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THE RIEMANN INTEGRAL USING ORDERED OPEN COVERINGS

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ABSTRACT. We define the Riemann integral for bounded functions defined on a general topological measure space. When the space is a compact metric space the integral is equivalent to the R-integral defined by Edalat using domain theory.

0. Introduction. Edalat [**1**] defined a Riemann type integral on a compact metric space, called the R-integral, using domain theory. The integral so defined has applications in various fields such as dynamic systems and chaos, and the work in [**1**] has also inspired other interesting research, see [**2, 3, 5**]. The main properties of this new integral among others are: (1) If the space is $[a, b]$, then this integral coincides with the ordinary Riemann integral;(2) a bounded function f is R-integrable if and only if it is continuous almost everywhere; (3) if f is R-integrable then it is also Lebesgue integrable and the value of the R-integral equals that of the Lebesgue integral of f. However, as the definition of the R-integral and most of the proofs in [**1**] rely heavily on very technical details of domain theory, this integral is hardly accessible to those who know little about domain theory. Furthermore, unlike the Riemann sum over a partition, the Riemann sum over a simple valuation, the key structure in defining the R-integral, lacks a clear geometric interpretation. In this paper we define a Riemann type integral with a domain-free approach. To make it easier to compare this integral with other known integrals we first introduce the more general $\mathcal{M}$-integral for a given collection $\mathcal{M}$ of some measurable subsets satisfying certain conditions. The integral introduced here is defined for bounded real valued functions on an arbitrary topological measure space X which need not be a compact metric space as required in [**1**]; it is a generalization of the Riemann integral on intervals; a function f is integrable if and only if it is continuous almost everywhere when

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the space is compact; when f is integrable it is Lebesgue integrable and the values for these two integrals equal. All these then imply that this integral is equivalent to the R-integral defined by Edalat when the space X is a compact metric space.

1. Ordered coverings.

Let X be a nonempty set and $\mathcal{M}$ be a collection of subsets of X satisfying

(M1) X and $\varnothing$ are in $\mathcal{M}$;

(M2) $A, B \in \mathcal{M}$ imply $A \cap B \in \mathcal{M}$.

Definition 1.1. An ordered $\mathcal{M}$-covering of X is an ordered tuple

$$\mathcal{A} = \langle A_1, A_2, \dots, A_N \rangle$$

of sets A_i in $\mathcal{M}$ such that $\cup_{i=1}^{N} A_i = X$. We also use the ordered chain

$$A_1 < A_2 < \cdots < A_N$$

to denote the above ordered covering. Here N could be any positive integer.

Put $\Delta_{\mathcal{M}} = \{\mathcal{A} : \mathcal{A}$ is an ordered $\mathcal{M}$-covering of X$\}$.

Remark 1.2. (1) The set A_i in an ordered covering could be empty.

(2) For any X and any $\mathcal{M}$, $\langle X \rangle$ is an ordered $\mathcal{M}$-covering.

Example 1.3. (1) If X is a topological space and $\mathcal{M}$ is the collection of all open sets then $\mathcal{M}$ satisfies (M1) and (M2). Such ordered $\mathcal{M}$-coverings will be called ordered open coverings of X. Ordered open coverings are used by Edalat in [**2**] to construct a sequence of simple valuations that approaches a given measure.

(2) If X is a measure space and $\mathcal{M}$ is the set of all measurable sets of X, then $\mathcal{M}$ satisfies (M1) and (M2). Such ordered $\mathcal{M}$-coverings are called ordered measurable coverings.

(3) Let X be a topological space and $\mathcal{M}$ the collection of all closed subsets of X. Then $\mathcal{M}$ satisfies (M1) and (M2).

(4) Let X be the set $\mathcal{R}$ of all real numbers. A subset A of X is said to be of density 1 at a point c if

$$\lim_{h\to 0^+} \frac{\mu(A\cap(c-h,c+h))}{2h} = 1.$$

Let $\mathcal{M}$ be the collection of all subsets A of X such that A is of density 1 at each point $c\in A$. Then $\mathcal{M}$ satisfies (M1) and (M2).

Definition 1.4. Let $\mathcal{A}=\langle A_1, A_2, \dots, A_N\rangle$ and $\mathcal{B}=\langle B_1, B_2, \dots, B_M\rangle$ be two ordered $\mathcal{M}$-coverings of X. Define $\mathcal{A}*\mathcal{B}$ to be the ordered covering in which the $\mathcal{M}$-sets are $A_i\cap B_j$, $1\le i\le N$, $1\le j\le M$, and $A_i\cap B_j < A_{i'}\cap B_{j'}$ if and only if either $i<i'$ or $i=i'$ and $j<j'$.

2. The Riemann sums over ordered coverings. We now define the lower and upper Riemann sums of a bounded function defined on a measure space and then use these to define the Riemann integral.

