

$$\begin{aligned} & [x^r T_{\mu,n} \varphi(x)]^2 \\ &= \int_0^x D [t^r T_{\mu,n} \varphi(t)]^2 dt \\ &= \int_0^x 2t^r T_{\mu,n} \varphi(t) \{ (r+n+\mu+1/2) t^{r-1} T_{\mu,n} \varphi(t) + t^r T_{\mu,n+1} \varphi(t) \} dt. \end{aligned}$$

Hence

$$(2.7) \quad \begin{aligned} \|x^r T_{\mu,n} \varphi(x)\|_\infty^2 &\leq C \{ (r+n+\mu+1/2) \|x^r T_{\mu,n} \varphi(x)\|_2 \\ &\quad \times \|x^{r-1} T_{\mu,n} \varphi(x)\|_2 + \|T_{\mu,n+1} \varphi\|_\infty \\ &\quad \times [\|x^{2r} T_{\mu,n} \varphi(x)\|_2 + \|x^{2r+1} T_{\mu,n} \varphi(x)\|_2] \} \end{aligned}$$

whenever  $r \in \mathbf{N}$  and  $n \in \mathbf{N}_0$ . Particularizing (2.7) for  $r = 2$  and  $n = m$ , by means of (2.5) and (2.6) we find that

$$(2.8) \quad \begin{aligned} \|x^2 T_{\mu,m} \varphi(x)\|_\infty^2 &\leq \{ P^m \|x^2 T_{\mu,m} \varphi(x)\|_2 \|x T_{\mu,m} \varphi(x)\|_2 \\ &\quad + C \|T_{\mu,m+1} \varphi\|_\infty \times [\|x^4 T_{\mu,m} \varphi(x)\|_2 \\ &\quad + \|x^5 T_{\mu,m} \varphi(x)\|_2] \} \\ &\leq C B_6^{2m} N_m^2 \end{aligned}$$

for some  $P > 0$ . Finally, upon multiplying and dividing the integrand in  $\|T_{\mu,m} \varphi\|_2$  by  $(1+x^2)^2$  and using (2.8) we arrive at

$$(2.9) \quad \begin{aligned} \|T_{\mu,m} \varphi\|_2 &\leq C \{ \|T_{\mu,m} \varphi\|_\infty + \|x^2 T_{\mu,m} \varphi(x)\|_\infty \} \\ &\leq C B_7^m N_m. \end{aligned}$$

From (2.5), (2.6) and (2.9) the desired conclusion follows.

To complete the proof, we shall establish that (iii) implies (i). Under (iii),  $\varphi \in \mathcal{H}_\mu$ , by Proposition 2.4, and hence  $h_\mu \varphi \in \mathcal{H}_\mu$  [17, Theorem 5.4-1]. The assumption that

$$\|x^m (h_\mu \varphi)(x)\|_\infty \leq C B^m N_m, \quad m \in \mathbf{N}_0,$$

gives

$$|(h_\mu\varphi)(x)| \leq C \exp \left\{ -N \left( \frac{x}{B} \right) \right\}, \quad x \in I,$$

where  $N = N(\rho)$  is the associated function of  $\{N_p\}_{p \in \mathbf{N}_0}$ . From the integral representation

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin x - \nu x) dx - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-(z \sinh \gamma + \nu\gamma)} d\gamma,$$

$$z \in I,$$

[9, Equation (9), p. 17], it follows that  $\sup_{z \in I} |J_\nu(z)| \leq C$ , with  $C > 0$  independent of  $\nu \geq 0$ . Using [17, Equation 5.4-(12)] and the fact that  $\{N_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$  we may write

