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# THE ASYMPTOTIC GROWTH OF EQUIVARIANT SECTIONS OF POSITIVE AND BIG LINE BUNDLES

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**1. Introduction.** Let $X$ be a complex manifold, say of dimension $n$, and $L$ an holomorphic line bundle on it. Suppose that a finite group $G$ acts holomorphically on $X$ and that this action lifts to a holomorphic action on $L$, so that for every $x \in X$ and $g \in G$ the induced map $L_x \to L_{gx}$ is complex linear. Then $G$ also acts on every tensor power $L^{\otimes k}$, and thus it acts linearly on the spaces $H^0(X, L^{\otimes k})$ of global holomorphic sections of $L^{\otimes k}$. Therefore, $H^0(X, L^{\otimes k})$ splits $G$-equivariantly in terms of the irreducible representations of $G$. Let $H^0(X, L^{\otimes k})_i$ be the equivariant summand corresponding to the $i$th irreducible representation. The first object of this paper is the asymptotic growth of the dimension of $H^0(X, L^{\otimes k})_i$ as $k \to +\infty$, in the following two situations: i) $X$ is complex projective and $L$ is ample, and ii) $X$ is complex projective and $L$ is *big*, that is, it has maximal Kodaira dimension. We shall then discuss a generalization of this to the symplectic category.

In spite of its very short and simple proof, perhaps the main result of this article is Theorem 3, which deals with case i). This case has been studied (in a broader algebraic formulation and with algebraic techniques) by various authors, see [**13, 6, 7**]; the two latter papers deal with the actions of reductive groups. It follows from this body of work that in our complex projective, ample situation $\dim H^0(X, L^{\otimes k})_i = a_i k^n + b_i k^{n-1} + O(k^{n-2})$, where $a_i$ and $b_i$ are described algebraically. Here we give, in terms of the Riemann-Roch polynomial of $L$, an asymptotic estimate which is in many cases more refined, if the dimension $d$ of the locus of points in $X$ with nontrivial stabilizer in $G$ is taken into account. More precisely, we give an explicit asymptotic expansion with a remainder which is $o(k^{d+1})$; this is thus strictly more informative (of course, in our complex geometric situation) if $d \leq n-3$, and equally fine if $d = n-2$.

Our new approach is analytic and based on the study of the Szegő kernel $\Pi$ of $L$; the key ingredient is the off-diagonal estimate discussed

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in [**8**]. The use of Szegő kernels to study asymptotic properties in algebraic geometry originates in [**18**] (see [**16**] for more results and references), and is founded on Boutet de Monvel and Sjöstrand's microlocal description of $\Pi$ [**5**].

Let us now come to a more detailed description of the results, starting from case ii) for expository reasons. In this case, as in the action free situation, there is generally no asymptotic expansion for $\dim H^0(X, L^{\otimes k})_i$, but only a much cruder description of the asymptotic growth of $\dim H^0(X, L^{\otimes k})_i$. Our arguments for this case are based on an equivariant version of the arguments in [**9**].

Thus, let $X$ be a smooth complex projective $n$-fold and $L$ a big line bundle on $X$, that is, having maximal Kodaira dimension: $\kappa(X, L) = n$. After Fujita, the *volume* $\upsilon(L) = \upsilon(X, L)$ is

$$\upsilon(X, L) = \limsup_{k\to\infty} \frac{n!}{k^n}\, h^0(X, \mathcal{O}_X(kL)).$$

If $L$ is ample, or more generally nef and big, $\upsilon(L) = (L^n)$, the self-intersection number of $L$. The volume of a general big line bundle has been studied in [**11**] and [**9**]; in particular, $\upsilon(L)$ has been given the following geometric interpretation (Proposition 3.6 of [**9**]): Let $(kL)^{[n]}$ be the moving self-intersection number of $kL$, that is, the number of intersection points away from the base locus of $n$ general divisors in the linear series $|kL|$. Then

$$\upsilon(L) = \limsup_{k\to\infty} \frac{(kL)^{[n]}}{k^n}.$$

Suppose now that a finite group $G$ acts holomorphically on $X$, and that the action linearizes to $L$. Let $V_1, \ldots, V_c$ be the irreducible linear representations of $G$. For every $k$, we have an induced linear action of $G$ on the space of global sections $H^0(X, L^{\otimes k})$, and therefore an essentially unique decomposition

$$H^0(X, L^{\otimes k}) = \bigoplus_{i=1}^{c} H_i^0(X, L^{\otimes k}), \tag{1}$$

where each summand $H_i^0(X, L^{\otimes k})$ is $G$-equivariantly isomorphic to a direct sum of copies of $V_i$. For each $i$, set $h_i^0(X, L^{\otimes k}) = \dim H_i^0(X, L^{\otimes k})$

and then define the $i$th equivariant volume of $L$ as

$$\upsilon_i(X, L) = \limsup_{k\to\infty} \frac{n!}{k^n}\, h_i^0(X, L^{\otimes k}). \tag{2}$$