In the following we assume that $(X,\mathcal{H},\mu)$ is a measure space with $\mu(X)=1$, and $\mathcal{M}$ is a collection of measurable subsets satisfying (M1) and (M2). Let $f: X\to\mathcal{R}$ be a bounded real valued function on X and $A\subseteq X$. Define

$$\inf\, f(A)=\inf\{f(x): x\in A\} \quad\text{and}\quad \sup\, f(A)=\sup\{f(x): x\in A\}.$$

We assume that $\inf\, f(\varnothing)=0$ and $\sup\, f(\varnothing)=0$.

Definition 2.1. Let $f: X\to\mathcal{R}$ be a bounded real valued function. For each $\mathcal{A}=\langle A_1, A_2, \dots, A_N\rangle\in\Delta_{\mathcal{M}}$, define

$$S^l(f,\mathcal{A})=\sum_{i=1}^N \mu(A_i^*)\inf\, f(A_i) \quad\text{and}\quad S^u(f,\mathcal{A})=\sum_{i=1}^N \mu(A_i^*)\sup\, f(A_i),$$

where $A_1^*=A_1$ and $A_i^*=A_i-\cup_{j<i}A_j$, $i=2,3,\dots,N$.

We call $S^l(f,\mathcal{A})$ and $S^u(f,\mathcal{A})$ the lower and upper Riemann sums of f over $\mathcal{A}$, respectively.

Lemma 2.2. *Let $\mathcal{A},\mathcal{B}\in\Delta_{\mathcal{M}}$. Then, for any bounded function $f: X\to\mathcal{R}$ we have*

$$S^l(f,\mathcal{A})\le S^l(f,\mathcal{A}*\mathcal{B})\le S^u(f,\mathcal{A}*\mathcal{B})\le S^u(f,\mathcal{A}),$$

and

$$S^l(f,\mathcal{B}) \le S^l(f,\mathcal{A}*\mathcal{B}) \le S^u(f,\mathcal{A}*\mathcal{B}) \le S^u(f,\mathcal{B}).$$

Proof. Let $\mathcal{A} = \langle A_1, A_2, \dots, A_N\rangle$ and $\mathcal{B} = \langle B_1, B_2, \dots, B_M\rangle$. Notice that $\mathcal{A}*\mathcal{B} = \{A_i \cap B_j\}$ in which $A_i \cap B_j < A_{i'} \cap B_{j'}$ if either $i < i'$, or $i = i'$ and $j < j'$. So we have

$$(A_k \cap B_l)^* = A_k \cap B_l - \bigcup_{(i,j)<(k,l)} A_i \cap B_j,$$

where $(i,j) < (k,l)$ if either $i < k$ or $i = k$ and $j < l$. Notice that $\cup_{1\le j\le M} B_j = X$, hence

$$\begin{aligned}\bigcup_{(i,j)<(k,l)} A_i \cap B_j &= \bigcup_{i<k}\left(\bigcup_{1\le j\le M}(A_i\cap B_j)\right) \cup \bigcup_{j<l}(A_k \cap B_j)\\ &= \bigcup_{i<k}\left(A_i \cap \left(\bigcup_{1\le j\le M} B_j\right)\right) \cup \left(A_k \cap \bigcup_{j<l} B_j\right)\\ &= \bigcup_{i<k} A_i \cup \left(A_k \cap \bigcup_{j<l} B_j\right).\end{aligned}$$

Hence

$$(A_k \cap B_l)^* = (A_k \cap B_l) - \left(\bigcup_{i<k} A_i \cup \left(A_k \cap \bigcup_{j<l} B_j\right)\right).$$

We first prove

$$S^l(f,\mathcal{A}) \le S^l(f,\mathcal{A}*\mathcal{B}) \le S^u(f,\mathcal{A}*\mathcal{B}) \le S^u(f,\mathcal{A}).$$

Now

$$\begin{aligned}&S^l(f,\mathcal{A}*\mathcal{B})\\ &= \sum_{1\le i\le N, 1\le j\le M} \mu((A_i\cap B_j)^*) \inf\, f(A_i \cap B_j)\\ &= \mu(A_1\cap B_1)\inf\, f(A_1\cap B_1) + \mu(A_1 \cap B_2 - A_1 \cap\, B_1)\\ &\qquad\qquad \times \inf\, f(A_1 \cap B_2) + \cdots\end{aligned}$$

$$+\mu\left(A_1\cap B_M - A_1\cap\bigcup_{j<M}B_j\right)\inf f(A_1\cap B_M)+\cdots$$
$$+\mu\left(A_k\cap B_1-\bigcup_{i<k}A_i\right)\inf f(A_k\cap B_1)$$
$$+\mu\left(A_k\cap B_2-\left(\left(A_k\cap B_1)\cup\bigcup_{i<k}A_i\right)\right)\inf f(A_k\cap B_2)+\cdots$$
$$+\mu\left(A_k\cap B_M-\left(\left(A_k\cap\bigcup_{j<M}B_j\right)\cup\bigcup_{i<k}A_i\right)\right)$$
$$\times\inf f(A_k\cap B_M)+\cdots$$
$$+\mu\left(A_N\cap B_1-\bigcup_{i<N}A_i\right)\inf f(A_N\cap B_1)$$
$$+\mu\left(A_N\cap B_2-\left((A_N\cap B_1)\cup\bigcup_{i<N}A_i\right)\right)\inf f(A_N\cap B_2)+\cdots$$
$$+\mu\left(A_N\cap B_M-\left(\left(A_N\cap\bigcup_{j<M}B_j\right)\cup\bigcup_{i<N}A_i\right)\right)$$
$$\times\inf f(A_N\cap B_M).$$