$$\begin{aligned} \left| x^{-1/2} T_{\mu,m} \varphi(x) \right| &= \left| x^{-1/2} T_{\mu,m} (h_\mu (h_\mu \varphi)(y))(x) \right| \\ &= \left| x^{-1/2} h_{\mu+m} (y^m (h_\mu \varphi)(y))(x) \right| \\ &= \left| \int_0^\infty J_{\mu+m}(xy) y^{m+1/2} (h_\mu \varphi)(y) dy \right| \\ &\leq C \int_0^\infty y^{m+1/2} \exp \left\{ -N \left( \frac{y}{B} \right) \right\} dy \\ &\leq C \int_0^\infty y^m (1+y) \exp \left\{ -N \left( \frac{y}{B} \right) \right\} dy \\ &\leq C \left\{ \left\| y^{2m} \exp \left\{ -N \left( \frac{y}{B} \right) \right\} \right\|_\infty^{1/2} + \|y^{2m+2}\| \right. \\ &\quad \left. \times \exp \left\{ -N \left( \frac{y}{B} \right) \right\} \right\|_\infty^{1/2} \int_0^\infty \exp \left\{ -\frac{N(y/B)}{2} \right\} dy \\ &\leq C \left( B^m N_{2m}^{1/2} + B^{m+1} N_{2m+2}^{1/2} \right) \\ &\leq C B_8^m N_m, \quad x \in I, \quad m \in \mathbf{N}, \end{aligned}$$

where  $C > 0$  does not depend on  $m \in \mathbf{N}$ . A corresponding bound is easily obtained for  $m = 0$ . Now it suffices to argue by induction exactly as in the proof that (ii) implies (i), this time taking as a starting point the estimates

$$\begin{aligned} \|x^k \varphi(x)\|_\infty &\leq C A^k M_k, \\ \left\| x^{-1/2} T_{\mu,m} \varphi(x) \right\|_\infty &\leq C B^m N_m, \quad k, m \in \mathbf{N}_0. \end{aligned}$$

Thus we are done.  $\square$

*Remark 2.6.* If  $\{M_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$  and  $\{N_p\}_{p \in \mathbf{N}_0} \in \mathfrak{S}$ , the techniques used in Theorem 2.5 allow to give similar characterizations for the spaces  $\mathcal{H}_{\mu, M_p}$  and  $\mathcal{H}_{\mu}^{N_p}$ .

By [10, Equation IV.2.1-(3)], particularizing  $M_k = k^{k\alpha}$ ,  $N_m = m^{m\beta}$  ( $\alpha, \beta > 0$ ,  $k, m \in \mathbf{N}_0$ ) yields as a corollary of Theorem 2.5 the next characterization for the spaces  $\mathcal{H}_{\mu, \alpha}^{\beta}$  introduced in [4], provided that  $\alpha + \beta \geq 1$ ; analogous results can be obtained for  $\mathcal{H}_{\mu, \alpha}$  and  $\mathcal{H}_{\mu}^{\beta}$ .

**Corollary 2.7.** *Let  $\alpha, \beta > 0$ ,  $\alpha + \beta \geq 1$ . Then  $\varphi \in \mathcal{H}_{\mu, \alpha}^{\beta}$  if and only if there exist positive constants  $a, b$  such that*

$$\left\| \exp \left\{ ax^{1/\alpha} \right\} \varphi(x) \right\|_{\infty} < +\infty, \quad \left\| \exp \left\{ bx^{1/\beta} \right\} (h_{\mu}\varphi)(x) \right\|_{\infty} < +\infty.$$

**3. The even functions in the spaces of type  $S$ .** We conclude by proving that the spaces  $\mathcal{H}_{\mu}$  and  $x^{\mu+1/2}Se$  coincide, and similarly for the spaces  $\mathcal{H}_{\mu, \alpha}^{\beta}$  and  $x^{\mu+1/2}Se_{\alpha}^{\beta}$ ,  $\alpha, \beta > 0$ ,  $\alpha + \beta \geq 1$ .