Here we shall study the volumes $\upsilon_i(L)$ and show, in particular, that

**Theorem 1.** *In the above situation suppose in addition that the action of $G$ on $X$ is faithful. Then for every $i = 1, \ldots, c$ we have*

$$\upsilon_i(L) = \frac{\dim(V_i)^2}{|G|}\ \upsilon(L).$$

For the trivial representation this has also been observed by Ein and Lazarsfeld. Furthermore, if $L$ is ample Theorem 1 follows from algebraic results of Howe [**13**]; more generally, if $L$ is big and numerically effective, that is, $L \cdot C \geq 0$ for every projective curve $C \subseteq X$, an algebro-geometric proof can be given applying the Riemann-Roch theorem on the quotient orbifold $X/G$ (I am indebted to M. Brion and J.-P. Demailly for pointing out these approaches to me). Here we follow, however, a different path, based first on an asymptotic estimate of the equivariant Szegő kernels in the positive case (see below for a precise definition), and next on an equivariant version of Fujita's approximation theorem to extend the result to arbitrary big line bundles. This has the following advantages: First, as we explain below, this approach applies very naturally to the context of almost complex quantization of compact symplectic manifolds. Second, in the ample case, it yields the lower order terms of the expansion of $h_i^0(X, L^{\otimes k})$ in decreasing powers of $k$, Theorem 3, in terms of the asymptotic expansion of the total Szegő kernel, and under suitable assumptions on the dimension of the locus of points having nontrivial stabilizer.

As hinted above, we can consider similarly defined invariants in the broader context of almost complex quantization. Namely, let $(X, \omega)$ be a compact symplectic manifold with $[\omega]/2\pi \in H^2(X, \mathbf{Z})$, and let $J$ be an almost complex structure on $X$ compatible with $\omega$. The pair $(\omega, J)$ fixes an Hermitian, hence a Riemannian, structure on $X$. Furthermore, by the integrality assumption on $\omega$, there exists an hermitian line bundle

$L$ on $X$, having a unitary connection $\nabla_L$ of curvature $-2\pi i\omega$. Let $H(X, L^{\otimes k}) \subset \mathcal{C}^\infty(X, L^{\otimes k})$ be the subspace of the space of smooth global sections of $L^{\otimes k}$ introduced and studied in [**2–4, 12, 16**]. When $J$ is integral, $\omega$ a Hodge form on $X$, $L$ an ample holomorphic line bundle, and $\nabla_L$ the unique unitary connection compatible with the holomorphic structure, $H(X, L^{\otimes k})$ is the usual space of holomorphic section of $L^{\otimes k}$. The dimension of $H(X, L^{\otimes k})$ is always given, for $k \gg 0$, by the Riemann-Roch formula, and the projective embeddings defined by the linear series $|H(X, L^{\otimes k})|$ have a good asymptotic behavior [**2, 16**].

Suppose now that the finite group $G$ acts faithfully on $X$ as a group of symplectomorphisms. We may choose in the above a $G$-invariant compatible almost complex structure $J$, and then all the construction can be made equivariantly. Thus $G$ acts linearly on $H(X, L^{\otimes k})$, and there is a direct sum decomposition as in (1): $H(X, L^{\otimes k}) = \oplus_i H_i(X, L^{\otimes k})$. Setting $h(X, L^{\otimes k}) = \dim H(X, L^{\otimes k})$, $h_i(X, L^{\otimes k}) = \dim H_i(X, L^{\otimes k})$, the $i$th volume of $L$ is then defined as in (2): $\upsilon_i(L) = \limsup_{k\to+\infty}(n!/k^n)h_i(X, L^{\otimes k})$. As explained in Remark 2, the proof of the following theorem is essentially the same as the proof of Theorem 4 below.

**Theorem 2.** *Let $(X, \omega)$ be a compact $2n$-dimensional symplectic manifold with $[\omega]/2\pi$ an integral cohomology class. Choose $L$, $J$ and $\nabla$ as described above. Suppose that the finite group $G$ acts faithfully as a group of symplectomorphisms on $X$, $J$ is $G$-invariant, and the action linearizes to $L$. Then*

$$\upsilon_i(L) = \frac{\dim(V_i)^2}{|G|}\int_X \omega^{\wedge n}.$$

Let us now dwell on the lower order terms of the expansion of the dimension of the covariant factors $H_i^0(X, L^{\otimes k})$, in the case where $X$ is a complex projective manifold and $L$ is ample. In the hypothesis of Theorem 1, let $V \subset X$ be the locus of points with non-trivial stabilizer. Then $V$ is a union of complex submanifolds of $M$; let $c$ be its complex codimension.

**Theorem 3.** *In the hypothesis of Theorem* 1*, and with the above notation, assume in addition that $L$ is ample. Suppose $s$ is an integer with $0 \leq s \leq n-1$ and $c > s$. Then*

$$h_i^0(X, L^{\otimes k}) - \frac{\dim(V_i)^2}{|G|}\, h^0(X, L^{\otimes k}) = o(k^{n-s})$$
*for every $i = 1, \ldots, c$.*

*If $V = \varnothing$, i.e., the action of $G$ on $X$ is free, then*

$$h_i^0(X, L^{\otimes k}) = \frac{\dim(V_i)^2}{|G|}\, h^0(X, L^{\otimes k})$$
*for every $i = 1, \ldots, c$ and $k \gg 0$.*

In particular, if $c > s$ we may compute the first $s$ terms in the asymptotic expansion of $H_i^0(X, L^{\otimes k})$ by integrating the first $s$ terms in the asymptotic expansion of the Szegő kernel of $L$ restricted to the diagonal, see below. In the projective case the first terms of this expansion have been explicitly computed by Lu [**14**].