For each $1\le k\le N$, we have

$$\mu\left(A_k\cap B_1-\bigcup_{i<k}A_i\right)\inf f(A_k\cap B_1)$$
$$+\mu\left(A_k\cap B_2-\left((A_k\cap B_1)\cup\bigcup_{i<k}A_i\right)\right)\inf f(A_k\cap B_2)+\cdots$$
$$+\mu\left(A_k\cap B_M-\left(\left(A_k\cap\bigcup_{j<M}B_j\right)\cup\bigcup_{i<k}A_i)\right)\inf f(A_k\cap B_M)$$
$$\ge\left[\mu\left(A_k\cap B_1-\bigcup_{i<k}A_i\right)+\mu\left(A_k\cap B_2-\left((A_k\cap B_1)\cup\bigcup_{i<k}A_i\right)\right)+\cdots\right.$$
$$\left.+\mu\left(A_k\cap B_M-\left(\left(A_k\cap\bigcup_{j<M}B_j\right)\cup\bigcup_{i<k}A_i\right)\right)\right]\inf f(A_k)$$

$$= \mu\left(\left(A_k \cap B_1 - \bigcup_{i<k} A_i\right) \cup \left(A_k \cap B_2 - \left((A_k \cap B_1) \cup \bigcup_{i<k} A_i\right)\right) \cup \cdots \right.$$
$$\left. \cup \left(A_k \cap B_M - \left(\left(A_k \cap \bigcup_{j<M} B_j\right) \cup \bigcup_{i<k} A_i\right)\right)\right) \inf\, f(A_k)$$
$$= \mu\left(A_k - \bigcup_{i<k} A_i\right) \inf\, f(A_k),$$

where the last and the second to the last equation follow from the fact that the sets

$$A_k \cap B_1 - \bigcup_{i<k} A_i, A_k \cap B_2 - \left(A_k \cap B_1 \cup \bigcup_{i<k} A_i\right), \dots ,$$
$$A_k \cap B_M - \left(\left(A_k \cap \bigcup_{j<M} B_j\right) \cup \bigcup_{i<k} A_i\right)$$

are pairwise disjoint and their union is $A_k - \bigcup_{i<k} A_i$.

Since $A_k - \cup_{i<k} A_i = A_k^*$, it then follows that

$$S^l(f, \mathcal{A} * \mathcal{B}) \ge S^l(f, \mathcal{A}).$$

Similarly we can prove

$$S^u(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{A}).$$

Now we prove

$$S^l(f, \mathcal{B}) \le S^l(f, \mathcal{A} * \mathcal{B}).$$

For each $1 \le l \le M$, the sum of the terms in $S^l(f, \mathcal{A} * \mathcal{B})$ involving B_l is

$$\mu((A_1 \cap B_l)^*) \inf\, f(A_1 \cap B_l) + \mu((A_2 \cap B_l)^*) \inf\, f(A_2 \cap B_l) + \cdots$$
$$+ \mu((A_N \cap B_l)^*) \inf\, f(A_N \cap B_l)$$
$$\ge [\mu((A_1 \cap B_l)^*) + \mu((A_2 \cap B_l)^*) + \cdots + \mu((A_N \cap B_l)^*)] \inf\, f(B_l).$$

In addition, $\mu((A_1 \cap B_l)^*) + \mu((A_2 \cap B_l)^*) + \cdots + \mu((A_N \cap B_l)^*)) = \mu((A_1 \cap B_l)^* \cup (A_2 \cap B_l)^* \cup \cdots \cup (A_N \cap B_l)^*)$ because $(A_1 \cap B_l)^*, (A_2 \cap B_l)^*, \dots , (A_N \cap B_l)^*$ are pairwise disjoint.

Notice that for any four sets A, B, C and D we have the equation $(A-B)\cap(C-D) = A\cap C - ((B\cap C)\cup D)$. Then for each $m \le N$,

$$\begin{aligned}(A_m \cap B_l)^* &= (A_m \cap B_l) - \left(\left(A_m \cap \bigcup_{j<l} B_j\right) \cup \bigcup_{i<m} A_i\right)\\ &= \left(B_l - \bigcup_{j<l} B_j\right) \cap \left(A_m - \bigcup_{i<m} A_i\right)\\ &= {B_l}^* \cap \left(A_m - \bigcup_{i<m} A_i\right).\end{aligned}$$

Hence

$$\begin{aligned}&(A_1 \cap B_l)^* \cup (A_2 \cap B_l)^* \cup \cdots \cup (A_N \cap B_l)^*\\ &\qquad = {B_l}^* \cap \bigcup_{i=1}^{N} \left(A_i - \bigcup_{j<i} A_j\right)\\ &\qquad = {B_l}^* \cap \bigcup_{i=1}^{N} A_i = {B_l}^* \cap X = {B_l}^*.\end{aligned}$$

Therefore, $S^l(f, \mathcal{A} * \mathcal{B}) \ge \sum_{j\le M} \mu(B_j^*) \inf\, f(B_j) = S^l(f, \mathcal{B})$. Similarly, we can show $S^u(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{B})$. The proof is complete.

