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WEIGHTED COMPOSITION OPERATORS ON NON-LOCALLY CONVEX WEIGHTED SPACES

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ABSTRACT. Let (A, τ) be a topological vector space, X and Y Hausdorff completely regular spaces and V and U Nachbin families on X and Y respectively. For a pair of maps $\varphi : Y \to X$ and $\psi : Y \to \mathcal{L}(A)$, $\mathcal{L}(A)$ being the vector space of continuous operators from A into itself, we study the conditions under which the corresponding weighted composition operator ψC_φ, assigning to each $f \in CV(X, A)$ the function $y \mapsto \psi_y(f \circ \varphi(y))$, maps a subspace E of $CV(X, A)$ continuously into another given subspace F of $CU(Y, A)$. We also examine when ψC_φ is bounded, (locally) equicontinuous or (locally) precompact from E into F.

1. Introduction. The weighted composition operators $uC_\varphi : f \mapsto uf \circ \varphi$ on the Banach algebra $C(K)$ of scalar-valued continuous functions on a compact space K were studied by Kamowitz in [**8**]; where $u \in C(K)$ and $\varphi : K \to K$ is a continuous self map on K. Since then, numerous papers were published in connection with the subject in the scalar case and in the vector-valued one [**6, 7, 10, 12, 15, 21, 22**], etc. In the scalar case, Singh and Summers [**21**] studied the composition operators C_φ on the Nachbin weighted spaces $CV(X)$ and $CV_0(X)$, X being a Hausdorff completely regular space and V a Nachbin family on X. The so-called extended composition operators between weighted spaces were the subject of [**14**].

Jeang and Wong [**7**] dealt with the weighted composition operators $uC_\varphi : f \mapsto uf \circ \varphi$ from $C_0(X)$ into $C_0(Y)$, where X and Y are Hausdorff locally compact spaces, $u \in C(Y)$ and φ a map from Y into X. For special function spaces, namely the Banach spaces of analytic functions on the unit disk, the multiplication operators were the subject of [**4**].

In the vector-valued setting, Jamison and Rajagopalan [**6**] considered the weighted composition operators ψC_φ on the Banach space $C(K, A)$,

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where K is a compact space, A a Banach space, φ a self map on X and ψ an $\mathcal{L}(A)$-valued function on K.

Such operator-valued weighted composition operators from the space $C_p(X, A)$ of all A-valued continuous functions f such that $f(X)$ is precompact were studied in [**22**] for an arbitrary completely regular space X and any locally convex space A.

For weighted spaces $CV(X, A)$ with A non-locally convex, the weighted composition operators were studied mainly in [**10, 12, 13**] and [**19**].

In this paper, we deal with weighted composition operators ψC_φ from a given subspace E of $CV(X, A)$ into another subspace F of some $CU(Y, A)$, A being an arbitrary Hausdorff topological vector space, Y a Hausdorff completely regular space and U a Nachbin family on Y. In Section 2 we produce some preliminaries and notations, while in Section 3, we characterize those weighted composition operators which map E continuously into $CU(Y, A)$, into $CU_0(Y, A)$ or into an arbitrary $F \subset CU(Y, A)$. Section 4 is devoted to the conditions under which ψC_φ is bounded, (locally) equicontinuous or (locally) precompact.

Note that, in most of the works on weighted spaces, essentiality as defined by Prolla in [**17**] plays an important role. Here, we release this condition and then cover many more situations.

2. Preliminaries.

Throughout this paper, A will be a Hausdorff topological vector space over the field $\mathbf{K}$ ($= \mathbf{R}$ or $\mathbf{C}$) and $\mathcal{N}$ (or $\mathcal{N}_A$ if any confusion might occur) the collection of all closed, shrinkable and circled 0-neighborhoods in A. This constitutes a fundamental system of 0-neighborhoods [**11**]. Recall that a subset G of A is shrinkable if $r\,\mathrm{cl}\,(G) \subset \mathrm{int}\,(G)$ for every $0 \le r < 1$, where $\mathrm{cl}\,(G)$ denotes the closure of G in A and $\mathrm{int}\,(G)$ its interior. For every $G \in \mathcal{N}$, P_G will be the gauge of G. This is

$$P_G(a) = \inf\{\alpha > 0 : a \in \alpha G\}, \quad a \in A.$$

It is clear that $P_G(\lambda a) = |\lambda| P_G(a)$ for every $a \in A$ and $\lambda \in \mathbf{K}$. Moreover, if $H \in \mathcal{N}$ enjoys $H + H \subset G$, then