**Proposition 3.1.** *The identity  $\mathcal{H}_{\mu} = x^{\mu+1/2}Se$  holds.*

*Proof.* It is known [16, Theorem 2.1] that an even function  $\psi \in C^{\infty}(\mathbf{R})$  belongs to  $Se$  if and only if

$$(3.1) \quad \left\| x^k \psi(x) \right\|_{\infty} < +\infty, \quad \left\| x^k (H_{\mu}\psi)(x) \right\|_{\infty} < +\infty, \quad k \in \mathbf{N}_0,$$

where  $H_{\mu}$  denotes the Hankel-type transformation

$$(H_{\mu}\psi)(x) = \int_0^{\infty} (xt)^{-\mu} J_{\mu}(xt) \psi(t) t^{2\mu+1} dt, \quad x \in I.$$

Observe that

$$(3.2) \quad (H_{\mu}\psi)(x) = x^{-\mu-1/2} h_{\mu}(t^{\mu+1/2} \psi(t))(x), \quad x \in I,$$

so that (3.1) is equivalent to

$$(3.3) \quad \begin{aligned} \left\| x^{k-\mu-1/2}(x^{\mu+1/2}\psi(x)) \right\|_{\infty} &< +\infty, \\ \left\| x^{k-\mu-1/2}h_{\mu}(t^{\mu+1/2}\psi(t))(x) \right\|_{\infty} &< +\infty, \end{aligned} \quad k \in \mathbf{N}_0.$$

We claim that (3.3) holds if and only if

$$(3.4) \quad \begin{aligned} \left\| x^k(x^{\mu+1/2}\psi(x)) \right\|_{\infty} &< +\infty, \\ \left\| x^k h_{\mu}(t^{\mu+1/2}\psi(t))(x) \right\|_{\infty} &< +\infty, \end{aligned} \quad k \in \mathbf{N}_0,$$

a condition in turn equivalent to the fact that  $x^{\mu+1/2}\psi(x) \in \mathcal{H}_{\mu}$ , by Proposition 2.4.

Indeed, if (3.3) is satisfied then so is (3.4), because to every  $k \in \mathbf{N}_0$  there corresponds  $r \in \mathbf{N}_0$  such that  $x^{k+\mu+1/2} \leq (1+x^2)^r$  ( $x \in I$ ). Conversely, assume that  $\psi$  satisfies (3.4) and let  $\varphi(x) = x^{\mu+1/2}\psi(x) \in \mathcal{H}_{\mu}$ . Then  $h_{\mu}\varphi \in \mathcal{H}_{\mu}$  as well, and (3.3) can be immediately deduced from (1.1).  $\square$

**Theorem 3.2.** For  $\alpha, \beta > 0$ ,  $\alpha + \beta \geq 1$ , we have

$$\mathcal{H}_{\mu, \alpha}^{\beta} = x^{\mu+1/2}Se_{\alpha}^{\beta}.$$

*Proof.* By [16, Theorem 2.5], an even function  $\psi \in C^{\infty}(\mathbf{R})$  belongs to  $Se_{\alpha}^{\beta}$  if and only if there exist  $a, b > 0$  such that

$$\left\| \exp \left\{ ax^{1/\alpha} \right\} \psi(x) \right\|_{\infty} < +\infty, \quad \left\| \exp \left\{ bx^{1/\beta} \right\} (H_{\mu}\psi)(x) \right\|_{\infty} < +\infty,$$

where  $H_{\mu}$  is related to  $h_{\mu}$  by (3.2). By [10, Equation IV.2.1-(3)], this means that some  $A, B > 0$  are such that

$$(3.5) \quad \begin{aligned} \left\| x^k \psi(x) \right\|_{\infty} &\leq CA^k k^{k\alpha} \\ \left\| x^m (H_{\mu}\psi)(x) \right\|_{\infty} &\leq CB^m m^{m\beta}, \end{aligned} \quad k, m \in \mathbf{N}_0.$$

Condition (3.5) is equivalent to

$$(3.6) \quad \begin{aligned} \left\| x^{k-\mu-1/2}(x^{\mu+1/2}\psi(x)) \right\|_{\infty} &\leq CA^k k^{k\alpha} \\ \left\| x^{m-\mu-1/2}h_{\mu}(t^{\mu+1/2}\psi(t))(x) \right\|_{\infty} &\leq CB^m m^{m\beta}, \end{aligned} \quad k, m \in \mathbf{N}_0.$$