*Notation.* We shall occasionally loosely shift from multiplicative to additive notation for line bundles. Furthermore, we shall generally identify without warning an invertible sheaf with the associated line bundle.

**2. Proofs.** To fix ideas, we focus on the complex projective case; the general almost complex case is discussed in Remark 2. To ease the exposition, in the following by a *$G$-line bundle* we shall mean a line bundle on $X$ to which the action of $G$ linearizes.

*Remark* 1. If $D$ is a $G$-invariant divisor on $X$ (not necessarily effective), then $\mathcal{O}_X(D)$ is a $G$-line bundle. In particular, for any line bundle $H$ on $X$, $\otimes_{g \in G} g^* H$ is a $G$-line bundle in a natural manner, (very) ample if so is $H$.

Before dealing with a general big line bundle, let us consider the special case where $L$ is ample.

**Theorem 4.** *Notation being as above (with $X$ a complex projective manifold of dimension $n$), assume again that the action of $G$ on $X$ is faithful and in addition that $L$ is an ample $G$-line bundle on $X$. Then for every $i = 1, \ldots, c$ we have*

$$\upsilon_i(L) = \lim_{k\to\infty} \frac{n!}{k^n}\, h_i^0\left(X, L^{\otimes k}\right) = \frac{\dim(V_i)^2}{|G|}\,(L^n).$$

This is a special case of Theorem 3 (the case $s = 0$). We prove it separately because it is just what is needed from the positive case to prove Theorem 1 in full generality, and furthermore its proof also establishes, with minor modifications, Theorem 2.

*Proof.* Let $h = h_L$ be an Hermitian metric on $L$ such that the curvature form $\omega$ of the unique compatible covariant derivative $\nabla_L$ on $L$ is Kähler. After averaging over $G$, we may assume that $h$, $\nabla_L$ and $\omega$ are $G$-invariant. Thus $\omega$ is a $G$-invariant Kähler form, inducing a $G$-invariant volume form $dx$ on $X$.

Let $L^* = L^{-1}$ be the dual line bundle, with the induced Hermitian structure and connection, and consider the unit disc bundle $L^* \supset \mathbf{S} \xrightarrow{\pi} X$. Let $i\alpha \in \Omega^1_{\mathbf{S}}(i\mathbf{R})$ be the connection form. Then $dp =: \pi^*(dx)/2\pi\wedge\alpha$ is a $G$-invariant volume form on $\mathbf{S}$. For every integer $k \geq 0$, denote by

$$P_k : L^2(\mathbf{S}) \longrightarrow \tilde{H}_k(\mathbf{S}) \cong H^0(X, L^{\otimes k})$$

the orthogonal projection onto the space of boundary values of holomorphic functions in the $k$th isotype with respect to the $S^1$-action. For each $i = 1, \ldots, c$, let

$$P_{k,i} : L^2(\mathbf{S}) \longrightarrow \tilde{H}_{k,i}(\mathbf{S}) \cong H^0_i(X, L^{\otimes k})$$

denote the orthogonal component onto the $i$th isotype of $\tilde{H}_k(\mathbf{S})$ with respect to the $G$-action. Also, let $\tilde{\Pi}_k$, $\tilde{\Pi}_{k,i} \in \mathcal{C}^\infty(\mathbf{S}\times\mathbf{S})$ be the Schwartz kernels of $\Pi_k$ and $\Pi_{k,i}$, respectively. Clearly, $\tilde{\Pi}_k = \sum_i \tilde{\Pi}_{k,i}$ and

$$H^0_i(X, A^{\otimes k}) = \int_{\mathbf{S}} \tilde{\Pi}_{k,i}(p,p)\, dp.$$

Now $\tilde{\Pi}_k(p,p)$ and $\tilde{\Pi}_{k,i}(p,p)$ descend to positive functions $\nu_k(x)$ and $\nu_{k,i}(x)$ on $X$, [**15, 16**]. Hence,

$$\upsilon_i(L) = \limsup_{k\to\infty} \frac{n!}{k^n} \int_X \nu_{k,i}(x)\, dx.$$

As in [**15**], we decompose $\nu_{k,i}(x)$ as the sum of two terms, the first being a multiple of $\nu_k(x)$ and the second growing at most as $k^{(n-1)/2}$. More precisely, if $G_x \subseteq G$ is the stabilizer subgroup of $x \in X$ and $p \in \mathbf{S}$ is any point lying over $x$, we have

(3)

$$\nu_{k,i}(x) = \frac{\dim(V_i)}{|G|} \cdot (\alpha_x^k, \chi_i)_{G_x} \cdot \nu_k(x) + \frac{\dim(V_i)}{|G|} \sum_{g\notin G_x} \overline{\chi}_i(g) \tilde{\Pi}_N(g^{-1}p, p).$$

Here notation is as follows: $\chi_i : G \to \mathbf{C}$ is the character of the irreducible representation $V_i$, $\alpha_x : G_x \to S^1 \subset \mathbf{C}^*$ is the unitary character describing the action of $G_x$ on $L(x)$ (the fiber of $L$ over $x$), and $(h,k)_{G_x} = \sum_{g\in G_x} f(g)\cdot \overline{k}(g)$ is the $L^2$-Hermitian product with respect to the counting measure on $G_x$. Furthermore, there exists $a > 0$ such that, setting $d_x = \min\{\mathrm{dist}\,(x, gx) : g \notin G_x\}$, the latter term is bounded above by $Ck^n e^{-a\sqrt{k}\, d_x}$, see Section 6 of [**8**].