$$P_G(a + b) \le P_H(a) + P_H(b), \quad a, b \in A.$$

A linear map $T : A \to A$ is continuous if, and only if, for every $G \in \mathcal{N}$, there is $H \in \mathcal{N}$ such that

$$P_G(T(a)) \leq P_H(a), \quad \text{for all } a \in A.$$

The algebra of all continuous operators from a topological vector space C into A is denoted by $\mathcal{L}(C, A)$. If $C = A$, we just write $\mathcal{L}(A)$. If $\mathcal{B}$ is a collection of subsets of C, we will denote by $\mathcal{L}_{\mathcal{B}}(C, A)$ the set of those $T \in \mathcal{L}(C, A)$ which are bounded on the members of $\mathcal{B}$. $\mathcal{L}_{\mathcal{B}}(C, A)$ will be equipped with the topology $\tau_{\mathcal{B}}$ of uniform convergence on the members of $\mathcal{B}$. A fundamental system of 0-neighborhoods for $\tau_{\mathcal{B}}$ is given by all the intersections of finitely many sets of the form

$$N(B, G) := \{T \in \mathcal{L}_{\mathcal{B}}(C, A) : T(B) \subset G\}, \quad G \in \mathcal{N}, \quad B \in \mathcal{B}.$$

If $\mathcal{B}$ consists of the finite (respectively bounded, precompact) sets, we will denote $\mathcal{L}_{\mathcal{B}}(C, A)$ by $\mathcal{L}_{\sigma}(C, A)$ (respectively $\mathcal{L}_{\beta}(C, A)$, $\mathcal{L}_c(C, A)$) and $\tau_{\mathcal{B}}$ by τ_{σ} (respectively τ_{β}, τ_c). When $C = A$, we drop it from the notations and write $\mathcal{L}_{\mathcal{B}}(A)$.

A Nachbin family on a Hausdorff completely regular space X is any collection V of non-negative upper semi-continuous functions on X such that, for every $x \in X$, some $v \in V$ exists so that $v(x) > 0$ and, for every $v_1, v_2 \in V$ and $\lambda > 0$, there is some $v \in V$ such that $\lambda v_i \leq v$, $i = 1, 2$. With such a family V is associated the so-called weighted space

$$CV(X, A) := \{f : X \longrightarrow A \text{ continuous}; (vf)(X) \text{ is bounded in } A, \text{ for all } v \in V\}.$$

This space is linearly topologized, see [**1**] and [**9**], by considering as a fundamental system of neighborhoods of zero all the sets of the form

$$B_{G,v} := \{f \in CV(X, A); (vf)(X) \subset G\},$$

G running over $\mathcal{N}$ and v over V. The gauge of such a set is denoted by $P_{G,v}$. This is

$$P_{G,v}(f) := \sup\{v(x)P_G(f(x)), x \in X\}, \quad f \in CV(X, A).$$

A remarkable subspace of $CV(X, A)$ is

$$CV_0(X, A) := \{f \in CV(X, A); v(P_G \circ f) \text{ vanishes at infinity}, \text{ for all } v \in V,\ G \in \mathcal{N}\}.$$

If we set for a non-negative function u on X and $\varepsilon > 0$,

$$N(u,\varepsilon) := \{x \in X : u(x) \geq \varepsilon\} \quad \text{and } N_u := \{x \in X : u(x) > 0\},$$

then a continuous function f belongs to $CV_0(X,A)$ if, and only if, for every $v \in V$, $G \in \mathcal{N}$ and $\varepsilon > 0$, the set $N(vP_G \circ f, \varepsilon)$ is compact. Finally, following [**1**], X is called a $V_{\mathbf{R}}$-space if a real-valued function on X is continuous whenever its restriction to each $N(v,\varepsilon)$ is, $v \in V$ and $\varepsilon > 0$, see [**1**–**3**] for more details.

