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ILLUSTRATION OF THE LOGARITHMIC DERIVATIVES BY EXAMPLES SUITABLE FOR CLASSROOM TEACHING

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ABSTRACT. In this contribution, the logarithmic derivatives of a matrix are illustrated by examples, which stem from dynamical systems representing problems for one- and multi-mass vibration models and which are suitable for classroom teaching. Further, historical remarks are made and links to the further development, namely the development of a differential calculus for norms of matrix and vector functions, are established.

1. Introduction. If A is a complex square matrix and $\|\cdot\|$ any matrix norm with $\|E\| = 1$, where E is the identity matrix, then the logarithmic derivative $\mu[A]$ is defined by $\mu[A] = \lim_{h\to 0+}(\|E + A\,h\| - 1)/h$. By means of $\mu[A]$, one obtains, e.g., $\|e^{A\,t}\| \le e^{\mu[A]\,t}$, $t \ge 0$, which improves the estimate $\|e^{A\,t}\| \le e^{\|A\|\,t}$, $t \ge 0$, since always $|\mu[A]| \le \|A\|$. The cited estimates play an important role in initial value problems of the form $\dot{x} = A\,x$, $x(0) = x_0$, because its solution is given by $x(t) = \Phi(t)\,x_0$ with $\Phi(t) = e^{A\,t}$. For the matrix sup-norms $\|\cdot\| = \|\cdot\|_p$ with $p \in \{1, 2, \infty\}$, in [**2**] and [**3**], formulae for $\mu_p^{(1)}[A] := \mu_p[A] = \lim_{h\to 0+}(\|E + A\,h\|_p - 1)/h$ are stated. Note that $\mu[A] = D_+\|\Phi(0)\|$. Now, in [**13**], the author has shown that the second right derivatives $\mu_p^{(2)}[A] = D_+^2\|\Phi(0)\|_p$, $p \in \{1, 2, \infty\}$, exist, and called them second logarithmic derivatives. Further, formulae for $\mu_p^{(2)}[A]$, $p \in \{1, 2, \infty\}$ are given in [**13**] and [**14**]. The motivation to investigate the logarithmic derivatives was the following: the graph of $y = \|\Phi(t)\|_\infty$ for the considered examples seems to have a right curvature. This is proven strictly if one can show that $D_+^2\|\Phi(0)\|_\infty < 0$. So, we had to prove the existence of $D_+^2\|\Phi(0)\|_\infty$ and to derive a formula for it, which was done in [**13**].

In this paper, we want to illustrate the logarithmic derivatives by examples suitable for classroom teaching.

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More precisely, the paper is structured as follows. Sections 2 and 3, respectively, review the definition of the logarithmic derivative and the formulae for it in the norms $\|\cdot\|_p$, $p \in \{1, 2, \infty\}$. In Section 4, examples are given using the system matrices A of a one-mass and of a multi-mass vibration model. In Section 5, applications to upper bounds on $y = \|\Phi(t)\|_\infty$ for the initial behavior of $t \geq 0$ are given. In Section 6, some historical remarks are made. Sections 4–6 could be used in classroom teaching. Section 7 describes the further development that has originated from the logarithmic derivatives. Finally, Section 8 is an Appendix containing another check of the formulae for $\mu_2^{(1)}[A]$ and $\mu_2^{(2)}[A]$ in the case of the one-mass model. This material can be used as an exercise for classroom teaching. The references [**4, 6, 8**–**10, 22, 23**] and [**25**–**28**] are given even though they are not directly used in this paper in order to provide the reader with some additional material helpful in the present subject.

2. Definition of the logarithmic derivatives. Let $A \in \mathbf{C}^{n\times n}$, respectively $A \in \mathbf{R}^{n\times n}$, and let $\Phi(t) = e^{A\,t}$, $t \geq 0$, be the associated *fundamental matrix*, cf. [**2**, p. 43], or *evolution*, cf. [**1**, p. 91]. As a preparation to the definition of the logarithmic derivatives, we need the following lemma, which states, loosely speaking, that for every $t_0 \geq 0$ and for $p \in \{1, 2, \infty\}$ the function $t \mapsto \|\Phi(t)\|_p$ is real analytic in some neighborhood $[t_0, t_0 + \Delta t_0]$.

Lemma 1. *Let $p \in \{1, 2, \infty\}$ and $t_0 \in \mathbf{R}_0^+$. Then, there exists a number $\Delta t_0 > 0$ and a function $t \mapsto \hat{\Phi}(t)$, which is real analytic on $[t_0, t_0 + \Delta t_0]$ such that $\hat{\Phi}(t) = \|\Phi(t)\|_p$ for $t \in [t_0, t_0 + \Delta t_0]$.*

Proof. See [**13**, Lemma 2.2] for $p = \infty$ and $t_0 = 0$, and also [**14**, Lemma 1] for $p = \infty$ and $p = 2$. The case $p = 1$ follows from the identity $\|\Phi(t)\|_1 = \|\Phi(t)^T\|_\infty = \|e^{A^T\,t}\|_\infty$. $\square$

We mention that we apply the case $t_0 = 0$, in this paper.

Due to Lemma 1, the following definition is meaningful.

Definition 2. Let $A \in \mathbf{C}^{n\times n}$, respectively $A \in \mathbf{R}^{n\times n}$. Further, let $\Phi(t) = e^{A\,t}$, $t \geq 0$, be the associated fundamental matrix. Then, the *kth logarithmic derivative* $\mu_p^{(k)}[A]$, $k = 0, 1, 2, \ldots$ of A in the norm $\|\cdot\|_p$, $p \in \{1, 2, \infty\}$, is defined as the kth right derivative of the function $t \mapsto \|\Phi(t)\|_p$ at $t = t_0 = 0$, that is,

$$\mu_p^{(k)}[A] := D_+^k \|\Phi(0)\|_p, \quad k = 0, 1, 2, \ldots .$$

Remark. According to [**21**, p. 71, Lemma 3',b)], in every matrix sup-norm, one has the representation

$$\mu[A] = D_+\|\Phi(0)\| = \lim_{h\to 0+} (\log \|e^{A\,h}\|)/h.$$

This seems to be the reason why $\mu[A]$ is called logarithmic derivative or logarithmic norm.

Remark. Whereas the first logarithmic derivative in every norm with $\|E\| = 1$ exists, nothing is known on higher logarithmic derivatives in the general case.

3. Formulae for the first two logarithmic derivatives. In this section we review the formulae for $\mu_p^{(k)}[A]$, $k = 1, 2$ with $p \in \{1, 2, \infty\}$.

3.1 Formulae for $p = \infty$. First, we state the formulae for complex and then for real $n \times n$-matrices A; in the latter case, a general formula for the logarithmic derivative of any order can be given.

(i) *A complex.* First, let $A \in \mathbf{C}^{n\times n}$. Further, as a preparation, define the following functionals on $\mathbf{C}^{n\times n}$:

(2)
$$\nu_{ij}[A] := \begin{cases} (\operatorname{Re} A_{ij} \operatorname{Re}(A^2)_{ij} + \operatorname{Im} A_{ij} \operatorname{Im}(A^2)_{ij})/|A_{ij}| & A_{ij} \neq 0, \\ |(A^2)_{ij}| & A_{ij} = 0, \end{cases}$$

(3)
$$\lambda_i^{(1)}[A] := \operatorname{Re} A_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}|,$$

(4)
$$\lambda_i^{(2)}[A] := \mathrm{Re}\,(A^2)_{ii} + (\mathrm{Im}\,A_{ii})^2 + \sum_{\substack{j=1 \\ i\neq j}}^{n} \nu_{ij}[A].$$

Then, one has the following theorem.