If, in particular, $G_x = \{e\}$, where $e \in G$ is the unit, the former term is $(\dim(V_i)^2/|G|) \cdot \nu_k(x)$. Let us now recall the following simple useful fact.

**Lemma 1.** *Suppose that the finite group $G$ acts faithfully and holomorphically on the connected projective manifold $X$. Then there is a nonempty Zariski dense open subset $U \subseteq X$ such that $G_x = \{e\}$ for every $x \in U$.*

Set $Z = X \setminus U$; thus $Z$ is a proper algebraic subvariety of $X$ of codimension, say, $c$. For $\varepsilon > 0$ let $V_\varepsilon \subseteq X$ be the $\varepsilon$-neighborhood of $Z$ in the geodesic distance associated to $\omega$. Then $V_\varepsilon$ has volume $\le C\varepsilon^{2c}$, where $C$ is a constant. On the other hand, by the above and the asymptotic expansion of $\nu_k(x)$ in Theorem 1 of [**18**], $n!\, k^{-n} \nu_{k,i}(x)$ is in any event a bounded function. Therefore,

$$\left| \frac{n!}{k^n} \int_X \nu_{k,i}(x)\, dx - \frac{n!}{k^n} \int_{X\setminus V_\varepsilon} \nu_{k,i}(x)\, dx \right| = \left| \frac{n!}{k^n} \int_{V_\varepsilon} \nu_{k,i}(x)\, dx \right| \le C\varepsilon^{2c}.$$

There exists $\delta = \delta_\varepsilon > 0$ such that $\operatorname{dist}(x, gx) > \delta$ if $x \in X \setminus V_\varepsilon$ and $g \neq e$. Hence

$$\frac{n!}{k^n}\left|\int_{X\setminus V_\varepsilon} \nu_{k,i}(x)\,dx - \frac{\dim(V_i)^2}{|G|}\int_{X\setminus V_\varepsilon} \nu_k(x)\,dx\right| \le Ce^{-\sqrt{k}\,\delta}.$$

Summing up,

$$\left|\frac{n!}{k^n}\int_X \nu_{k,i}(x)\,dx - \frac{n!}{k^n}\cdot\frac{\dim(V_i)^2}{|G|}\int_X \nu_k(x)\,dx\right| \le C(\varepsilon^{2c} + e^{-\sqrt{k}\,\delta}).$$

Since $n!\,k^{-n}\int_X \nu_k(x)\,dx \to \operatorname{vol}(L) = (L^n)$, the statement follows by taking $k \gg 1/\varepsilon \gg 0$.

*Remark* 2. The asymptotic expansions used in the proof also hold in almost complex quantization [**2, 15, 16**], but the off-diagonal estimate on the $k$th Fourier coefficient of the Szegő kernel from [**8**] has been proved only in the complex projective case. However, in the more general almost complex case we still have the estimate

$$|\Pi_k(x,y)| \le C\nu_k(x)e^{-k\operatorname{dist}(x,y)^2/2} + O(k^{(n-1)/2}),$$

from [**2**] and [**16**], which is still enough to prove the theorem. Furthermore, since $G$ preserves the Riemannian structure on $X$ associated to $\omega$ and $J$, in place of Lemma 1 we may as well use Theorem 8.1 on page 213 of [**17**]: the set of all $x \in X$ with non-trivial stabilizer is a finite collection of proper submanifolds (the action being faithful). The same argument, with minor changes, thus also proves Theorem 2.

**Lemma 2.** *In the same hypothesis, let $A$ and $B$ be $G$-line bundles on $X$, with $A$ ample. Then $H^0_i(X, A^{\otimes m}\otimes B) \neq 0$ for every $i = 1, \dots, c$ and $m \gg 0$.*

*Proof.* Let $H$ be any ample $G$-line bundle on $X$. Then, perhaps after replacing $H$ by $H^{\otimes k|G|}$ for some fixed $k \gg 0$, we may assume that the linear series $\left|H^0(X,H)^G\right|$ (corresponding to the subspace of $G$-invariant sections of $H$) is base point free. In fact, by [**15**], for any $i = 1, \dots, c$ and $k \gg 0$ the base locus of $\left|H^0_i(X, H^{\otimes k})\right|$ is contained in the locus $\{x \in X : (\alpha_x^k, \chi_i)_{G_x} = 0\}$ (notation here is as in the proof of Theorem 4, with $L = H$). If $|G|$ divides $k$ then $(\alpha_x^k, \chi_i) = \sum_{g \in G_x} \overline{\chi}_i(g)$; if $V_i$ is the trivial representation this is $|G_x| \neq 0$.