Henceforth, X and Y will be Hausdorff completely regular spaces and V and U Nachbin families on X and Y respectively. A linear map T from $CV(X,A)$ into $CU(Y,A)$ is continuous if, and only if, for all $u \in U$, $G \in \mathcal{N}$, there exists $v \in V$, $H \in \mathcal{N}$:

$$P_{G,u}(T(f)) \leq P_{H,v}(f), \quad f \in CV(X,A).$$

The set of all A-valued functions on Y will be denoted by $\mathcal{F}(Y,A)$ while $C(Y,A)$ will be that of all the continuous ones. With arbitrary maps $\psi : Y \to \mathcal{L}(A)$ and $\varphi : Y \to X$ is associated the linear map ψC_φ defined from $CV(X,A)$ into $\mathcal{F}(Y,A)$ by $\psi C_\varphi(f)(y) = \psi_y(f(\varphi(y)))$. This map will be called the weighted composition operator associated with ψ and φ. From now on, E will be a linear subspace of $CV(X,A)$ and $\operatorname{coz}(E)$ its cozero set. This is:

$$\operatorname{coz}(E) := \{x \in X; f(x) \neq 0 \text{ for some } f \in E\}.$$

We will also consider the sets:

$$\begin{aligned} Y_{E,\varphi} &:= \{y \in Y : \varphi(y) \in \operatorname{coz}(E)\} = \varphi^{-1}(\operatorname{coz}(E)), \\ Y_{E,\varphi,\psi} &:= \operatorname{coz}(\psi C_\varphi(E)). \end{aligned}$$

The set $Y_{E,\varphi}$, respectively $Y_{E,\varphi,\psi}$, is an open subset of Y whenever $C_\varphi(E) \subset C(Y,A)$, respectively $\psi C_\varphi(E) \subset C(Y,A)$, where C_φ is the composition operator $f \mapsto f \circ \varphi$. Finally, E will be said to satisfy the property (M) if, for every $a \in A$, $G \in \mathcal{N}$ and $f \in E$, the function $P_G \circ f \otimes a : x \mapsto P_G(f(x))a$ belongs to E. It is easily seen that, whenever E satisfies (M), the following equality holds:

$$Y_{E,\varphi,\psi} = Y_{E,\varphi} \cap \operatorname{coz}(\psi).$$

The spaces $CV(X, A)$ itself and $CV_0(X, A)$ as well as many other subspaces of $CV(X, A)$ satisfy (M).

3. Continuous weighted composition operators. In this section we study the continuity of ψC_φ from E into a subspace F of $CU(Y, A)$. Since $\psi C_\varphi(f)$ must then be continuous on Y for every $f \in E$, we first provide instances in which this is realized. For this purpose, let γ be a property a net from Y may or may not satisfy. Any net satisfying γ will be called a γ-net. A function from Y into an arbitrary topological space is γ-continuous if, for every $y \in Y$ and every γ-net $(y_i)_i$ converging to y, the net $(f(y_i))_i$ converges to $f(y)$. We will say that Y is a $\gamma_{\mathbf{R}}$-space, if every γ-continuous function from Y into $\mathbf{R}$ or into any completely regular space is continuous. At this point, it is worthwhile to recall that the classical $k_{\mathbf{R}}$-spaces as well as the sequential, the pseudo-compact and the $V_{\mathbf{R}}$-spaces enter in this category, see [**15**] for more details. For such a property γ, denote by $\mathcal{L}_\gamma(A)$ the algebra consisting of all continuous operators on A which are bounded on the converging γ-nets of A, endowed with the topology of uniform convergence on such nets. In the sequel, we will assume, in addition, that every constant net is a γ-net and that γ is defined also in A and is conserved by A-valued continuous functions.

Proposition 1. *Let $f : X \to A$ be a map such that $C_\varphi(f) \in C(Y, A)$. Under each of the following conditions $\psi C_\varphi(f)$ belongs to $C(Y, A)$:*

1. *Y is a $\gamma_{\mathbf{R}}$-space for some γ and ψ maps Y continuously into $\mathcal{L}_\gamma(A)$.*

2. *ψ maps Y continuously into $\mathcal{L}_\sigma(A)$ and every $y \in Y$ possesses some neighborhood whose image by ψ is equicontinuous on A.*

Proof. Let $y_0 \in Y$ and $G \in \mathcal{N}$ be given. Choose $H \in \mathcal{N}$ so that $H + H \subset G$.