Theorem 3 ($p = \infty$, A complex). *Let $A \in \mathbf{C}^{n\times n}$, let $I_0 := \{1,\dots,n\}$ and I_1 be the set of all indices $i_1 \in I_0$ where $\lambda_i^{(1)}[A]$ attains its maximum, i.e.,*

$$I_1 := \Big\{ i_1 \in I_0 \,|\, \lambda_{i_1}^{(1)}[A] = \max_{i\in I_0} \lambda_i^{(1)}[A] \Big\}.$$

Then

$$\mu_\infty^{(1)}[A] = \max_{i\in I_0} \lambda_i^{(1)}[A], \tag{5}$$

$$\mu_\infty^{(2)}[A] = \max_{i\in I_1} \lambda_i^{(2)}[A], \tag{6}$$

where $\lambda_i^{(1)}[A]$ and $\lambda_i^{(2)}[A]$ are given by equations (3) *and* (4), *respectively.*

Proof. See [**13**, pp. 385–386]. □

(ii) *A real.* Now, let $A \in \mathbf{R}^{n\times n}$. In this case, a unified formula for all logarithmic derivatives can be given. For this, define the following sign functionals:

$$s_{ij}^{(1)}(A) := \mathrm{sgn}\,(A_{ij}) \tag{7}$$

and

$$s_{ij}^{(k)}(A) := \begin{cases} \mathrm{sgn}\,(A_{ij}) & A_{ij} \neq 0 \\ \mathrm{sgn}\,((A^2)_{ij}) & A_{ij} = 0,\ (A^2)_{ij} \neq 0 \\ \mathrm{sgn}\,((A^3)_{ij}) & A_{ij} = 0,\ (A^2)_{ij} = 0,\ (A^3)_{ij} \neq 0 \\ \vdots & \\ \mathrm{sgn}\,((A^k)_{ij}) & (A^l)_{ij} = 0,\ l = 1,\dots,k-1 \end{cases} \tag{8}$$

for $k = 2, 3, \ldots$. Relation (8) can also be written as

$$s_{ij}^{(k)}(A) = \begin{cases} s_{ij}^{(k-1)}(A) & s_{ij}^{(k-1)}(A) \neq 0 \\ \operatorname{sgn}\left((A^k)_{ij}\right) & s_{ij}^{(k-1)}(A) = 0 \end{cases} \tag{9}$$

for $k = 2, 3, \ldots$. With these sign functionals, define the further functionals

$$\lambda_i^{(k)}[A] = (A^k)_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} s_{ij}^{(k)}(A)\,(A^k)_{ij}, \tag{10}$$

$k = 1, 2, \ldots$. Then, the logarithmic derivatives for *real matrices* read as follows

Theorem 4 ($p = \infty$, A real). *Let $A \in \mathbf{R}^{n \times n}$, $I_0 = \{1, \ldots, n\}$ and I_k be the set of all indices $i_k \in I_{k-1}$ where $\lambda_i^{(k)}[A]$ from equation* (10) *attains its maximum, i.e.,*

$$I_k := \Big\{ i_k \in I_{k-1} \,|\, \lambda_{i_k}^{(k)}[A] = \max_{i \in I_{k-1}} \lambda_i^{(k)}[A] \Big\},$$

$k = 1, 2, \ldots$. *Then, the logarithmic derivatives are given by*

$$\mu_\infty^{(k)}[A] = \max_{i \in I_{k-1}} \lambda_i^{(k)}[A], \tag{11}$$

$k = 1, 2, \ldots$.

Proof. See [**13**, p. 388]. $\square$

3.2 Formulae for $p = 1$. This case is reduced to the case $p = \infty$. We have

Theorem 5 ($p = 1$, A complex or real). *Let $A \in \mathbf{C}^{n \times n}$ or $A \in \mathbf{R}^{n \times n}$. Then,*

$$\mu_1^{(k)}[A] = \mu_\infty^{(k)}[A^T], \ k = 1, 2, \ldots. \tag{12}$$

Proof. Formula (12) follows from the relation $\|\Phi(t)\|_1 = \|\Phi(t)^T\|_\infty = \|e^{A^T t}\|_\infty$. $\square$

3.3 Formulae for $p = 2$. For $p = 2$, we have unified formulae for the complex and real cases.

Theorem 6. *Let $A \in \mathbf{C}^{n\times n}$ or $A \in \mathbf{R}^{n\times n}$. Then, the following formulae hold*:

$$\mu_2^{(1)}[A] = \lambda_{\max}\left(\frac{A^* + A}{2}\right) \tag{13}$$

and

$$\mu_2^{(2)}[A] = \lambda_{\max}^2\left(\frac{A^* + A}{2}\right). \tag{14}$$

In particular, the second logarithmic derivative in the spectral norm is always nonnegative.

Proof. See [**14**, Theorem 6, p. 10]. $\square$

4. Examples. In this section, we illustrate the logarithmic derivatives by examples, which can be used in classroom teaching. In addition, we cast more light on some details. That is, for the one-mass model and $p = 2$, we carry out a check of $\mu_2^{(k)}[A]$, $k = 1, 2$, and Lemma 1, and for the multi-mass model and $p = \infty$, we derive $\mu_\infty^{(k)}[A]$, $k = 1, 2$, in a less formal way, which is quite helpful for the understanding.

4.1 One-mass vibration model. (0) *The model.* We consider the one-mass vibration model without damping in Figure 1.

The associated initial value problem is given by

$$m\ddot{y} + k\,y = 0, \quad y(0) = y_0,\ \dot{y}(0) = \dot{y}_0,$$

or, in the *state space description*,

$$\dot{x}(t) = A\,x(t), \quad x(0) = x_0,$$

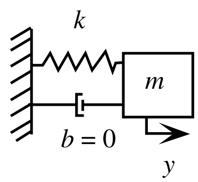

FIGURE 1. One-mass vibration model.

with the *displacement vector* $x = [y, z]^T,\ z = \dot{y}$ and the *system matrix*

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{15}$$

with $\omega^2 = k/m$.

Here, $\Phi(t)$ can be determined explicitly; it is given by

$$\Phi(t) = \begin{bmatrix} \cos \omega t & (1/\omega) \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}, \tag{16}$$

cf. [**17**, pp. 76–77].

(i) $p = \infty$. We obtain

$$A^{2j+1} = (-1)^j\, \omega^{2j}\, A, \quad j = 0, 1, 2, \dots \tag{17}$$

$$A^{2j} = (-1)^j\, \omega^{2j}\, E, \quad j = 0, 1, 2, \dots \tag{18}$$

and

$$s_{12}^{(k)}(A) = 1,\ s_{21}^{(k)}(A) = -1, \quad k = 1, 2, \dots . \tag{19}$$

Case 1. $\omega \le 1$. In this case,

$$\|\Phi(t)\|_\infty = |\cos \omega t| + \frac{1}{\omega}\, |\sin \omega t| \tag{20}$$

and

$$I_1 = \{1\} \quad \text{as well as} \quad I_1 = I_2 = I_3 = \cdots . \tag{21}$$

So, because of (17)–(19) and (21), we have

$$\mu_\infty^{(2j)}[A] \,=\, \mu_\infty^{(2j+1)}[A] \,=\, (-1)^j\,\omega^{2j}, \quad j=0,1,2,\dots\,. \tag{22}$$

In particular,

$$\left.\begin{aligned}\mu_\infty^{(1)}[A] &= 1,\\ \mu_\infty^{(2)}[A] &= -\omega^2,\end{aligned}\right. \quad \text{for } \omega \le 1. \tag{23}$$

Check of formulae (22) *and Lemma* 1. We want to check formulae (22) and Lemma 1. Since $\|\Phi(t)\|_\infty$ is analytic in $0 \le t \le t_1$ with sufficiently small $t_1 > 0$, we have the series expansion