Let then $V_1, \dots, V_{n-1} \in \left|H^0(X, H)^G\right|$ be general divisors; their complete intersection is a smooth $G$-invariant curve $C \subseteq X$. As $A$ is ample, for $m \gg 0$ the restriction

$$H^0(X, A^{\otimes m} \otimes B) \longrightarrow H^0(C, A^{\otimes m} \otimes B \otimes \mathcal{O}_C)$$

is a surjective $G$-equivariant linear map. Hence for every $i = 1, \dots, c$, we have surjective maps

$$H_i^0(X, A^{\otimes m} \otimes B) \longrightarrow H_i^0(C, A^{\otimes m} \otimes B \otimes \mathcal{O}_C).$$

On the other hand, $A^{\otimes m} \otimes B \otimes \mathcal{O}_C = W_m^{\otimes d_m}$, where $W_m$ is a line bundle of degree one on $C$ (hence ample) and $d_m = m(A \cdot C) + (B \cdot C)$. In other words, for $m \gg 0$ and every $i$ there are surjections

$$H_i^0(X, A^{\otimes m} \otimes B) \longrightarrow H_i^0(C, W_m^{\otimes d_m}).$$

Since $\mathrm{Pic}^1(C)$ is compact, for any $\varepsilon > 0$ there is a uniform estimate

$$h_i^0(C, W_m^{\otimes d_m}) \geq d_m \left( \frac{\dim(V_i)^2}{|G|} - \varepsilon \right)$$

for $m \geq m_\varepsilon$. This completes the proof. □

**Lemma 3.** *Let $X$ be a smooth complex projective $n$-fold, $G$ a finite group acting holomorphically on $X$ and $L$ a big $G$-line bundle on $X$. Then for any $G$-line bundle $H$ on $X$ and any $\varepsilon > 0$ there exists $m_0 \in \mathbf{N}$ such that for every integer $m \geq m_0$ and every $i = 1, \dots, c$ we have*

$$\upsilon_i\Big(L^{\otimes m} \otimes H^{-1}\Big) \geq m^n \big(\upsilon_i(L) - \varepsilon\big).$$

*Proof.* This extends to our setting Lemma 3.5 in [**9**], and the proof only requires some slight modifications to the argument given there. Fix

$i \in \{1, \ldots, c\}$ and $\varepsilon > 0$. By definition, there is a sequence $k_\nu \uparrow +\infty$ such that

$$h_i^0(X, L^{\otimes k_\nu}) \geq \frac{k_\nu^n}{n!}\left(\upsilon_i(L) - \frac{\varepsilon}{2}\right).$$

Fix $m \gg 0$ and set $\ell_\nu = [k_\nu/m]$, $r_\nu = k_\nu - \ell_\nu m$ so that, in additive notation,

$$\ell_\nu(mL - H) = k_\nu L - (r_\nu L + \ell_\nu H).$$

After replacing $H$ by $H \otimes E$ for a suitably positive $G$-line bundle $E$ on $X$, we may suppose that $H$ is a very ample $G$-line bundle. Furthermore, as in the proof of Lemma 2, perhaps after replacing $H$ by a suitably large tensor power of $H^{\otimes |G|}$, we may also assume that $\left|H^0(X, H)^G\right|$ is base point free. Choose a smooth divisor $D \in \left|H^0(X, H)^G\right|$. Since $D$ is a $G$-invariant submanifold of $X$, perhaps after a change of linearization we may also suppose that the bundle action of $G$ on $H$ is the natural action on $\mathcal{O}_X(D)$ induced by the action on $k(X)$. Thus for every $G$-line bundle $A$ on $X$ we have a $G$-equivariant isomorphism

$$H^0(X, A(-D)) = H^0(X, A \otimes \mathcal{I}_D) \cong H^0(X, A \otimes H^{-1}),$$

where $\mathcal{I}_Z \subseteq \mathcal{O}_X$ denotes the ideal sheaf of a closed subscheme $Z \subseteq X$. Furthermore, for every $i = 1, \ldots, c$ the short exact sequence of sheaves

$$0 \longrightarrow A(-D) \longrightarrow A \longrightarrow A \otimes \mathcal{O}_D \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow H_i^0(X, A(-D)) \longrightarrow H_i^0(X, A) \to H_i^0(D, A \otimes \mathcal{O}_D). \tag{4}$$

Finally, by the statement of Lemma 2 we may also assume that

$$\left|H^0(X, H^{\otimes b} \otimes L^{-r})^G\right| \neq \varnothing,$$

for all integers $0 \leq r \leq m-1$ and $b \geq 1$. If $\sigma \in H^0(X, H^{\otimes m} \otimes L^{-r_\nu})^G$ is non-zero, in additive notation tensor product by $\sigma$ determines injections

$$H_i^0\big(X, L^{\otimes k_\nu}(-(\ell_\nu + m)D)\big) \hookrightarrow H_i^0\big(X, L^{\otimes m\ell_\nu}(-\ell_\nu D)\big)$$

for every $\nu$; therefore,

$$h_i\big(X, \mathcal{O}_X(\ell_\nu(mL - h))\big) \geq h_i^0\big(X, \mathcal{O}_X(k_\nu L - (\ell_\nu + m)h)\big).$$