1. Since ψ_{y_0} is continuous, there exists $K \in \mathcal{N}$ with $K \subset H$ and $\psi_{y_0}(K) \subset H$. In order to show that $\psi C_\varphi(f)$ is continuous, it suffices to show that it is γ-continuous. Let then $(y_i)_{i\in I}$ be a γ-net converging to y_0. Since $f \circ \varphi$ is continuous, the net $(f \circ \varphi(y_i))_i$ is a γ-net and the set

$$N(C, K) := \{T \in \mathcal{L}_\gamma(A) : T(C) \subset K\}$$

is a 0-neighborhood in $\mathcal{L}_\gamma(A)$, where $C := \{f \circ \varphi(y_i), i \in I\}$. Then there is a neighborhood Ω of y_0 such that, for every $y \in \Omega$, $(\psi_y - \psi_{y_0}) \in N(C, K)$. Therefore

$$(\psi_y - \psi_{y_0})(f \circ \varphi(y_i)) \in K, \quad i \in I, \quad y \in \Omega.$$

But y_i tends to y_0. Then there exists $i_1 \in I$ so that $y_i \in \Omega$ whenever $i \geq i_1$. As $f \circ \varphi$ is continuous, there is some $i_2 \in I$ with

$$f \circ \varphi(y_i) - f \circ \varphi(y_0) \in K, \quad \text{for all } i \geq i_2.$$

Now, for $i \in I$ larger than both i_1 and i_2, we have

$$\begin{aligned}\psi_{y_i}(f \circ \varphi(y_i)) - \psi_{y_0}(f \circ \varphi(y_0)) &= (\psi_{y_i} - \psi_{y_0})(f \circ \varphi(y_i)) \\ &\quad + \psi_{y_0}\left(f \circ \varphi(y_i) - f \circ \varphi(y_0)\right) \\ &\in K + K \subset G.\end{aligned}$$

whence the γ-continuity of $\psi C_\varphi(f)$ at y_0 and then everywhere.

2. Let Ω_1 be a neighborhood of y_0 so that $\psi(\Omega_1)$ is equicontinuous on A. Then there exists $K \in \mathcal{N}$ contained in H and satisfying $\psi_y(K) \subset H$ for all $y \in \Omega_1$. Since $f \circ \varphi$ and ψ are continuous at y_0, there exists another neighborhood Ω_2 of y_0 such that:

$$(f \circ \varphi(y) - f \circ \varphi(y_0)) \in K \quad \text{and} \quad (\psi_y - \psi_{y_0})(f \circ \varphi(y_0)) \in K, \quad y \in \Omega_2.$$

For every $y \in \Omega := \Omega_1 \cap \Omega_2$, one has

$$\begin{aligned}\psi_y\left(f \circ \varphi(y)\right) - \psi_{y_0}\left(f \circ \varphi(y_0)\right) &= \psi_y\left(f \circ \varphi(y) - f \circ \varphi(y_0)\right) \\ &\quad + \left(\psi_y - \psi_{y_0}\right)\left(f \circ \varphi(y_0)\right) \\ &\in \psi_y(K) + K \\ &\subset H + H \subset G.\end{aligned}$$

Whence the continuity of $\psi C_\varphi(f)$. □

According to Proposition 1, $\psi C_\varphi(f)$ belongs to $C(Y, A)$ whenever $C_\varphi(f)$ and $\psi : Y \to \mathcal{L}_\beta(A)$ are continuous and Y is a $k_{\mathbf{R}}-$, a $b_{\mathbf{R}}-$, a pseudo-compact, or a sequential space.

The fact that $C_\varphi(E) \subset C(Y,A)$, respectively $\psi C_\varphi(E) \subset C(Y,A)$, forces φ, respectively ψ, to be continuous on $Y_{E,\varphi}$, as we will see next.

Proposition 2. 1. *If E is a $C_b(X)$-module and $C_\varphi(E) \subset C(Y,A)$, then φ is continuous on $Y_{E,\varphi}$.*

2. *If E satisfies (M) and $C_\varphi(E) \cup (\psi C_\varphi)(E) \subset C(Y,A)$, then ψ is σ-continuous on $Y_{E,\varphi}$.*

Proof. Let $y_0 \in Y_{E,\varphi}$. Then there are $f_0 \in E$ and $K \in \mathcal{N}$ such that $P_K(f_0(\varphi(y_0))) = 1$.