(24)

$$\begin{aligned}\|\Phi(t)\|_{\infty,\text{series}} &= \sum_{j=0}^{\infty} \mu_\infty^{(j)}[A]\,\frac{t^j}{j!}\\ &= 1 + 1\,\frac{t}{1!} - \omega^2\,\frac{t^2}{2!} - \omega^2\,\frac{t^3}{3!} + \omega^4\,\frac{t^4}{4!} + \omega^4\,\frac{t^5}{5!}\\ &\quad - \omega^6\,\frac{t^6}{6!} - \omega^6\,\frac{t^7}{7!} + \omega^8\,\frac{t^8}{8!} + \omega^8\,\frac{t^9}{9!}\\ &\quad - \omega^{10}\,\frac{t^{10}}{10!} - \omega^{10}\,\frac{t^{11}}{11!} + \cdots\\ &= 1 - \omega^2\,\frac{t^2}{2!} + \omega^4\,\frac{t^4}{4!} - \omega^6\,\frac{t^6}{6!} + \omega^8\,\frac{t^8}{8!} - \omega^{10}\,\frac{t^{10}}{10!} \pm \cdots\\ &\quad + \Big(t - \omega^2\,\frac{t^3}{3!} + \omega^4\,\frac{t^5}{5!} - \omega^6\,\frac{t^7}{7!} + \omega^8\,\frac{t^9}{9!} - \omega^{10}\,\frac{t^{11}}{11!} \pm \cdots\Big)\\ &= \cos\omega\,t + \frac{1}{\omega}\,\sin\omega\,t =: \hat\Phi(t).\end{aligned}$$

Let $t_1 = \pi/(2\,\omega)$. Then,

$$\begin{gathered}\hat\Phi(t) = \|\Phi(t)\|_{\infty,\text{series}} = |\cos\omega\,t| + \frac{1}{\omega}\,|\sin\omega\,t| = \|\Phi(t)\|_\infty,\\ 0 \le t \le t_1.\end{gathered} \tag{25}$$

Case 2. $\omega \ge 1$. In this case,

$$\|\Phi(t)\|_\infty = \omega\,|\sin\,\omega t| + |\cos\,\omega t| \tag{26}$$

and

$$I_1 = \{2\} \quad \text{as well as} \quad I_1 = I_2 = I_3 = \cdots . \tag{27}$$

Because of (17)–(19) and (27), we have

$$\begin{aligned} \mu_\infty^{(2j+1)}[A] &= (-1)^j \, \omega^{2j+2}, \quad j = 0, 1, 2, \dots \\ \mu_\infty^{(2j)}[A] &= (-1)^j \, \omega^{2j}, \quad j = 1, 2, \dots \end{aligned} \tag{28}$$

In particular,

$$\left. \begin{aligned} \mu_\infty^{(1)}[A] &= \omega^2, \\ \mu_\infty^{(2)}[A] &= -\omega^2, \end{aligned} \right. \quad \text{for } \omega \ge 1. \tag{29}$$

Check of formulae (28) *and Lemma* 1. We want to check also formulae (28) and Lemma 1. For sufficiently small $t_1 > 0$, for $0 \le t \le t_1$, we have

(30)

$$\begin{aligned} \|\Phi(t)\|_{\infty,\text{series}} &= \sum_{j=0}^{\infty} \mu_\infty^{(j)}[A] \, \frac{t^j}{j!} \\ &= 1 + \omega^2 \, \frac{t}{1!} - \omega^2 \, \frac{t^2}{2!} - \omega^4 \, \frac{t^3}{3!} + \omega^4 \, \frac{t^4}{4!} + \omega^6 \, \frac{t^5}{5!} - \omega^6 \, \frac{t^6}{6!} \pm \cdots \\ &= 1 - \omega^2 \, \frac{t^2}{2!} + \omega^4 \, \frac{t^4}{4!} - \omega^6 \, \frac{t^6}{6!} \pm \cdots \\ &\quad + \omega \left(\omega \, t - \omega^3 \, \frac{t^3}{3!} + \omega^5 \, \frac{t^5}{5!} \mp \cdots \right) \\ &= \cos \omega \, t + \omega \, \sin \omega \, t =: \hat{\Phi}(t). \end{aligned}$$

Again let $t_1 = \pi/(2\,\omega)$. Then,

$$\begin{aligned} \hat{\Phi}(t) = \|\Phi(t)\|_{\infty,\text{series}} = |\cos \omega \, t| + \omega \, |\sin \omega \, t| = \|\Phi(t)\|_\infty, \\ 0 \le t \le t_1. \end{aligned} \tag{31}$$

Combining the two cases, i.e., (23) and (29), and adding the formula for $\mu_\infty^{(0)}[A] := \|\Phi(0)\|_\infty$, we obtain

$$\begin{aligned} \mu_\infty^{(0)}[A] &= 1 \\ \mu_\infty^{(1)}[A] &= \max\{1, \omega^2\} \\ \mu_\infty^{(2)}[A] &= -\omega^2. \end{aligned} \tag{32}$$

(ii) $p = 2$. We have

$$A^* = A^T = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} \tag{33}$$

and thus

$$B_1 = A^* + A = (1-\omega^2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{34}$$

Consequently,

$$\lambda_{\max}(B_1) = \lambda_1(B_1) = \lambda(T) = \lambda_1 = |1-\omega^2| \tag{35}$$

and therefore

$$\mu_2^{(1)}[A] = \frac{|1-\omega^2|}{2}, \qquad \mu_2^{(2)}[A] = \frac{(1-\omega^2)^2}{4}. \tag{36}$$

Special case $\omega = 1$. Here,

$$\mu_2^{(1)}[A] = 0, \qquad \mu_2^{(2)}[A] = 0. \tag{37}$$

Limit case $\omega = 0$. For $\omega \to 0$ in $\lambda_1 = |1-\omega^2|$, we obtain

$$\mu_2^{(1)}[A] = \frac{1}{2}, \qquad \mu_2^{(2)}[A] = \frac{1}{4}. \tag{38}$$

Check of formulae (36). We want to check formulae (36) by expanding $\|\Phi(t)\|_2$. From (16), we obtain

(39)

$$\begin{aligned} \Psi(t) &= \Phi^*(t)\,\Phi(t) \\ &= \begin{bmatrix} \cos^2\omega t + \omega^2 \sin^2\omega t & ((1/\omega)-\omega)\sin\omega t + \cos\omega t \\ ((1/\omega)-\omega)\sin\omega t + \cos\omega t & \cos^2\omega t + (1/\omega^2)\sin^2\omega t \end{bmatrix}. \end{aligned}$$

Now, according to [**24**, p. 107, (21) and (25)],

$$\|\Phi(t)\|_2 = \sqrt{\lambda_{\max}(\Psi(t))}. \tag{40}$$

Equation (39) entails

$$
\begin{aligned}
\lambda_1 &= \lambda_1(\omega > 0) = \lambda_{\max}(\Psi(t)) \\
&= \frac{(\omega^2 + (1/\omega^2))\, \sin^2 \omega\, t + 2\, \cos^2 \omega\, t}{2} \\
&\quad + \sqrt{\left[\frac{(\omega^2 + (1/\omega^2))\, \sin^2 \omega\, t + 2\, \cos^2 \omega\, t}{2}\right]^2 - 1}.
\end{aligned}
\tag{41}
$$