Now we set $A = L^{\otimes k_\nu}(-jD)$ in (4) and proceed inductively as in *loc. cit.*, Lemma 3.5. More precisely, for any $s > 0$ the exact sequences

$$0 \longrightarrow L^{\otimes k_\nu}(-(j+1)D) \longrightarrow L^{\otimes k_\nu}(-jD) \longrightarrow L^{\otimes k_\nu}(-jD) \otimes \mathcal{O}_D \longrightarrow 0$$

for $0 \leq j < s$ imply

$$\begin{aligned} h_i^0\big(X, L^{\otimes k_\nu}(-sD)\big) &\geq h_i\big(X, L^{\otimes k_\nu}\big) - \sum_{0 \leq j < s} h_i^0\big(D, L^{\otimes k_\nu}(-jD) \otimes \mathcal{O}_D\big) \\ &\geq h_i^0\big(X, L^{\otimes k_\nu}\big) - s\, h^0(D, L^{\otimes k_\nu} \otimes \mathcal{O}_D) \\ &\geq \frac{k_\nu^n}{n!}\left(\upsilon_i(L) - \frac{\varepsilon}{2}\right) - sCk_\nu^{n-1}. \end{aligned}$$

The statement follows as in [**9**] by letting $\ell_\nu \gg m \gg 1$.

In particular, if $L$ is any big $G$-line bundle on $X$, for any $\varepsilon > 0$ there is $m_0 \in \mathbf{N}$ such that

$$\upsilon_i(L) \geq m^{-n}\upsilon_i(mL) \geq \upsilon_i(L) - \varepsilon \tag{5}$$

for every integer $m \geq m_0$. Unless $V_i$ is the trivial representation, the spaces $H_i^0(X, L^{\otimes k})$ do not form a graded linear series. Therefore homogeneity of $\upsilon_i$ does not follow directly from Lemma 3.4 of [**10**].

**Lemma 4.** *Let $L$ be an nef and big $G$-line bundle on $X$. Then*

$$\upsilon_i(L) = \frac{\dim(V_i)^2}{|G|}\,(L^n).$$

*Proof.* By the definition of $\upsilon_i$, $\sum_i \upsilon_i(L) \geq \upsilon(L)$. Fix rational numbers $\varepsilon, \delta > 0$ and let $A$ be an ample $G$-line bundle on $X$. Let $r \gg 0$ be an integer such that $r\delta \in \mathbf{N}$. By choosing $r$ sufficiently large and divisible, we may assume that there exists a $G$-invariant non-zero section $\sigma \in H^0\big(X, \mathcal{O}_X(r\delta A)\big)^G$. Tensor product by $\sigma$ determines for every $i$ injective maps

$$H_i^0(X, \mathcal{O}_X(rL)) \hookrightarrow H_i^0(X, \mathcal{O}_X(r(L + \delta A)).$$

Since $L + \delta A$ is ample, for $r \gg 0$ we have

$$\frac{\dim(V_i)^2}{|G|}\, r^n\big((L+\delta A)^n\big) = \upsilon_i\big(r(L+\delta A)\big) \geq \upsilon_i(rL) \geq r^n(\upsilon_i(L) - \varepsilon),$$

and taking $\varepsilon$ and $\delta$ arbitrarily small we conclude that

$$\frac{\dim(V_i)^2}{|G|}\,(L^n) \geq \upsilon_i(L)$$

for every $i$. Thus, since $\sum_i \dim(V_i)^2 = |G|$,

$$\upsilon(L) = (L^n) = \sum_i \frac{\dim(V_i)^2}{|G|}\,(L^n) \geq \sum_i \upsilon_i(L) \geq \upsilon(L).$$

This implies the statement.

**Lemma 5.** *Let $L$ be any big $G$-line bundle on $X$. Then $\upsilon_i(L) > 0$ for every $i = 1, \dots, c$.*

*Proof.* Fix $m \gg 0$ with $\dim \phi_m(X) = n$, where

$$\phi_m : X - \longrightarrow \mathbf{P}H^0(X, L^{\otimes m})^*$$

is the rational map associated to the linear series $|L^{\otimes m}|$. Let $\psi : X' \to X$ be a $G$-equivariant resolution of singularities of $|L^{\otimes m}|$ [**1**]. Then

$$|\psi^*(L^{\otimes m})| = |M| + F,$$

where $F$ is the fixed divisor of $|\psi^*(L^{\otimes m})|$ and $M$ is a base point free (hence nef) big $G$-line bundle on $X'$. For every $k$ we have $G$-equivariant injective maps

$$H^0\big(X', M^{\otimes k}\big) \longrightarrow H^0\big(X', \psi^*(L^{\otimes mk})\big) \cong H^0(X, L^{\otimes mk}).$$

Therefore, $\upsilon_i(L) \geq m^{-n}\upsilon_i(L^{\otimes m}) \geq m^{-n}\upsilon_i(M)$ for every $i$. The statement then follows from the nef and big case of Lemma 4.

