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ON APPROXIMATION FOR FUNCTIONS OF TWO VARIABLES ON A TRIANGULAR DOMAIN

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ABSTRACT. The problem of approximation of continuous functions of two variables by Bernstein-Chlodowsky polynomials on a triangular domain is studied. Moreover, the problem of weighted approximations of continuous functions of two variables by a sequences of linear positive operators is discussed.

1. Introduction. The aim of this paper is to study the problem of the approximation of functions of two variables by means of Bernstein-Chlodowsky polynomials in a triangular domain.

There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations.

We refer to papers [**6**–**8**] and to monograph [**1**].

On the other hand, Bernstein-Chlodowsky polynomials have not been studied so well and we don't know of papers devoted to the two-dimensional case.

Some generalization of these polynomials in the one-dimensional case may be found in [**4, 5**].

We will prove theorems on the weighted approximation of continuous functions by Bernstein-Chlodowsky polynomials of two variables.

Also, some problems of weighted approximation of functions of two variables by linear positive operators are discussed at the end of this paper, as an analog of the one-dimensional results, established in [**2, 3**].

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2. Main results. Let (b_n) be a sequence of positive numbers with the properties

$$\lim_{n\to\infty} b_n = \infty, \quad \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{1}$$

For any positive $a > 0$ we denote by Δ_a the triangular domain

$$\Delta_a = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq a\}$$

and by Δ_{b_n} the corresponding triangle with $a = b_n$.

For $(x,y) \in \Delta_{b_n}$ we can introduce the Bernstein-Chlodowsky polynomials for a function f of two variables as follows:

$$\begin{aligned} B_n(f;x,y) = {} & \sum_{k=0}^{n} C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \\ & \times \sum_{j=0}^{k} f\left(\frac{k-j}{n}\, b_n, \frac{j}{n}\, b_n\right) C_k^j \left(\frac{x}{b_n}\right)^{k-j} \left(\frac{y}{b_n}\right)^{j}. \end{aligned} \tag{2}$$

Note that a special case of this type of Bernstein polynomial, which may be obtained from (2) by the formal substitution $b_n = 1$, was considered by Stancu [**7**].

We shall study approximative properties of polynomials (2) for a continuous function f.

Note also that by the properties (1), the triangular domain Δ_{b_n} extends to the infinite quadrant $x \geq 0$, $y \geq 0$ as $n \to \infty$ and therefore in effect we established a theorem on the approximation of continuous functions by polynomials (2) on an unbounded set.

At first we can prove the following simple result.

Theorem 1. *For any fixed positive a, the relation*

$$\lim_{n\to\infty} \max_{(x,y)\in\Delta_a} |B_n(f;x,y) - f(x,y)| = 0$$

holds for all functions f which are continuous in $x \geq 0$, $y \geq 0$ and satisfy the condition

$$|f(x,y)| \leq M_f(1 + x^2 + y^2), \tag{3}$$

where M_f is a constant depending on the function f only.

Proof. Let $f_{k,m}(t,\tau) = t^k \tau^m$ Then simple calculations show that

$$B_n(f_{0,0}; x, y) = 1 \tag{4}$$

$$B_n(f_{1,0}; x, y) = x \tag{5}$$

$$B_n(f_{0,1}; x, y) = y \tag{6}$$

$$B_n(f_{2,0}; x, y) = x^2 + \frac{x(b_n - x)}{n} \tag{7}$$

$$B_n(f_{0,2}; x, y) = y^2 + \frac{y(b_n - y)}{n}. \tag{8}$$

Now we can apply the Korovkin's type approximation theorem, proved in [**9**] (the statement in [**9**] is given only for bounded functions but remains true under the condition (3)).

Given a triangular region Δ_a, no matter how large, for some n, Δ_{b_n} will contain Δ_a and therefore the theorem gives a solution of the approximation problem for closed subset of Δ_a. This gives the proof. □

We make some remarks about this theorem.

It is seen that for a large n the triangle Δ_{b_n} coincides with any triangle Δ_a, and therefore this theorem gives a solution of approximation problems of continuous functions, satisfying (3) only in any closed subset of Δ_{b_n}.