Note that (3.6) suffices for  $\varphi(x) = x^{\mu+1/2}\psi(x)$ ,  $x \in I$ , to lie in  $\mathcal{H}_{\mu,\alpha}^{\beta}$ . Indeed, the same reason why (3.4) follows from (3.3) also proves

$$(3.7) \quad \begin{aligned} \left\| x^k \varphi(x) \right\|_{\infty} &\leq CA^k k^{k\alpha} \\ \left\| x^m (h_{\mu}\varphi)(x) \right\|_{\infty} &\leq CB^m m^{m\beta}, \end{aligned} \quad k, m \in \mathbf{N}_0,$$

so that  $\varphi \in \mathcal{H}_{\mu,\alpha}^{\beta}$ , by Theorem 2.5.

Conversely, assume  $\varphi \in \mathcal{H}_{\mu,\alpha}^{\beta}$ , and let  $\psi(x) = x^{-\mu-1/2}\varphi(x)$ ,  $x \in I$ . Then

$$\left\| x^k T_{\mu,m}\varphi(x) \right\|_{\infty} = \left\| x^{m+k+\mu+1/2}(x^{-1}D)^m \psi(x) \right\|_{\infty} \leq CA^k B^m k^{k\alpha} m^{m\beta},$$

$$k, m \in \mathbf{N}_0,$$

by Proposition 2.3. Using [17, Equation 5.4-(12)], it is easy to see that

$$\begin{aligned} \left\| x^m T_{\mu,k}(h_{\mu}\varphi)(x) \right\|_{\infty} &= \left\| x^{m+k+\mu+1/2}(x^{-1}D)^k (H_{\mu}\psi)(x) \right\|_{\infty} \\ &\leq C\tilde{A}^k \tilde{B}^m k^{k\alpha} m^{m\beta}, \quad k, m \in \mathbf{N}_0. \end{aligned}$$

In particular,

$$\begin{aligned} \left\| x^{k+\mu+1/2}\psi(x) \right\|_{\infty} &\leq CA^k k^{k\alpha}, \\ \left\| x^{m+\mu+1/2}(H_{\mu}\psi)(x) \right\|_{\infty} &\leq C\tilde{B}^m m^{m\beta}, \quad k, m \in \mathbf{N}_0. \end{aligned}$$

Now

$$\begin{aligned} |x^k \psi(x)| &= \left| x^{-\mu-1/2}(x^{k+\mu+1/2}\psi(x)) \right| \\ &\leq \left| x^{k+\mu+1/2}\psi(x) \right| \leq CA^k k^{k\alpha}, \quad k \in \mathbf{N}_0, \quad x \geq 1, \end{aligned}$$

whence

$$\sup_{x \geq 1} \left| \exp \left\{ ax^{1/\alpha} \right\} \psi(x) \right| < +\infty$$

for some  $a > 0$ . As

$$\sup_{0 < x < 1} \left| \exp \left\{ ax^{1/\alpha} \right\} \psi(x) \right| < +\infty,$$

we obtain

$$\left\| \exp \left\{ ax^{1/\alpha} \right\} \psi(x) \right\|_{\infty} < +\infty.$$

Similarly, it can be proven that there exists  $b > 0$  for which

$$\left\| \exp \left\{ bx^{1/\beta} \right\} (H_{\mu}\psi)(x) \right\|_{\infty} < +\infty.$$

From [16, Theorem 2.5] we conclude that  $\psi \in Se_{\alpha}^{\beta}$ .  $\square$

*Remark 3.3.* If  $\alpha, \beta > 0$ , corresponding results can be obtained connecting the space  $\mathcal{H}_{\mu, \alpha}$ , respectively  $\mathcal{H}_{\mu}^{\beta}$ , to the space  $Se_{\alpha}$ , respectively  $Se^{\beta}$ .

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