**Lemma 6.** *Let $L$ be a big $G$-line bundle on $X$. Then for every $i = 1, \dots, c$ and every $m \gg 0$ there exists $D \in |H_i^0(X, L^{\otimes m})|$ which can be written as $D = A + E$, where $A \subset X$ is a $G$-invariant ample divisor and $E \subset X$ is an effective divisor such that $\mathcal{O}_X(E)$ is an effective $G$-line bundle with $E \in |H_i^0(X, \mathcal{O}_X(E))|$.*

*Proof.* Fix a $G$-invariant very ample smooth divisor $A \subseteq X$, and let $\sigma \in H^0(X, \mathcal{O}_X(A))^G$ be a section with $A = \mathrm{div}\,(\sigma)$. Consider the short exact sequence

$$0 \longrightarrow H_i^0\big(X, L^{\otimes m}(-A)\big) \xrightarrow{\otimes\sigma} H_i^0(X, L^{\otimes m}) \longrightarrow H_i^0(A, L^{\otimes m} \otimes \mathcal{O}_A).$$

By Lemma 5, $H_i^0(X, L^{\otimes m}) = O(m^n)$; since $H_i^0(A, L^{\otimes m} \otimes \mathcal{O}_A) \leq Cm^{n-1}$, we conclude that $H_i^0(X, L^{\otimes m}(-A)) \neq 0$ for $m \gg 0$.

By taking $V_i$ to be the trivial representation, we see in particular that there exists $D \in |L|$ of the form $D = A + E$, where $A$ and $E$ are $G$-invariant $\mathbf{Q}$-divisors, with $A$ ample and $E$ effective. If $m \in \mathbf{N}$ is such that $mA$ and $mE$ are integral, we obtain $G$-invariant injections $H^0(X, \mathcal{O}_X(mkA)) \to H^0(X, L^{\otimes km})$ for every $k$, whence

$$\begin{aligned} \upsilon_i(L) &\geq m^{-n}\upsilon_i(L^{\otimes m}) \geq m^{-n}\upsilon_i\big(\mathcal{O}_X(mA)\big) \\ &= m^{-n}\frac{\dim(V_i)^2}{|G|}\big((mA)^n\big) = \frac{\dim(V_i)^2}{|G|}\,(A^n). \end{aligned}$$

Similarly, of course, $\upsilon(L) \geq (A^n)$.

Theorem 1 is now a consequence of the following equivariant version of a theorem of Fujita [**9, 11**].

**Theorem 5.** *Let $X$ be a smooth projective $n$-fold, $G$ a finite group acting holomorphically and faithfully on $X$. Let $L$ be a big $G$-line bundle on $X$. Fix $\varepsilon > 0$. Then there exists a $G$-equivariant birational modification, depending on $\varepsilon$,*

$$\mu : X' \longrightarrow X$$

*and a decomposition $\mu^*(L) \equiv E + A$, where $E$ and $A$ are $G$-invariant $\mathbf{Q}$-divisor on $X'$, with $E$ effective and $A$ ample, such that $(A^n) \geq \upsilon(X, L) - \varepsilon$ and $\dim(V_i)^2/|G|(A^n) \geq \upsilon_i(X, L) - \varepsilon$.*

*Proof.* By Fujita's theorem, for every $\varepsilon > 0$ there exist a birational modification $\mu : X' \to X$ and a decomposition $\mu^*(L) \equiv A + E$, with $\mathbf{Q}$-divisors $A$ and $E$, ample and effective respectively, such that $(A^n) \geq \upsilon(X, L) - \varepsilon$. Thus, in order to ensure the second inequality for every $i$ we need only give an equivariant version of the proof in [**9**], Theorem 3.2.

To this end, it suffices to produce for every $\varepsilon > 0$ a birational modification $\mu : X' \to X$ and a decomposition $\mu^*(L) \equiv A + E$, where $A$ and $E$ are $G$-invariant $\mathbf{Q}$-divisors on $X'$, $A$ is big and nef and $E$ is effective, satisfying the stated numerical conditions. In fact, by Lemma 6, $A \equiv A' + D$, where $A'$ and $D$ are ample and effective $G$-invariant divisors, respectively. Therefore, for any $\delta \in \mathbf{Q}_+$ we have $A + E \equiv A'' + F$, where $A'' = (1-\delta)A + \delta A'$ is ample, $F = E + \delta D$ is effective and $((A'')^n)$ approximates $(A^n)$ as closely as desired.

Let $B$ be a $G$-line bundle on $X$, so positive that $R =: K_X \otimes B^{\otimes(n+1)}$ is very ample and $H(X, R)^G \neq \{0\}$, Lemma 2. Fix a nonzero section $\sigma \in H(X, R)^G$ and set $M_m = L^{\otimes m} \otimes R^{-1}$. Then for $m$ sufficiently large $M_m$ is a big $G$-line bundle and $\upsilon_i(M_m) \geq m^n(\upsilon_i(L) - \varepsilon)$, Lemma 3. Tensoring with $\sigma^{\otimes \ell}$ determines for every $\ell \geq 1$ and $i = 1, \ldots, c$, injective linear maps

$$H^0_i\left(X, \mathcal{O}_X(M_m^{\otimes \ell})\right) \longrightarrow H^0_i\big(X, \mathcal{O}_X(L^{\otimes \ell m})\big),$$

whence $\upsilon_i(L^{\otimes m}) \geq \upsilon(M_m)$. Summing up,

$$m^n \upsilon_i(L) \geq \upsilon_i(L^{\otimes m}) \geq \upsilon_i(M_m) \geq m^n\big(\upsilon_i(L) - \varepsilon\big).$$

Let us now consider the asymptotic multiplier ideal [**9**]

$$\mathcal{J} = \mathcal{J}(X, ||M_m||).$$

Then $\mathcal{J} = \mathcal{J}(|kM_m|/k) = \mathcal{J}(\mathfrak{b}_{kM_m}/k)$ for $k \gg 0$, where $\mathfrak{b}_{kM_m} \subset \mathcal{O}_X$ is the base ideal of the linear series $\left|M_m^{\otimes k}\right|$. As $M_m$ is a $G$-line bundle, $\mathcal{J}$ is a $G$-invariant ideal sheaf. Let $\mu : X' \to X$ be a $G$-equivariant log-resolution of $\mathcal{J}$ [**1**], so that $\mu^*\mathcal{J} = \mathcal{O}_{X'}(-E_m)$ for some $G$-invariant effective divisor $E_m$ on $X'$.