We expand this up to the third order because we want to obtain also the third logarithmic derivative. For sufficiently small $t \geq 0$,

$$
\begin{aligned}
\lambda_1(\omega > 0) &= 1 + |1 - \omega^2|\, t + (1 - \omega^2)^2\, \frac{t^2}{2} \\
&\quad + 3\, |1 - \omega^2|\, \left[\frac{(1 - \omega^2)^2}{4} - \frac{\omega^2}{3}\right] \frac{t^3}{3!} + O(t^4) \\
&= \kappa_0 + \kappa_1\, t + \kappa_2\, \frac{t^2}{2} + \kappa_3\, \frac{t^3}{3!} + O(t^4).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\Phi(t)\|_2 = \sqrt{\lambda_{\max}(\Psi(t))} &= \mu_2^{(0)}[A] + \mu_2^{(1)}[A]\, t \\
&\quad + \mu_2^{(2)}[A]\, \frac{t^2}{2!} + \mu_2^{(3)}[A]\, \frac{t^3}{3!} + O(t^4)
\end{aligned}
$$

with $\mu_2^{(0)}[A] = 1$, $\mu_2^{(1)}[A] = (1/2)\, \kappa_1 = |1 - \omega^2|/2$, $\mu_2^{(2)}[A] = (1/2)\, \kappa_2 - (1/4)\, \kappa_1^2 = (1 - \omega^2)^2/4$, cf. (36), and

$$
\mu_2^{(3)}[A] = \frac{3}{8}\, \kappa_1^3 - \frac{3}{4}\, \kappa_1\, \kappa_2 + \frac{1}{2}\, \kappa_3.
$$

Limit case. $\omega = 0$ (cf. [**13**, p. 384]). Letting $\omega \to 0$ in $\lambda_1(\omega > 0)$ results in

$$
\lambda_1(\omega = 0) = \sup_{\|x\|_2 = 1} (\Psi(t)\, x, x) = 1 + \frac{1}{2}\, t^2 + t\, \sqrt{1 + \frac{t^2}{4}}, \quad t \geq 0.
$$

Thus, for sufficiently small $t_1 > 0$,

$$
\|\Phi(t)\|_2 = \Big[\sup_{\|x\|_2 = 1} (\Psi(t)\, x, x)\Big]^{1/2} = 1 + \frac{1}{2}\, t + \frac{1}{4}\, \frac{t^2}{2} + O(t^4), \quad 0 \leq t \leq t_1.
$$

Consequently,

$$\mu_2^{(1)}[A] = \frac{1}{2} = \lambda_{\max}\left(\frac{A^*+A}{2}\right),$$
$$\mu_2^{(2)}[A] = \frac{1}{4} = \left[\lambda_{\max}\left(\frac{A^*+A}{2}\right)\right]^2,$$

but

$$\mu_2^{(3)}[A] = 0 \neq \frac{1}{8} = \left[\lambda_{\max}\left(\frac{A^*+A}{2}\right)\right]^3.$$

4.2 Multi-mass vibration model. We consider the multi-mass vibration model in Figure 2.

The associated initial value problem is given by

$$M\,\ddot{y} + B\,\dot{y} + K\,y = 0, \quad y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0,$$

where $y = [y_1, \dots, y_n]^T$ and

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_n \end{bmatrix},$$

$$B = \begin{bmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & b_2+b_3 & -b_3 & & & \\ & -b_3 & b_3+b_4 & -b_4 & & \\ & \ddots & \ddots & \ddots & & \\ & & -b_{n-1} & b_{n-1}+b_n & -b_n \\ & & & -b_n & b_n+b_{n+1} \end{bmatrix},$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & -k_3 & k_3+k_4 & -k_4 & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{bmatrix}.$$

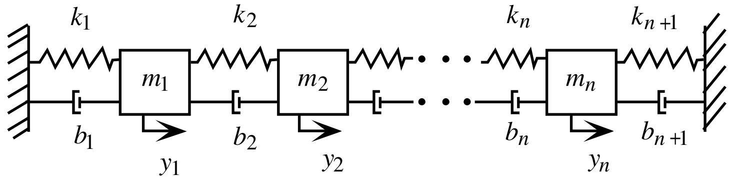

FIGURE 2. Multi-mass vibration model.

The *displacement vector* is $x = [y^T, z^T]^T$, $z = \dot{y}$, and the *system matrix* A has the form

$$A = \left[\begin{array}{c|c} 0 & E \\ \hline -M^{-1}K & -M^{-1}B \end{array}\right].$$

As of now, we specify the values as

$$m_j = 1, \quad j = 1, \ldots, n$$
$$k_j = 1, \quad j = 1, \ldots n+1$$

and

$$b_j = \begin{cases} 1/2 & \text{if } j \text{ even} \\ 1/4 & \text{if } j \text{ odd.} \end{cases}$$

Then,

$$M = E,$$

$$B = \begin{bmatrix} 3/4 & -1/2 & & & & \\ -1/2 & 3/4 & -1/4 & & & \\ & -1/4 & 3/4 & -1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1/4 & 3/4 & -1/2 \\ & & & & -1/2 & 3/4 \end{bmatrix}$$

(if n is even), and

$$K = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

So, for $n = 5$ and $m = 2\,n = 10$, we obtain

$$A = \left[\begin{array}{ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
-2 & 1 & 0 & 0 & 0 & -3/4 & 1/2 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 1/2 & -3/4 & 1/4 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 1/4 & -3/4 & 1/2 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 1/2 & -3/4 & 1/4 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1/4 & -3/4
\end{array}\right]$$

and

$$A^2 = \left[\begin{array}{ccccc|ccccc}
-2 & 1 & 0 & 0 & 0 & -3/4 & 1/2 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 1/2 & -3/4 & 1/4 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 1/4 & -3/4 & 1/2 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 1/2 & -3/4 & 1/4 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1/4 & -3/4 \\
\hline
2 & -7/4 & 1/2 & 0 & 0 & -19/16 & 1/4 & 1/8 & 0 & 0 \\
-7/4 & 9/4 & -5/4 & 1/4 & 0 & 1/4 & -9/8 & 5/8 & 1/8 & 0 \\
1/4 & -5/4 & 9/4 & -7/4 & 1/2 & 1/8 & 5/8 & -9/8 & 1/4 & 1/8 \\
0 & 1/2 & -7/4 & 9/4 & -5/4 & 0 & 1/8 & 1/4 & -9/8 & 5/8 \\
0 & 0 & 1/4 & -5/4 & 7/4 & 0 & 0 & 1/8 & 5/8 & -11/8
\end{array}\right]$$

This leads to the results listed in Table 1. As can be seen,

$$I_1 = \{7, 8, 9\} \quad \text{and} \quad I_2 = \{8\} \tag{43}$$

as well as

$$\mu_\infty^{(1)}[A] = Z_7' = Z_8' = Z_9' = 4 \quad \text{and} \quad \mu_\infty^{(2)}[A] = Z_8'' = -\frac{36}{8} = -4.5.$$

TABLE 1. Quantities for the calculation of $\mu_\infty^{(1)}[A]$ and $\mu_\infty^{(2)}[A]$.

i	$A_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} s_{ij}^{(1)}(A)\, A_{ij} = Z_i'$
1	1
2	1
3	1
4	1
5	1
6	$-3/4+2+1+1/2=2.75$
7	$-3/4+1+2+1+1/2+1/4=4$
8	$-3/4+1+2+1+1/4+1/2=4$
9	$-3/4+1+2+1+1/2+1/4=4$
10	$-3/4+1+2+1/4=2.5$
i	$(A^2)_{ii} + \sum_{j=1 j\neq i}^{n} s_{ij}^{(2)}(A)\,(A^2)_{ij} = Z_i''$
1	$-2+1-3/4+1/2=-1.25$
2	$-2+1+1+1/2-3/4+1/4=0$
3	$-2+1+1+1/4-3/4+1/2=0$
4	$-2+1+1+1/2-3/4+1/4=0$
5	$-2+1+1/4-3/4=-1.5$
6	$-19/16-2-7/4+1/2+1/4+1/8=-65/16$
7	$-9/8-7/4-9/4-5/4+1/4+1/4+5/8+1/8=-41/8$
8	$-9/8+1/4-5/4-9/4-7/4+1/2+1/8+5/8+1/4+1/8=-36/8$
9	$-9/8+1/2-7/4-9/4-5/4+1/8+1/4+5/8=-39/8$
10	$-11/8+1/4-5/4-7/4+1/8+5/8=-27/8$

Further, we remark that

$$\begin{aligned}\mu_\infty^{(2)}[A] &= (A^2)_{i_2 i_2} + \sum_{\substack{j=1 \\ j\neq i_2}}^{n} s_{i_2 j}^{(2)}(A)\,(A^2)_{i_2 j} \\ &= -\frac{36}{8} \neq \max_{i=1,\dots,n}\left\{(A^2)_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} s_{ij}^{(2)}(A)\,(A^2)_{ij}\right\} = 0.\end{aligned}$$

By (44), we have

$$\begin{aligned}\|\Phi(t)\|_\infty &\doteq \mu_\infty^{(0)}[A] + \mu_\infty^{(1)}[A]\, t + \mu_\infty^{(2)}[A]\, \frac{t^2}{2} \\ &= 1 + 4\, t - 4.5\, \frac{t^2}{2} \\ &= 1 + 4\, t - 2.25\, t^2, \quad 0 \le t \le t_1\end{aligned}$$

for sufficiently small $t_1 > 0$.