On the other hand, the relations (7) and (8) show that the approximation of the infinitely differentiable function $f^*(x,y) = x^2 + y^2$ on the entire triangular domain Δ_{b_n} by polynomials (2) is impossible, since

$$\max_{(x,y)\in\Delta_a} |B_n(f^*; x, y) - f^*(x, y)| = \frac{b_n^2}{2n},$$

and the righthand side, in general, does not tend to zero as $n \to \infty$, by (1).

For a sequence b_n with the properties (1) we can however prove the following weighted theorem.

Theorem 2. *The relation*

$$\lim_{n\to\infty} \sup_{(x,y)\in\Delta_{b_n}} \frac{|B_n(f; x, y) - f(x, y)|}{(1 + x^2 + y^2)^{1+\alpha}} = 0$$

holds for any continuous function f, satisfying (3) *and for any positive* α.

Proof. For a given $\varepsilon > 0$ we can choose a large $a > 0$ so that the inequality

$$\frac{1}{1+x^2+y^2} < \varepsilon \tag{9}$$

holds for $x + y > a$.

Since by (1) the sequence b_n/n is bounded, say $b_n/n < C$, the properties (4), (7) and (8) show that, for any function f satisfying (3)

$$\begin{aligned}|B_n(f;x,y)| &\leq M_f\left(1+x^2+y^2+\frac{x(b_n-x)+y(b_n-y)}{n}\right)\\ &\leq M_f(1+2C)(1+x^2+y^2).\end{aligned}$$

Setting $\widetilde{M} = M_f(1+2C)$, we obtain for any continuous function f, satisfying (3), and all $x, y \geq 0$ the inequality

$$|B_n(f;x,y)| \leq \widetilde{M_f}(1+x^2+y^2) \tag{10}$$

where $\widetilde{M_f}$ is independent of n.

Now, for any continuous function f, satisfying (3) and any $\alpha > 0$ we can write

$$\begin{aligned}&\sup_{(x,y)\in\Delta_{b_n}} \frac{|B_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\alpha}}\\ &\leq \sup_{(x,y)\in\Delta_a} |B_n(f;x,y)-f(x,y)| + \sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a} \frac{|B_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\alpha}}\\ &= I_n'(f) + I_n''(f)\end{aligned} \tag{11}$$

The first term $I_n'(f)$ tends to zero as $n \to \infty$ by Theorem 1.

Consider the term $I_n''(f)$.

We obtain by (3) and (10),

$$\begin{aligned} I_n''(f) &\le \sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a} \frac{|B_n(f;x,y)-f(x,y)|}{(1+x^2+y^2)^{1+\alpha}} \\ &\le (\widetilde{M_f}+M_f) \sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a} \frac{1}{(1+x^2+y^2)^{\alpha}} \end{aligned}$$

and according to (9)

$$I_n''(f) < (\widetilde{M_f}+M_f)\varepsilon^{\alpha}$$

where $\alpha > 0$ and ε is an arbitrary positive number.

Therefore, the righthand side of the inequality (11) tends to zero as $n \to \infty$ which gives the proof.

Note that it is interesting to prove or to refuse this result in the case of $\alpha = 0$.

In the one-dimensional case, it is known [**2, 3**] that in general a Korovkin's type theorem does not hold for a sequence of linear positive operators acting on the linear normed space of continuous functions, satisfying the inequality $|f(x)| \le M_f(1+x^2)$.

The relations (4)–(8) show that Bernstein-Chlodowsky polynomials (2), have the properties

$$\text{(12)} \qquad \lim_{n\to\infty} \sup_{(x,y)\in\Delta_{b_n}} \frac{|B_n(f_{k,m};x,y)-f_{k,m}(x,y)|}{1+x^2+y^2} = 0, \quad \le k+m \le 2$$

where, as above, $f_{k,m}(x,y) = x^k y^m$ and k, m are nonnegative integers.

The construction of the counterexample given in [**2, 3**] may be extended easily to the two-dimensional case to show that, in general, the approximation of continuous functions, satisfying (3), by linear positive operators with the properties (12), is impossible if $\alpha = 0$.

To formulate this result in a more precise form we need the following notations.