Since

$$L^{\otimes m} \otimes \mathcal{J}(||M_m||) = M_m \otimes K_X \otimes B^{\otimes(n+1)} \otimes \mathcal{J}(||M_m||),$$

is globally generated by Theorem 1.8 of [**9**], so is the $G$-line bundle $A_m =: \mu^*(L^{\otimes m})(-E_m)$ on $X'$. Since all the sheaves involved are $G$-sheaves and $\sigma$ and $E_m$ are $G$-invariant, using Theorem 1.8 (iii) of *loc. cit.*, subadditivity and tensor product by $\sigma^{\otimes \ell}$ we have a chain of $G$-equivariant inclusions

$$\begin{aligned} H^0\big(X, M_m^{\otimes \ell}\big) &\cong H^0\big(X, M_m^{\otimes \ell} \otimes \mathcal{J}(||M^{\otimes \ell}||)\big) \subseteq H^0(X, M_m^{\otimes \ell} \otimes \mathcal{J}^\ell) \\ &\subseteq H^0(X, L^{\otimes m\ell} \otimes \mathcal{J}^\ell) \subseteq H^0(X', L^{\otimes m\ell}(-\ell E_m)) \\ &= H^0(X', A_m^{\otimes \ell}). \end{aligned}$$

Thus $A_m$ is an nef and big $G$-line bundle on $X'$; we have

$$(A_m^n) = \upsilon(X', A_m) \geq \upsilon(X, M_m) \geq m^n\big(\upsilon(L) - \varepsilon\big)$$

and

$$\frac{\dim(V_i)^2}{|G|}\,(A_m^n) = \upsilon_i(X', A_m) \geq \upsilon_i(X, M_m) \geq m^n\big(\upsilon_i(L) - \varepsilon\big)$$

for every $i = 1, \ldots, c$. Now we are done. We need only choose some $D_m \in \big|H(X', A_m)^G\big|$ and set $A = D_m/m$, $E = E_m/m$.

**3. Proof of Theorem 3.** By Theorem 8.1 on page 213 of [**17**], $V$ is a union of submanifolds. If $H \subset X$ is a subgroup and $V_H \subset X$ is the submanifold of the points fixed by $H$, around any $p \in V_H$ there are local coordinates in terms of which every $g \in H$ is a linear transformation, and therefore $V_H$ is a linear subspace. Therefore, $H$ acts freely on the unit sphere bundle of the normal bundle of $V_H$, and this implies that there is $a > 0$ such that $\operatorname{dist}(gx, x) \geq a \operatorname{dist}(x, V_H)$, for any $x$ sufficiently close to $V_H$ and every $g \in H\backslash\{e\}$. It follows, in the notation of the theorem, that there exists $a > 0$ such that if $x \notin V_\varepsilon$ for sufficiently small $\varepsilon > 0$, then $\operatorname{dist}(gx, x) \geq a\varepsilon$ for $g \neq e$ ($e$ is the neutral element of $G$). The constants involved in the coming estimates will be allowed to vary from line to line without mention.

In view of the above and the off-diagonal estimate on the Szegő kernel discussed in Section 6 of [**8**], if $x \notin V_\varepsilon$ and $g \neq e$, then

$$|\Pi_k(gx, x)| \leq Ck^n e^{-a\sqrt{k}\,\varepsilon}.$$

Arguing as in the proof of Theorem 1, we have the following estimates:

$$\begin{aligned}\left|h_i^0(X, L^{\otimes k}) - \frac{\dim(V_i)^2}{|G|}\, h^0(X, L^{\otimes k})\right| \\ &= \left|\int_X \left(\nu_{k,i}(x) - \frac{\dim(V_i)^2}{|G|}\,\nu_k(x)\right) dx\right| \\ &\le \int_{V_\varepsilon} \left|\nu_{k,i}(x) - \frac{\dim(V_i)^2}{|G|}\,\nu_k(x)\right| dx \\ &\quad + \int_{X\setminus V_\varepsilon} \left|\nu_{k,i}(x) - \frac{\dim(V_i)^2}{|G|}\,\nu_k(x)\right| dx \\ &\le Ck^n(\varepsilon^{2c} + e^{-a\sqrt{k}\,\varepsilon}).\end{aligned}$$

If now $\alpha \in (s/c, 1)$ and $\varepsilon = k^{-\alpha/2}$, we have $k^n(\varepsilon^{2c} + e^{-a\sqrt{k}\,\varepsilon}) = o(k^{n-s})$ as $k \to \infty$.

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