Less formal derivation of $\mu_\infty^{(1)}[A]$ *and* $\mu_\infty^{(2)}[A]$. The formulae (5) and (6) for $\mu_\infty^{(1)}[A]$ and $\mu_\infty^{(2)}[A]$ are based on [**13**, Lemma 2.1]. We give here a less formal and more intuitive argument for the derivation of these quantities, which was the first step in the development of the differential calculus for norms of vector and matrix functions.

We start with the representation

$$\|\Phi(t)\|_\infty = \max_{i=1,\dots,n} \varphi_i(t), \tag{45}$$

where

$$\varphi_i(t) = \sum_{j=1}^{n} |\Phi_{ij}(t)|. \tag{46}$$

(Here, the index $n = 10$ instead of $m = 2\, n = 2 \cdot 5 = 10$ is used.)

For sufficiently small $t_1 > 0$, the function $\varphi_i(t)$ can be expanded in a series. Namely, using

(47)

$$|a| = \sqrt{(\operatorname{Re} a)^2 + (\operatorname{Im} a)^2} = |\operatorname{Re} a| \sqrt{1 + \left(\frac{\operatorname{Im} a}{\operatorname{Re} a}\right)^2}\,, \quad \operatorname{Re} a \ne 0, \quad a \in \mathbf{C},$$

and

$$\sqrt{1+x} = 1 + \frac{1}{2}\, x - \frac{1}{8}\, x^2 + \frac{1}{16}\, x^3 \mp \cdots\,, \quad |x| < 1, \tag{48}$$

we obtain

$$
\begin{array}{lclclcl}
\varphi_1(t) & = & \boxed{1} & + & 1\cdot t & -1.25\cdot t^2/2 & + \cdots \\
\varphi_2(t) & = & \boxed{1} & + & 1\cdot t & 0\cdot t^2/2 & + \cdots \\
\varphi_3(t) & = & \boxed{1} & + & 1\cdot t & 0\cdot t^2/2 & + \cdots \\
\varphi_4(t) & = & \boxed{1} & + & 1\cdot t & 0\cdot t^2/2 & + \cdots \\
\varphi_5(t) & = & \boxed{1} & + & 1\cdot t & -1.5\cdot t^2/2 & + \cdots \\
\varphi_6(t) & = & \boxed{1} & + & 2.75\cdot t & -4.0625\cdot t^2/2 & + \cdots \\
\varphi_7(t) & = & \boxed{1} & + & \boxed{4}\cdot t & -5.125\cdot t^2/2 & + \cdots \\
\varphi_8(t) & = & \boxed{1} & + & \boxed{4}\cdot t & \boxed{-4.5}\cdot t^2/2 & + \cdots \\
\varphi_9(t) & = & \boxed{1} & + & \boxed{4}\cdot t & -4.875\cdot t^2/2 & + \cdots \\
\varphi_{10}(t) & = & \boxed{1} & + & 2.5\cdot t & -3.375\cdot t^2/2 & + \cdots
\end{array}
\tag{49}
$$

for $0 \le t \le t_1$. Let $t_1 > 0$ be so small that, in $\varphi_i(t)$, the constant term is larger than the linear term and the linear term itself is larger than the quadratic term; for this, e.g., take $t_1 = 10^{-3}$. By comparing the constant terms only, all $\varphi_i(t)$, $i = 1, \dots, n$ are candidates for the maximum. Comparing the linear terms, only $\varphi_7(t)$, $\varphi_8(t)$ and $\varphi_9(t)$ remain as candidates for the maximum. Finally, comparing the quadratic terms of $\varphi_7(t)$, $\varphi_8(t)$ and $\varphi_9(t)$, we find that $\varphi_8(t)$ must be the maximum, i.e.,

$$\|\Phi(t)\|_\infty = \varphi_8(t) = 1 + 4\cdot t - 4.5\cdot t^2/2 + \cdots, \quad 0 \le t \le t_1, \tag{50}$$

for sufficiently small $t_1 > 0$, say, $t_1 = 10^{-3}$. From this, it follows that $\mu_\infty^{(1)}[A] = 4$ and $\mu_\infty^{(2)}[A] = -4.5$. For comparison reasons, we state that $\|A\|_\infty = 5.5$ so that indeed $|\mu_\infty^{(1)}[A]| \le \|A\|_\infty$.

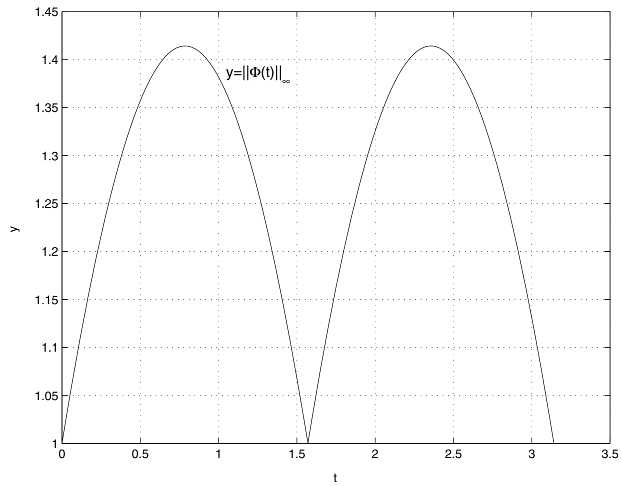

FIGURE 3. Graph of $\|\Phi(t)\|_\infty$ for one-mass vibration model with $\omega = 1$, plotted with $\Delta t = \pi/100$.

5. Applications to upper bounds. For the sake of brevity, we restrict ourselves to the case $p = \infty$. Applications for $p = 2$ can be found in [**14**]. Particularly, for a numerical example, see [**14**, p. 14].

(i) *Oscillation behavior of* $\|\Phi(t)\|_\infty$. The fundamental matrix (evolution) $\Phi(t) = e^{At}$ is the unique solution to the initial value problem

$$\dot{\Phi}(t) = A\,\Phi(t), \quad \Phi(0) = E,$$

cf. [**2**, p. 43]. Setting

$$\Phi(t) = [\varphi_1(t), \dots, \varphi_n(t)],$$

this is equivalent to

$$\dot{\varphi}_j(t) = A\,\varphi_j(t), \quad \varphi_j(0) = e_j, \quad j = 1, \dots, n,$$

where e_j is the jth unit vector. Hence, if A describes a vibration problem, the oscillation character of the solutions $\varphi_j(t)$, $j = 1, \dots, n$ is somehow inherited by $\|\Phi(t)\|_\infty$.

(ii) *Plots of* $\|\Phi(t)\|_\infty$ *for the vibration models.* We illustrate the oscillation character by the examples of subsections 4.1 and 4.2.