Corollary 2.3. *For any* $\mathcal{A}, \mathcal{B} \in \Delta_{\mathcal{M}}$, $S^l(f, \mathcal{A}) \le S^u(f, \mathcal{B})$.

Proof. This follows from

$$S^l(f, \mathcal{A}) \le S^l(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{B}).$$

Definition 2.4. Let $\mathcal{M}$ be a collection of measurable sets of X satisfying (M1) and (M2). For any bounded function $f : X \to \mathcal{R}$ define

$$\begin{aligned}(\mathcal{M}) \int_{-} f d\mu &= \mathrm{Sup}\,\{S^l(f, \mathcal{A}) : \mathcal{A} \in \mathcal{M}\},\\ (\mathcal{M}) \int^{-} f d\mu &= \mathrm{Inf}\,\{S^u(f, \mathcal{A}) : \mathcal{A} \in \mathcal{M}\}.\end{aligned}$$

Remark 2.5. (1) From Corollary 2.3 it follows immediately that

$$(\mathcal{M})\int_{-} f\,d\mu \le (\mathcal{M})\int^{-} f\,d\mu.$$

(2) If $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then obviously

$$(\mathcal{M}_1)\int_{-} f\,d\mu \le (\mathcal{M}_2)\int_{-} f\,d\mu \le (\mathcal{M}_2)\int^{-} f\,d\mu \le (\mathcal{M}_1)\int^{-} f\,d\mu.$$

(3) If in an ordered covering $\mathcal{A} = \langle A_1, A_2, \dots , A_N\rangle$, A_i is contained in the union of those A_j with $j < i$, then we can remove A_i from $\mathcal{A}$ without effecting the values of the Riemann sums. In particular we can always remove the empty set from $\mathcal{A}$.

3. The $\mathcal{M}$-integral. Now we can define a Riemann type integral for each $\mathcal{M}$ satisfying the conditions (M1) and (M2) which includes both the Riemann integral and the Lebesgue integral as special cases when the functions considered are bounded.

Definition 3.1. Given a collection $\mathcal{M}$ of measurable sets satisfying the conditions (M1) and (M2). A bounded real valued function $f : X \to \mathcal{R}$ is called $\mathcal{M}$-integrable if

$$(\mathcal{M})\int_{-} f\,d\mu = (\mathcal{M})\int^{-} f\,d\mu.$$

In this case we call $(\mathcal{M})\int_{-} f d\mu = (\mathcal{M})\int^{-} f d\mu$ the $\mathcal{M}$-integral of f on X and denote it by

$$(\mathcal{M})\int f\,d\mu.$$

Corollary 3.2. *If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then by Remark* 2.5*, every $\mathcal{M}_1$-integrable function is also $\mathcal{M}_2$-integrable, and in this case*

$$(\mathcal{M}_1)\int f\,d\mu = (\mathcal{M}_2)\int f\,d\mu.$$

For any scalar k and any two functions f and g we have

$$(\mathcal{M})\int_{-} k\,f\,d\mu = k(\mathcal{M})\int_{-} f\,d\mu, \quad (\mathcal{M})\int^{-} k\,f\,d\mu = k(\mathcal{M})\int^{-} f\,d\mu$$

and

$$\begin{aligned}(\mathcal{M})\int_{-} f\,d\mu + (\mathcal{M})\int_{-} g\,d\mu \\ &\leq (\mathcal{M})\int_{-} (f+g)\,d\mu \leq (\mathcal{M})\int^{-} (f+g)\,d\mu \\ &\leq (\mathcal{M})\int^{-} f\,d\mu + (\mathcal{M})\int^{-} g\,d\mu.\end{aligned}$$

From these we obtain

Corollary 3.3. *If f and g are $\mathcal{M}$-integrable functions and k is any scalar, then both kf and $f+g$ are $\mathcal{M}$-integrable, and in these cases*

$$(\mathcal{M})\int (f+g)\,d\mu = (\mathcal{M})\int f\,d\mu + (\mathcal{M})\int g\,d\mu,$$
$$(\mathcal{M})\int k\,f\,d\mu = k(\mathcal{M})\int f\,d\mu.$$

The following lemma can be verified directly.

Lemma 3.4. *Let $f : X \to \mathcal{R}$ be any bounded function. Then the following statements are equivalent*:

(1) *The function f is $\mathcal{M}$-integrable.*

(2) *For any $\varepsilon > 0$ there exists $\mathcal{A} \in \Delta_{\mathcal{M}}$ such that*

$$S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) < \varepsilon.$$

(3) *There is a number b such that for any $\varepsilon > 0$ there exists $\mathcal{A} = \langle A_1, A_2, \dots, A_N\rangle \in \Delta_{\mathcal{M}}$ such that*

$$\left|\sum_{i=1}^{N} \mu(A_i^*) f(\xi_i) - b\right| < \varepsilon$$

holds for arbitrary points $\xi_i \in A_i$, $i = 1, 2, \dots, N$.