Let

$$R_2^{++} = \{(x,y)\colon x \ge 0, y \ge 0\} \text{ and } \rho(x,y) = 1+x^2+y^2.$$

We denote by $B_\rho(R_2^{++})$ the space of all functions defined in R_2^{++} and satisfying the inequality (3) and by $C_\rho(R_2^{++})$ the space of all continuous functions belonging to $B_\rho(R_2^{++})$.

Obviously, we can introduce the norm

$$\|f\|_\rho = \sup_{x\geq 0, y\geq 0} \frac{|f(x,y)|}{1+x^2+y^2},$$

and therefore B_ρ and C_ρ are linear normed spaces.

The two-dimensional analog of the theorem, given in [**2, 3**] is as follows.

Theorem 3. *There exists a sequence of linear positive operators L_n, acting from $C_\rho(R_2^{++})$ to $B_\rho(R_2^{++})$ and satisfying the conditions*

$$\lim_{n\to\infty} \|L_n f_{k,m} - f_{k,m}\|_\rho = 0, \quad 0\leq k+m\leq 2, \tag{13}$$

and there exists a function $f^* \in C_\rho(R_2^{++})$ such that

$$\overline{\lim_{n\to\infty}} \|L_n f^* - f^*\|_\rho \geq \frac{1}{4} \tag{14}$$

Proof. Consider a sequence of operators L_n, acting on the functions $f \in C_\rho(R_2^{++})$ by the formulas

$$L_n(f;x,y) = \begin{cases} f(x,y) + \dfrac{1+x^2+y^2}{1+4n^2}[f(x+1,y+1)-f(x,y)] & \text{if } (x,y)\in\Delta_n, \\ f(x,y) & \text{if } (x,y)\in R_2^{++}\setminus\Delta_n. \end{cases}$$

Obviously, L_n is the sequence of linear positive operators, acting from $C_\rho(R_2^{++})$ to $B_\rho(R_2^{++})$.

To verify (13) we see that, for functions $f_{0,0}(x,y)=1$, $f_{1,0}(x,y)=x$ and $f_{0,1}(x,y)=y$

$$\begin{aligned}
L_n(f_{0,0};x,y)-f_{0,0}(x,y)&=0\\
L_n(f_{1,0};x,y)-f_{1,0}(x,y)&=\frac{1+x^2+y^2}{1+4n^2}\\
L_n(f_{0,1};x,y)-f_{0,1}(x,y)&=\frac{1+x^2+y^2}{1+4n^2}
\end{aligned}$$

where $(x,y)\in\Delta_n$.

Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\|L_n(f_{0,1})-f_{0,1}\|_\rho&=0\\
\lim_{n\to\infty}\|L_n(f_{1,0})-f_{1,0}\|_\rho&=0.
\end{aligned}$$

Finally, for a function $g(x,y)=f_{2,0}(x,y)+f_{0,2}(x,y)$, we have

$$L_n(g;x,y)-g(x,y)=\frac{1+x^2+y^2}{1+4n^2}\,[2x+2y+2]$$

which gives

$$\lim_{n\to\infty}\|L_ng-g\|_\rho=0.$$

Therefore, all the conditions (13) hold.

Consider now a function

$$f^*(x,y)=(x^2+y^2)\cos\pi(x+y),$$

which obviously belongs to $C_\rho(R_2^{++})$.

For this function, if $(x,y)\in\Delta_n$,

$$\begin{aligned}
L_n(f^*;x,y)=f^*(x,y)-{}&\frac{1+x^2+y^2}{1+4n^2}\\
&\times\Big[\big((x+1)^2+(y+1)^2\big)\cos\pi(x+y)+(x^2+y^2)\cos\pi(x+y)\Big].
\end{aligned}$$

Therefore,

$$\sup_{(x,y)\in\Delta_n} \frac{|L_n(f^*;x,y)-f^*(x,y)|}{1+x^2+y^2}$$
$$\geq \frac{1}{1+4n^2}\,|\cos \pi n| \left[2\left(\frac{n}{2}+1\right)^2+\frac{2n^2}{4}\right]$$
$$= \frac{2+2n+2n^2}{1+4n^2}$$

which gives (14), and the proof is completed. $\square$

This theorem shows that in general the approximation statement given in Theorem 2 does not hold for all linear positive operators, satisfying (13).

However, this problem is still open for polynomials (2).

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