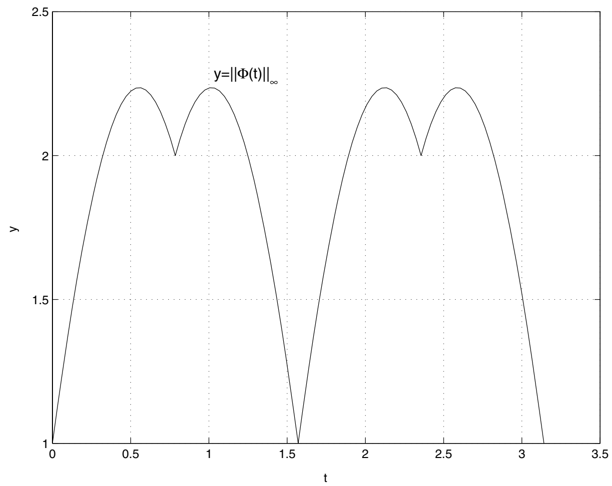

FIGURE 4. Graph of $\|\Phi(t)\|_\infty$ for one-mass vibration model with $\omega = 2$, plotted with $\Delta t = \pi/100$.

One-mass model. $\|\Phi(t)\|_\infty$ is shown for some cases in Figures 3 and 4.

Multi-mass vibration model. Here, $\Phi(t) = e^{At}$ is computed by the MATLAB routine 'expm1.' $\|\Phi(t)\|_\infty$ is shown for some cases in Figures 5 and 6.

(iii) *Tangent at $t_0 = 0$ as upper bound on $\|\Phi(t)\|_\infty$.*

One-mass vibration model. In this case, for sufficiently small $t_1 > 0$, we obtain

$$\begin{aligned}\|\Phi(t)\|_\infty &\doteq 1 + \mu_\infty^{(1)}[A]\, t + \mu_\infty^{(2)}[A]\, \frac{t^2}{2} \\ &= 1 + \max\{1, \omega^2\}\, t - \omega^2\, \frac{t^2}{2}, \quad 0 \le t \le t_1.\end{aligned}$$

Since $\mu_\infty^{(2)}[A] < 0$, for sufficiently small $t_1 > 0$,

$$\|\Phi(t)\|_\infty \ \le\ 1 + \mu_\infty^{(1)}[A]\, t, \quad 0 \le t \le t_1.$$

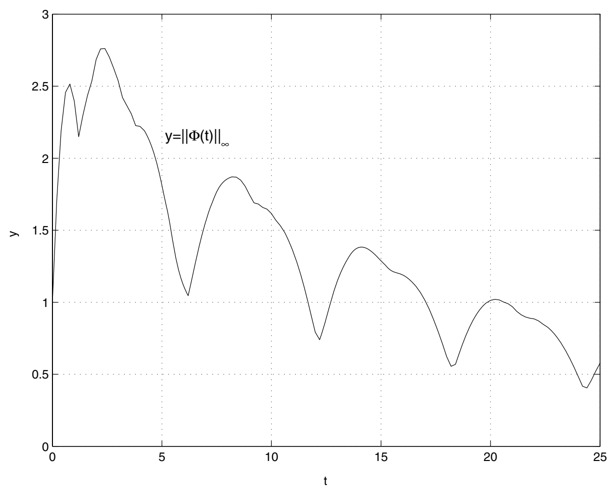

FIGURE 5. Graph of $\|\Phi(t)\|_\infty$ for multi-mass vibration model with $n=5$, resp. $m=10$, plotted with $\Delta t = 0.2$.

From the plots above, it is clear that this bound also holds globally, that is, for all values $0 \le t < \infty$.

In this special case, a strict proof for this can be given. Namely, from (16) we infer

$$\|\Phi(t)\|_\infty = \begin{cases} |\cos\omega\, t| + (1/\omega)\,|\sin\omega\, t|, & \omega \le 1, \\ \omega\,|\sin\omega\, t| + |\cos\omega\, t|, & \omega \ge 1. \end{cases}$$

Case 1. $\omega \le 1$. Here,

$$\|\Phi(t)\|_\infty = |\cos\omega\, t| + \left|\frac{\sin\omega\, t}{\omega\, t}\right| t \le 1 + t = 1 + \mu_\infty^{(1)}[A]\, t, \quad t > 0,$$

since $\mu_\infty^{(1)}[A] = \max\{1, \omega^2\}$.

Case 2. $\omega \ge 1$. In this case,

$$\|\Phi(t)\|_\infty = \omega^2\, t \left|\frac{\sin\omega\, t}{\omega\, t}\right| + |\cos\omega\, t| \le 1 + \omega^2\, t = 1 + \mu_\infty^{(1)}[A]\, t, \quad t > 0,$$

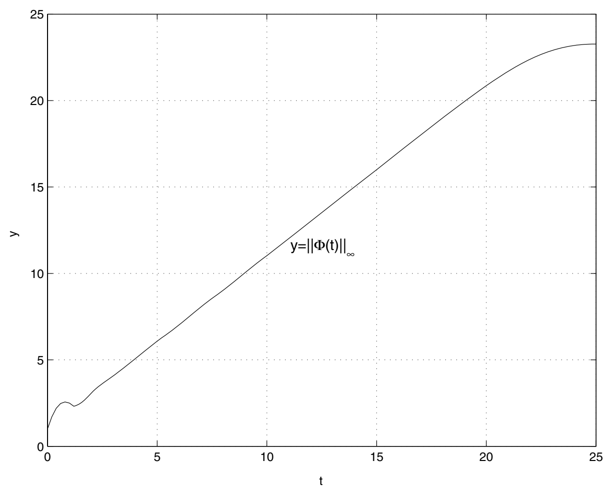

FIGURE 6. Graph of $\|\Phi(t)\|_\infty$ for multi-mass vibration model with $n=50$, resp. $m=100$, plotted with $\Delta t = 0.2$.

since $\mu_\infty^{(1)}[A] = \max\{1, \omega^2\}$. For $t = 0$, these estimates remain valid. $\square$

In Figures 3–6, we have not shown the plots of the tangents. Such plots can be found in [**14**].

Multi-mass vibration model. Here, for sufficiently small $t_1 > 0$, we obtain

$$\|\Phi(t)\|_\infty \doteq 1 + \mu_\infty^{(1)}[A]\, t + \mu_\infty^{(2)}[A]\, \frac{t^2}{2} = 1 + 4\,t - 2.25\,t^2, \quad 0 \le t \le t_1.$$

Since $\mu_\infty^{(2)}[A] < 0$, for sufficiently small $t_1 > 0$,

$$\|\Phi(t)\|_\infty \le 1 + \mu_\infty^{(1)}[A]\, t, \quad 0 \le t \le t_1.$$

From the plots in Figures 5 and 6, it is clear that this bound also holds globally, that is, for all values $0 \le t < \infty$. A strict proof of this is not yet available, however. Particularly, in Figure 6, it is clear that the tangent at $t_0 = 0$ is extremely better than $y = e^{\mu_\infty[A]\,t} = e^{4\,t}$. We mention that, for Figures 3–6, MATLAB was used.

6. Historical remarks. In this section, we make some historical remarks on the *logarithmic derivative* or *logarithmic norm.* Eltermann [**5**], Lozinskiĭ [**16**], and Dahlquist [**3**] seem to be the first to have used the logarithmic derivative.