(4) *For any* $\varepsilon > 0$ *there is* $\mathcal{A} = \langle A_1, A_2, \dots, A_N \rangle \in \Delta_{\mathcal{M}}$ *such that*

$$\sum_{i=1}^{N} \mu(A_i^*)\,\omega(f, A_i) < \varepsilon,$$

where $\omega(f, A_i)$ *is the oscillation of* f *on* A_i.

4. The Lebesgue integral for bounded functions.

In this section we consider the $\mathcal{L}$-integral where $\mathcal{L}$ is the set of all measurable sets of X. It turns out with no surprise that this is exactly the Lebesgue integral.

The Lebesgue integral of a bounded real valued function can be defined in various equivalent ways. Here we adopt the following definition. For the case when $X = [a, b]$ see [**4**, Definition 3.6].

Let $s : X \to \mathcal{R}$ be a measurable function. The function s is a simple function if it has a finite range, equivalently, if there are pairwise disjoint measurable sets $E_1, E_2, \dots, E_n$ of X which form a covering of X and $s = \sum_{k=1}^{n} c_k \chi_{E_k}$, where χ_{E_k} is the characteristic function of E_k. The Lebesgue integral of the simple function $s = \sum_{k=1}^{n} c_k \chi_{E_k}$ is defined by

$$\int s\, d\mu = \sum_{k=1}^{n} c_k\, \mu(E_k).$$

Definition 4.1. Let f be a bounded measurable function on X. The lower and the upper Lebesgue integrals of f are defined by

$$\underline{\int} f = \sup\Big\{ \int \phi\, d\mu : \phi \le f \text{ is a simple function}\Big\},$$
$$\overline{\int} f = \inf\Big\{ \int \psi\, d\mu : \psi \ge f \text{ is a simple function}\Big\}.$$

If these two integrals are equal, then f is called Lebesgue integrable on X and the common value is denoted by $(L)\int_X f\, d\mu$, or simply $\int f\, d\mu$.

Lemma 4.2. *A bounded function f is Lebesgue integrable if and only if for any $\varepsilon > 0$ there is an ordered measurable covering $\mathcal{A} = \langle E_1, E_2, \dots, E_n \rangle$ of X such that*

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

Proof. Suppose the condition is satisfied. For any $\varepsilon > 0$, let $\mathcal{A} = \langle E_1, E_2, \dots, E_n \rangle$ be an ordered measurable covering satisfying

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

If necessary we can replace $\mathcal{A}$ by the ordered measurable covering $\mathcal{B}^*$ obtained by removing the empty sets from the covering $\mathcal{A}^* = \langle E_1, E_2 - E_1, \dots, E_k - \cup_{j<k} E_j, \dots, E_n - \cup_{j<n} E_j \rangle$. This is possible because $S^l(f, \mathcal{A}) \le S^l(f, \mathcal{A}^*) \le S^u(f, \mathcal{A}^*) \le S^u(f, \mathcal{A})$, and $S^l(f, \mathcal{B}^*) = S^l(f, \mathcal{A}^*), S^u(f, \mathcal{B}^*) = S^u(f, \mathcal{A}^*)$. Thus we can assume the sets E_i are pairwise disjoint and nonempty. Define two simple functions ψ and ϕ as follows:

$$\psi = \sum_{i=1}^{n} s_i \chi_{E_i}, \qquad \phi = \sum_{i=1}^{n} l_i \chi_{E_i},$$

where $s_i = \sup f(E_i)$, $l_i = \inf f(E_i)$. Obviously $\phi \le f \le \psi$, and $\int \phi \, d\mu = S^l(f, \mathcal{A}), \int \psi \, d\mu = S^u(f, \mathcal{A})$. This then deduces that $\int^- f - \int_- f \le S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon$. Thus f is Lebesgue integrable.

Conversely if f is Lebesgue integrable, then for any $\varepsilon > 0$ there are simple functions ϕ and ψ such that

$$\phi = \sum c_k \chi_{E_k} \le f \le \psi = \sum s_i \chi_{B_i}$$

and

$$\int \psi \, d\mu - \int \phi \, d\mu < \varepsilon.$$

Let $\mathcal{A}$ be the ordered measurable covering formed by the pairwise disjoint sets $E_k \cap B_i$ in any fixed order. Then one easily verifies that $\int \phi \, d\mu \le S^l(f, \mathcal{A}) \le S^u(f, \mathcal{A}) \le \int \psi \, d\mu$, hence

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) \le \int \psi \, d\mu - \int \phi \, d\mu < \varepsilon.$$

Corollary 4.3. *A bounded function f is Lebesgue integrable if and only if it is $\mathcal{L}$-integrable. In this case the values for the two integrals are equal.*

Since $\mathcal{L}$ is the largest collection of measurable sets satisfying the conditions (M1) and (M2), by Corollary 3.2 we deduce the following.