(i) As indicated in [**7**, p. 61], in 1955, Eltermann [**5**, p. 499] has applied the formula for the logarithmic norm $\mu_\infty[A]$ (without calling it such) for real matrices A having real eigenvalues $\lambda(A)$ in the estimate

$$\lambda(A) \le \max_{i=1,\dots,n} \Bigg(A_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}| \Bigg). \tag{51}$$

(ii) In 1958, for $A \in \mathbf{C}^{n\times n}$ and matrix norms $\|\cdot\|$ with $\|E\| = 1$, Lozinskiĭ [**16**, pp. 57–58] shows that the limit

$$\mu[A] = \lim_{h\to 0+} \frac{\|E + A\,h\| - 1}{h} \tag{52}$$

exists and calls it *logarithmic norm.* Due to

$$\Phi(h) = E + A\,h + A^2\,\frac{h^2}{2!} + \cdots, \quad h > 0,$$

it is clear that

$$\mu[A] = \lim_{h\to 0+} \frac{\|\Phi(h)\| - 1}{h}. \tag{53}$$

The following norm-like properties of the logarithmic norm are proven in [**16**, p. 58]:

$$\begin{gathered} \mu[A+B] \le \mu[A] + \mu[B] \\ \mu[\alpha\,A] = \alpha\,\mu[A], \quad \alpha \ge 0 \\ |\mu[B] - \mu[A]| \le \|B - A\| \end{gathered} \tag{54}$$

and in [**16**, p. 59], for eigenvalues $\lambda(A)$ of A,

$$\operatorname{Re}\lambda(A) \le \mu[A]. \tag{55}$$

Further, in [**16**, pp. 59–60], the following formulae for $A \in \mathbf{C}^{n\times n}$ are given:

$$\begin{aligned}
&\mu_\infty^{(1)}[A] = \max_{i=1,\dots,n} \Bigg\{ \operatorname{Re} A_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}| \Bigg\},\\
(56)\qquad &\mu_1^{(1)}[A] = \max_{j=1,\dots,n} \Bigg\{ \operatorname{Re} A_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} |A_{ij}| \Bigg\},\\
&\mu_2^{(1)}[A] = \lambda_{\max}\left(\frac{A^* + A}{2}\right).
\end{aligned}$$

In addition, Lozinskiĭ uses the logarithmic norm in the error estimate of the numerical integration of ordinary differential equations.

(iii) In 1959, Dahlquist publishes his dissertation of 1958 in [**3**]. The above formulae for $\mu_\infty^{(1)}[A]$, $\mu_1^{(1)}[A]$, and $\mu_2^{(1)}[A]$ are stated in [**3**, p. 11] without proof. Dahlquist also uses the directional derivative of vector norms. For y, $v \in \mathbf{C}^n$ and $h > 0$, it is shown that

$$(57)\qquad \lambda[y; v] := \lim_{h\to 0+} \frac{\|y + h\,v\| - \|y\|}{h}$$

exists.

In the existence proof of (52) and (57), the *convexity property* of norms is used.

(iv) In [**2**, p. 58], for continuous matrix functions $A(t)$ and the differential system $\dot{x} = A\,x$, the inequality

$$(58)\qquad D_+\|x(t)\| \le \mu[A(t)]\,\|x(t)\|, \quad t \ge 0,$$

is proven, where (57) is applied to $y = x(t)$. Also, for constant A, the estimate

$$(59)\qquad \|e^{A\,t}\| \le e^{\mu[A]\,t}, \quad t \ge 0,$$

is derived in [**2**, p. 58].

(v) In [**21**, pp. 69–71], an important application of the logarithmic norm is given. For this, consider the linear system of ordinary differential equations

$$(60)\qquad \dot{x} = A(t)\,x + g(t), \quad t \ge 0,$$

with continuous $A(t) \in \mathbf{R}^{n\times n}$, $g(t) \in \mathbf{R}^n$, $t \geq 0$. Classical estimates of the solution of (60) can be obtained from the differential inequality

$$D_+\|x(t)\| \leq \|A(t)\|\,\|x(t)\| + \|g(t)\|, \quad t \geq 0. \tag{61}$$

Better estimates can be inferred from

$$D_+\|x(t)\| \leq \mu[A(t)]\,\|x(t)\| + \|g(t)\|, \quad t \geq 0. \tag{62}$$

More precisely, from [**21**, Lemma 1, p. 71], we state the following lemma.

Lemma 7. *Let $x(t)$ be a solution of $\dot{x} = A(t)\,x + g(t)$, $t \geq 0$. Then, $\|x(t)\| \leq \xi(t)$, where the scalar function ξ satisfies the differential equation $\dot{\xi} = \mu[A(t)]\,\xi + \|g(t)\|$, $t \geq 0$, with the initial condition $\xi(0) = \|x(0)\|$.*

The above historical remarks are by no means exhaustive. So, for the interested reader, the references [**4, 7, 8–10, 19, 20**] and [**22, 23**] are added, in which more information on the logarithmic derivatives can be found.

7. Further development. By inspection of Figures 3–6, it is evident that the tangent of $y = \|\Phi(t)\|_\infty$ at $t_0 = 0$ is only a good upper bound for sufficiently small values of t, i.e., for the initial behavior.

For large values of t, i.e., for the asymptotic behavior of $y = \|\Phi(t)\|_\infty$, and more generally for any matrix norm $\|\cdot\|$, $\|\Phi(t)\|$ can be estimated from above by

$$\|\Phi(t)\| \leq M_\varepsilon\, e^{(\nu[A]+\varepsilon)\,t}, \quad t \geq 0, \tag{63}$$

where $\varepsilon > 0$ is given, M_ε is some constant and

$$\nu[A] := \max_{j=1,\dots,n} \operatorname{Re}\lambda_j(A), \tag{64}$$

is the *spectral abscissa* of A, with $\lambda_j(A)$, $j = 1, \dots, n$ being the eigenvalues of A, see [**2**, p. 56].

To obtain the minimal M_ε in (63), in [**14**], the author has developed a differential calculus for the functions $t \mapsto \|\Phi(t)\|_p$, $p \in \{1, 2, \infty\}$. The optimal constants $M_{\varepsilon,p}$ in (63) for the norms $p \in \{1, 2, \infty\}$ are then determined by the conditions

$$\begin{aligned} \|\Phi(t_c)\|_p &= M_{\varepsilon,p}\, e^{(\nu[A]+\varepsilon)\, t_c}, \\ D_+ \|\Phi(t_c)\|_p &= M_{\varepsilon,p}\, (\nu[A] + \varepsilon)\, e^{(\nu[A]+\varepsilon)\, t_c}, \end{aligned}$$

where t_c is the point of contact between $y = \|\Phi(t)\|_p$ and the upper bound $y = M_{\varepsilon,p}\, e^{(\nu[A]+\varepsilon)\, t}$.

We want to point out that the differential calculus developed in [**13–15**] extends the notion of the logarithmic derivative in two directions:

- not only the first derivative $D_+^1 \|\Phi(0)\|_p$ is considered, but all derivatives $D_+^k \|\Phi(0)\|_p$, $k = 1, 2, \ldots$ for $p \in \{1, 2, \infty\}$ and
- not only $t_0 = 0$ is considered, but any $t_0 > 0$, that is, $D_+^k \|\Phi(t_0)\|_p$, $k = 1, 2, \ldots$ are defined.

For more details on this and the plots of the upper bounds for the asymptotic behavior as well as the differential calculus for norms of vector functions, the reader is referred to [**14**] and [**15**].

Appendix

In subsection 4.1, for the one-mass model, we have checked formulae (13) and (14) for $\mu_2^{(1)}[A]$ and $\mu_2^{(2)}[A]$ by expanding $\|\Phi(t)\|_2$ in a series using the explicit representation of $\Phi(t)$ in (16) and formula (40).

In this Appendix, we want to make a second check. Since this second check, which can be used as an Exercise for classroom teaching, is not elementary, we have postponed it to the Appendix.