Corollary 4.4. *Let $\mathcal{M}$ be a collection of measurable sets satisfying the conditions* (M1) *and* (M2). *If a bounded function f is $\mathcal{M}$-integrable it is also Lebesgue integrable, and in this case*

$$(\mathcal{M}) \int f \, d\mu = (L) \int f \, d\mu.$$

5. The R-integral. In this section we consider an integral for bounded real valued functions defined on a topological space X equipped with a normed Borel measure μ, that is $\mu(X) = 1$. Let $\mathcal{O}$ be the collection of all open sets of X. The $\mathcal{O}$-integrable functions will be called R-integrable functions. We shall prove that the R-integral is a generalization of the Riemann integral on intervals.

An ordered $\mathcal{O}$-covering of X is called an ordered open covering.

Let $f : X \to \mathcal{R}$ be a bounded function. Recall that, for each subset A of X, the oscillation of f on A is defined by

$$\omega(f, A) = \sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\},$$

and for each point $a \in X$, the oscillation of f at a is defined by

$$\omega(f, a) = \inf\{\omega(f, U) : \text{U is an open neighborhood of } a\}.$$

It is well known that f is continuous at a if and only if $\omega(f, a) = 0$. For each $\varepsilon > 0$, the set $D(f; \varepsilon) = \{x : \omega(f, x) \geq \varepsilon\}$ is a closed subset of X, and the set of discontinuity points of f, denoted by $D(f)$, is

$$D(f) = \bigcup_{n=1}^{+\infty} D(f; 1/n).$$

A function f is said to be continuous almost everywhere if $\mu(D(f)) = 0$.

Lemma 5.1. *If a function f is R-integrable, then f is continuous almost everywhere.*

Proof. Suppose, on the contrary, $\mu(D(f)) \neq 0$. Then $\mu(D(f; 1/n)) \neq 0$ for some n. Now for any ordered open covering $\mathcal{A} = \langle A_1, A_2, \dots, A_N \rangle$ of X,

$$\begin{aligned} S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) &= \sum_{k=1}^{N} \mu(A_k^*)\,\omega(f, A_k) \\ &\geq \sum_{k=1}^{N} \mu(D(f; 1/n) \cap A_k^*)\,\omega(f, A_k). \end{aligned}$$

Notice that $D(f; 1/n) \cap A_k^* \subseteq D(f; 1/n) \cap A_k$. If $\mu(D(f; 1/n) \cap A_k^*) \neq 0$, then $D(f; 1/n) \cap A_k \neq \varnothing$. Since A_k is open, it follows that $\omega(f, A_k) \geq 1/n$, thus $\mu(D(f; 1/n) \cap A_k^*)\,\omega(f, A_k) \geq \mu(D(f; 1/n) \cap A_k^*) 1/n$. If $\mu(D(f; 1/n) \cap A_k^*) = 0$, then trivally $\mu(D(f; 1/n) \cap A_k^*)\,\omega(f, A_k) = \mu(D(f; 1/n) \cap A_k^*) 1/n$.

Hence we have

$$\begin{aligned} \sum_{k=1}^{N} \mu(D(f; 1/n) \cap A_k^*)\,\omega(f, A_k) &\geq \sum_{k=1}^{N} \mu(D(f; 1/n) \cap A_k^*)\,\frac{1}{n} \\ &= \frac{1}{n} \sum_{k=1}^{N} \mu(D(f; 1/n) \cap A_k^*) = \frac{1}{n}\,\mu(D(f; 1/n)). \end{aligned}$$

The last equation follows from the fact that the A_k^*'s are pairwise disjoint and their union is X. This contradicts the assumption that f is R-integrable. Hence $\mu(D(f)) = 0$.

For the converse conclusion to be true we need the measure to have the following property:

For any measure zero set A and any $\varepsilon > 0$, there is an open set U, such that

$$(*) \qquad A \subseteq U \quad \text{and} \quad \mu(U) < \varepsilon.$$

Lemma 5.2. *Let X be a compact Hausdorff space with a normed Borel measure μ satisfying the condition $(*)$. If f is bounded and continuous almost everywhere, then f is R-integrable.*

Proof. Assume that f is continuous almost everywhere and $|f(x)| \leq B$ for all $x \in X$, where B is a positive number. Now for each $\varepsilon > 0$, by condition $(*)$ we can choose an open set A_1 containing $D(f)$ such that $\mu(A_1) < (\varepsilon/4B)$. As a closed subset of X, $F = X - A_1$ is compact, and f is continuous at every point in F. Thus there is an open covering of F, say $\{A_2, A_2, \ldots, A_N\}$ such that $\omega(f, A_k) < (\varepsilon/2)$ for each $k = 2, 3, \ldots, N$. Put $\mathcal{A} = \langle A_1, A_2, \ldots, A_N\rangle$. Then

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) = \sum_{k=1}^{N} \mu(A_k^*)\omega(f, A_K) < \frac{\varepsilon}{4B}\, 2B + \frac{\varepsilon}{2}\sum_{k=2}^{N} \mu(A_k^*) \leq \varepsilon.$$

Hence f is R-integrable.

Theorem 5.3. *Let X be a compact Hausdorff space with a normed Borel measure satisfying the condition $(*)$. Then a bounded function is R-integrable if and only if it is continuous almost everywhere.*

It is well known that a bounded function defined on an interval $[a, b]$ is Riemann integrable if and only if it is continuous almost everywhere. And in this case the Riemann integral and the Lebesgue integral of f are equal. The Lebesgue measure μ on $[a, b]$ satisfies the condition $(*)$. Thus combining the above results we obtain the following corollary which shows that the R-integral is a generalization of the Riemann integral.

Corollary 5.4. *A bounded function f on $[a, b]$ is Riemann integrable if and only if it is R-integrable. And in this case the values of the two integrals of f are equal.*

Remark 5.5. (1) Let $X = [a, b]$ and $\mathcal{I} = \{[c, d] : a \leq c \leq d \leq b\} \cup \{\varnothing\}$. Then $\mathcal{I}$ satisfies the conditions (M1) and (M2) and we can prove that $\mathcal{I}$-integral also coincides with the Riemann integral.

(2) Suppose $\mathcal{B}$ is a basis of a topological space X which includes X and $\varnothing$, so $\mathcal{B}$ satisfies (M1) and (M2). It is natural to ask if $\mathcal{B}$-integral

is equivalent to the R-integral. Since $\mathcal{B} \subseteq \mathcal{O}$, by Corollary 3.2 if f is $\mathcal{B}$-integrable, then it is R-integrable and the values of the two integrals of f are equal. Now suppose f is R-integrable and X is a compact Hausdorff space with a normed Borel measure satisfying the condition $(*)$. Then f is continuous almost everywhere by Theorem 5.3. Let B be a bound of f. For each $\varepsilon > 0$ choose $n > 0$ with $1/n < (\varepsilon/2)$. Then there is an open set U of X with $\mu(U) < (\varepsilon/4B)$ and $D(f;(1/n)) \subseteq U$. There exist $U_1, U_2, \ldots, U_m \in \mathcal{B}$ such that $D(f;(1/n)) \subseteq U_1 \cup U_2 \cdots \cup U_m \subseteq U$ because $D(f;(1/n))$ is a closed subset of the compact space X and $\mathcal{B}$ is a basis. Let $W = U_1 \cup U_2 \cdots \cup U_m$. Now for each $x \in W^c$ we have $\omega(f,x) < (1/n)$, so there exists an open neighborhood V of x such that $\omega(f,V) < (1/n)$, and this V can be chosen from $\mathcal{B}$. Since W^c is compact it follows that there are $U_{m+1}, \ldots, U_N \in \mathcal{B}$ such that $W^c \subseteq \cup_{i=m+1}^N U_i$ and $\omega(f,U_i) < (1/n)$ for each $i = m+1, \ldots, N$. Let $\mathcal{A} = \langle U_1, U_2, \ldots, U_N \rangle$. Then $\mathcal{A}$ is an ordered $\mathcal{B}$-covering, and we have the following equations and inequalities:

$$
\begin{aligned}
S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) &= \sum_{i=1}^{N} \mu(U_i^*)\, \omega(f,U_i) \\
&= \sum_{i=1}^{m} \mu(U_i^*)\, \omega(f,U_i) + \sum_{i=m+1}^{N} \mu(U_i^*)\, \omega(f,U_i) \\
&\leq 2B \sum_{i=1}^{m} \mu(U_i^*) + \frac{1}{n} \sum_{i=m+1}^{N} \mu(U_i^*) \\
&\leq 2B\mu(U) + \frac{\varepsilon}{2}\, \mu(X) \leq 2B\, \frac{\varepsilon}{4B} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Hence f is $\mathcal{B}$-integrable.

Remark 5.6. In [**1**] Edalat defines a Riemann type integral on compact metric spaces, also called R-integral, by using domain theory. He also proves that a bounded function f is R-integrable if and only if it is continuous almost everywhere [**1**, Theorem 6.5], and in this case R-integral of f is equal to the Lebesgue integral of f [**1**, Theorem 7.2]. Thus when X is a compact metric space then our R-integral is equivalent to Edalat's R-integral, and the values of the two integrals coincide for every integrable function.

6. Computability of R-integral. Compared with the Lebesgue integral, a distinct virtue of the Riemann integral is its computability, as was pointed out by Edalat in [**1**]. In terms of the definition given in this paper, computability means that one can choose a fixed countable collection $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of ordered open coverings such that for each R-integrable function f, we have

$$\int f\,d\mu = \lim_{n\to\infty} S^l(f,\mathcal{A}_n) = \lim_{n\to\infty} S^u(f,\mathcal{A}_n).$$

For compact metric spaces, Edalat has proved the computability of R-integral by using the domain theory. Here we provide an elementary proof for this fact.

In the following we assume that X is a compact metric space with a normed Borel measure μ satisfying the condition $(*)$.

The main step in the proof is to show that if f is R-integrable then for any $\varepsilon > 0$ there is $\delta > 0$ such that for each ordered open covering $\mathcal{A} = \langle A_1, A_2, \dots, A_N\rangle$, if $\dim(A_i) < \delta$ for $i = 1, 2, \dots, N$, then $S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) < \varepsilon$, where $\dim(A_i) = \sup\{d(x,y) : x, y \in A_i\}$.