One has

$$\Psi(t) = \Phi(t)^* \Phi(t).$$

Thus,

$$\lambda_{\max}(\Psi(t)) = \lambda_{\max}\left(E + B_1\, t + B_2\, \frac{t^2}{2!} + \cdots\right)$$

with

$$B_1 = A^* + A,$$
$$B_2 = {A^*}^2 + 2\, A^* A + A^2,$$

so that

$$\lambda_{\max}(\Psi(t)) = 1 + \lambda_{\max}\left(B_1\; + B_2\, \frac{t}{2} + \cdots\right) t.$$

Set

$$\begin{aligned} T(t) &:= B_1\; + B_2\, \frac{t}{2} + \cdots \\ &= T + T^{(1)}\, t + \cdots \end{aligned}$$

with

$$T := B_1,$$
$$T^{(1)} := \frac{1}{2}\, B_2.$$

Then,

$$\lambda_{\max}(\Psi(t)) = 1 + \lambda_{\max}(T(t)).$$

According to [**14**, pp. 8–10], for sufficiently small $t \geq 0$,

$$\lambda_{\max}(T(t)) = \lambda_1(B_1) + \max_{i=1,\ldots,m} \nu_i^{(1)}\left(P\, \frac{1}{2}\, B_2\, P\right) t + o(t),$$

where $\lambda_1 = \lambda_1(B_1) = \lambda_1(T) = \lambda_{\max}(T)$ is the largest eigenvalue of B_1, $P = P(\lambda_1) = P(\lambda_{\max})$ the eigenprojection associated with λ_1, m the dimension of algebraic eigenspace $M = P\, X$ with $X = \mathbf{C}^n$, and $\nu_i^{(1)}$ are the repeated eigenvalues of $P\,(1/2)\, B_2\, P$ in the subspace M. Consequently, for sufficiently small $t_1 > 0$,

$$\lambda_{\max}(\Psi(t)) = \kappa_0 + \kappa_1\, t + \kappa_2\, \frac{t^2}{2} + o(t^2), \quad 0 \leq t \leq t_1,$$

with

$$\begin{aligned} \kappa_0 &= 1, \\ \kappa_1 &= \lambda_1(B_1) = \lambda_1(A^* + A), \\ \kappa_2 &= \max_{i=1,\ldots,m} \nu_i^{(1)}\left(P\, \frac{1}{2}\, B_2\, P\right). \end{aligned}$$

Next, we show that

$$\kappa_2 = \kappa_1^2.$$

Determination of the eigenprojection P.

(i) *First way.* Here, we use the eigenvector $v^{(1)}$ of the eigenvalue λ_1. From $B_1\, v^{(1)} = \lambda_1\, v^{(1)}$, we obtain

$$v^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{if} \quad 0 < \omega < 1, \quad \text{and}$$
$$v^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{if} \quad 1 < \omega < \infty.$$

From [**18**, p. 234 and pp. 236–238], we infer

$$P = v^{(1)} \cdot {v^{(1)}}^T$$

and therefore

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{if} \quad 0 < \omega < 1, \quad \text{and}$$
$$P = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{if} \quad 1 < \omega < \infty.$$

Remark. The *special case* $\omega = 1$ and the *limit case* $\omega = 0$ will be discussed later.

(ii) *Second way.* According to [**11**, p. 66],

$$P = -\frac{1}{2\,\pi\, i} \int_\Gamma R(\zeta)\, d\,\zeta,$$

where

$$R(\zeta) = (T - \zeta)^{-1} = (B_1 - \zeta)^{-1}$$

is the resolvent of $T = B_1$ and Γ is a sufficiently small positively oriented circle about z_0 where $z_0 = 1 - \omega^2$ if $0 < \omega < 1$ and

$z_0 = -(1-\omega^2)$ if $1 < \omega < \infty$. We derive P only for the case $z_0 = 1-\omega^2$. The case $z_0 = -(1-\omega^2)$ is left to the reader. We have

$$R(\zeta) = \frac{1}{\zeta^2 - (1-\omega^2)^2} \begin{bmatrix} -\zeta & -(1-\omega^2) \\ -(1-\omega^2) & -\zeta \end{bmatrix}.$$

Using the partial-fraction decomposition

$$R(\zeta)_{12} = -\frac{1-\omega^2}{\zeta^2 - (1-\omega^2)^2} = \frac{1}{2}\,\frac{1}{\zeta + (1-\omega^2)} - \frac{1}{2}\,\frac{1}{\zeta - (1-\omega^2)},$$

we obtain (cf. [**12**, p. 43])

$$P_{12} = -\frac{1}{2\,\pi\,i}\int_\Gamma R(\zeta)_{12}\,d\,\zeta = \frac{1}{2}\left(\frac{1}{2\,\pi\,i}\int_\Gamma \frac{d\,\zeta}{\zeta - z_0}\right) = \frac{1}{2}.$$

Further,

$$R(\zeta)_{11} = -\frac{\zeta}{\zeta^2 - (1-\omega^2)^2} = -\frac{1}{\zeta + (1-\omega^2)} - \frac{1-\omega^2}{\zeta^2 - (1-\omega^2)^2}.$$

Hence,

$$P_{11} = -\frac{1}{2\,\pi\,i}\int_\Gamma R(\zeta)_{11}\,d\,\zeta = -\frac{1}{2\,\pi\,i}\int_\Gamma R(\zeta)_{12}\,d\,\zeta = \frac{1}{2}.$$

Therefore, for $0 < \omega < 1$ again

$$P = P(\lambda_1) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Determination of $P\,T^{(1)}\,P = P\,((1/2)\,B_2)\,P$. Since

$$B_2 = A^{*2} + 2\,A^*\,A + A^2 = \;2\begin{bmatrix} -\omega^2\,(1-\omega^2) & 0 \\ 0 & 1-\omega^2 \end{bmatrix},$$

it follows in both cases that

$$P\left(\frac{1}{2}\,B_2\right)P = \frac{1}{4}\,(1-\omega^2)^2\,P.$$

For the *eigenvalues* of $P\,T^{(1)}\,P = P\,((1/2)\,B_2)\,P$, we have

$$\nu_1^{(1)} = 0,$$
$$\nu_2^{(1)} = \tfrac{1}{2}\,(1-\omega^2)^2,$$

and for the *algebraic eigenspace*,

$$M^{(1)} := P(\lambda_1)\,X = [v^{(1)}].$$

Eigenvalue expansion of $\lambda_1(E+B_1\,t+B_2\,(t^2/2)+\cdots)$. For sufficiently small $t_1 > 0$,

$$\begin{aligned}\lambda_1\left(B_1 + t\,\frac{B_2}{2}\right) &= \lambda_1(B_1) + \nu_2^{(1)}\left(P\,\frac{1}{2}\,B_2\,P\right)t + o(t)\\ &= |1-\omega^2| + (1-\omega^2)^2\,\frac{t}{2} + o(t), \quad 0 \le t \le t_1,\end{aligned}$$

and therefore

$$\lambda_1\left(E + B_1\,t + B_2\,\frac{t^2}{2} + \cdots\right) = 1 + |1-\omega^2|\,t + (1-\omega^2)^2\,\frac{t^2}{2} + o(t^2),$$
$$0 \le t \le t_1,$$

so that

$$\kappa_0 = 1,$$
$$\kappa_1 = |1-\omega^2|,$$
$$\kappa_2 = (1-\omega^2)^2 = \kappa_1^2.$$

Finally, this implies

$$\mu_2^{(1)}[A] = \frac{1}{2}\,\kappa_1 = \frac{|1-\omega^2|}{2} = \lambda_{\max}\left(\frac{A^*+A}{2}\right),$$

$$\mu_2^{(2)}[A] = \frac{1}{2}\left(\kappa_2 - \frac{1}{2}\,\kappa_1^2\right) = \frac{(1-\omega^2)^2}{4} = \left[\lambda_{\max}\left(\frac{A^*+A}{2}\right)\right]^2.$$

Special case. $\omega = 1$. Here, $B_j = 0$, $j = 1, 2, \ldots$ so that

$$\Phi(t) = E.$$

Thus,

$$\mu_2^{(k)}[A] = 0, \quad k = 1, 2, \ldots .$$

Limit case. $\omega = 0$. Here,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and therefore

$$B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{as well as} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$B_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad j \geq 3.$$

From the above formulae for $\omega > 0$, we obtain for $\omega \to 0$:

$$\mu_2^{(1)}[A] = \frac{1}{2},$$
$$\mu_2^{(2)}[A] = \frac{1}{4} = \left(\frac{1}{2}\right)^2.$$

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