To prove the main result we need the following lemma.

Lemma 6.1. *Let $f : X \to \mathcal{R}$ be a real valued function defined on a compact metric space and $\omega(f,x) < \delta$ hold for all $x \in X$. Then there is an $\varepsilon > 0$ such that*

$$|f(x) - f(y)| \le \delta$$

whenever $d(x,y) < \varepsilon$.

Lemma 6.2. *Let $f : X \to \mathcal{R}$ be R-integrable. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that for any ordered open covering $\mathcal{A} = \langle A_1, A_2, \dots, A_N\rangle$ if $\dim(\mathcal{A}_i) < \delta$ for each $i = 1, 2, \dots, N$, then*

$$S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) < \varepsilon.$$

Proof. Suppose $|f(x)| \le B$ for all $x \in X$. By Lemma 5.1, f is continuous almost everywhere. Choose a number $r > 0$ with $r < (\varepsilon/2)$. The set $D(f;r) = \{x \in X : \omega(f,x) \ge r\}$ is closed and has zero measure.

Take an open set $U \supseteq D(f;r)$ such that $\mu(U) < (\varepsilon/4B)$. Since X is compact there is an open set V satisfying

$$D(f;r) \subseteq V \subseteq \mathrm{cl}\,(V) \subseteq U, \mathrm{cl}\,(V) \neq U,$$

where $\mathrm{cl}\,(V)$ is the closure of V. Let $\delta_1 = \inf\{d(x,y) : x \in \mathrm{cl}\,(V), y \in X - U\}$. Then $\delta_1 > 0$ and $\omega(f,x) < r$ for all $x \notin V$. By Lemma 6.1, it follows that there exists $\delta_2 > 0$ such that for any $x, y \in V^c$, if $d(x,y) < \delta_2$ then $|f(x) - f(y)| \leq r < (\varepsilon/2)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now suppose $\mathcal{A} = \langle A_1, A_2, \dots, A_N \rangle$ is an ordered open covering such that $\dim(A_i) < \delta$ for $i = 1, 2, \dots, N$. Then each A_i is either contained in U or is contained in V^c. Assume that $A_{i_1}, A_{i_2}, \dots, A_{i_m}$ are contained in U and the rest of them are contained in V^c. Then

$$\begin{aligned} S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) &= \sum_{j=1}^{m} \mu({A_{i_j}}^*)(\sup f(A_{i_j}) - \inf f(A_{i_j})) \\ &\quad + \sum_{k \neq i_j} \mu(A_k^*)(\sup f(A_k) - \inf f(A_k)) \\ &\leq 2B \sum_{j=1}^{m} \mu({A_{i_j}}^*) + \frac{\varepsilon}{2} \sum_{k \neq i_j} \mu((A_k)^*). \end{aligned}$$

Note that the sets ${A_i}^*$ are pairwise disjoint sets, so

$$\sum_{j=1}^{m} \mu(A_{i_j}^*) = \mu(\cup_{j=1} A_{i_j}^*) \leq \mu(U).$$

Similarly

$$\sum_{k \neq i_j} \mu(A_k^*) \leq \mu(V^c).$$

Hence

$$S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) \leq 2B\mu(U) + \frac{\varepsilon}{2}\,\mu(V^c) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete.

Theorem 6.3. *For each $n \in \mathbf{N}$ choose an ordered open covering $\mathcal{A}_n$ such that each A_i in $\mathcal{A}_n$ has diameter less than $1/n$. Then a bounded function f is R-integrable if and only if*

$$\lim_{n\to\infty} S^l(f, \mathcal{A}_n) = \lim_{n\to\infty} S^u(f, \mathcal{A}_n),$$

and in this case

$$\int f\, d\mu = \lim_{n\to\infty} S^l(f, \mathcal{A}_n) = \lim_{n\to\infty} S^u(f, \mathcal{A}_n).$$

Proof. By Lemma 3.4 the condition is evidently sufficient. The necessity follows from Lemma 6.2. The equations

$$\int f\, d\mu = \lim_{n\to\infty} S^l(f, \mathcal{A}_n) = \lim_{n\to\infty} S^u(f, \mathcal{A}_n)$$

obviously hold. □

Remark 6.4. Since X is a compact metric space, for each $n > 0$ there exists an ordered open covering $\mathcal{A}_n$ such that for each A_i in $\mathcal{A}$, $\dim(A_i) < (1/n)$. Also by Lemma 2.2, if we define $\mathcal{B}_{n+1} = \mathcal{A}_{n+1} * \mathcal{B}_n$ for $n = 1, 2, \ldots$, then $\{\mathcal{B}_n\}_{n=1}^{\infty}$ is a sequence of ordered open coverings that can replace $\{\mathcal{A}_n\}_{n=1}^{\infty}$. In addition, for each bounded real valued function f we have two monotone sequences $S^l(f, \mathcal{B}_n) \nearrow$ and $S^u(f, \mathcal{B}_n) \searrow$, which converge to the same number when f is R-integrable.

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