1. Let Ω be a neighborhood of $\varphi(y_0)$ and choose $g \in C_b(X)$ so that $g(\varphi(y_0)) = 1$, $0 \le g \le 1$ and $\operatorname{supp} g \subset \Omega$. Since E is a $C_b(X)$-module, gf_0 belongs to E. Then $[g(P_K \circ f_0)] \circ \varphi$ is continuous on Y. Hence

$$\Lambda := \left\{ y \in Y : \frac{1}{2} < g(\varphi(y)) P_K(f_0(\varphi(y))) < \frac{3}{2} \right\}$$

is open and contains y_0. Since $\operatorname{supp} g \subset \Omega$, $\varphi(y) \in \Omega$ for all $y \in \Lambda$. Hence $\varphi(\Lambda) \subset \Omega$, whereby φ is continuous at y_0 and then on the whole $Y_{E,\varphi}$.

2. Let $G \in \mathcal{N}$ and $a \in A$. We have to find a neighborhood Ω of y_0 so that $\psi_y(a) - \psi_{y_0}(a) \in G$ for all $y \in \Omega$. Consider $H \in \mathcal{N}$ with $H + H + H \subset G$. The continuity of $y \mapsto P_K(f_0(\varphi(y)))$ yields a neighborhood Ω_1 of y_0 such that

$$\left| \frac{1}{P_K(f_0(\varphi(y)))} - 1 \right| < \frac{1}{2}, \quad y \in \Omega_1.$$

By our assumption, $P_K \circ f_0 \otimes a$ belongs to E and $y \mapsto \psi_y(P_K(f_0(\varphi(y)))a)$ is continuous at y_0. Hence some neighborhood $\Omega \subset \Omega_1$ exists so that

$$\psi_y(P_K(f_0(\varphi(y)))a) - \psi_{y_0}(P_K(f_0(\varphi(y_0)))a) \in \frac{1}{P_H(\psi_{y_0}(a)) + 1} H, \quad \forall\, y \in \Omega.$$

Hence

$$\begin{aligned}\psi_y(a) - \psi_{y_0}(a) &= \frac{1}{P_K(f_0(\varphi(y)))} \\ &\quad \times [\psi_y(P_K(f_0(\varphi(y))a)) - \psi_{y_0}(P_K(f_0(\varphi(y_0))a))] \\ &\quad + \left[\frac{1}{P_K(f(\varphi(y)))} - 1\right]\psi_{y_0}(a) \\ &\in 2H + H \subset H + H + H \subset G\end{aligned}$$

Hence ψ is continuous at y_0 and then on $Y_{E,\varphi}$. □

The continuity of ψC_φ from E into $CU(Y, A)$ does not imply that of φ on $Y_{E,\varphi}$ in general. Such a situation occurs for example if $\psi : Y \to \mathcal{L}(A)$ is a constant function with a non one-to-one value T and $E := \{f \in CV(X, A) : f(X) \subset \ker T\}$. Then E is a $C_b(X)$-module and $\operatorname{coz}(E) = \operatorname{coz}(CV(X))$. Moreover, $\psi C_\varphi = 0$ is continuous. But φ need not be, since it is arbitrary. Notice that E does not enjoy the property (M).

Now, we are going to characterize the continuous operators ψC_φ from a subspace E of $CV(X, A)$ into $CU(Y, A)$.

Theorem 3. *Assume that $E \subset CV(X, A)$ is a $C_b(X)$-module satisfying (M) and that $\psi C_\varphi(E) \subset C(Y, A)$. Then ψC_φ is continuous from E into $CU(Y, A)$ if, and only if, the following condition holds. For all $G \in \mathcal{N}$, $u \in U$, there exists $H \in \mathcal{N}$, $v \in V$:*

$$(1) \qquad u(y)P_G(\psi_y(a)) \le v(\varphi(y))P_H(a), \quad \textit{for all } a \in A, \quad y \in Y_{E,\varphi}.$$

Proof. Necessity. Since $\psi C_\varphi : E \to CU(Y, A)$ is continuous, for every $G \in \mathcal{N}$ and $u \in U$, there exist $H \in \mathcal{N}$ and $v \in V$ such that

$$P_{G,u}(\psi C_\varphi(f)) \le P_{H,v}(f), \quad f \in E.$$

Then for every $y \in Y$, one has

$$u(y)P_G(\psi_y(f(\varphi(y))) \le \sup\{v(x)P_H(f(x)),\ x \in X\}.$$

Let y_0 be given in $Y_{E,\varphi}$. There is some $f \in E$ such that $f(\varphi(y_0)) \neq 0$. Then we may (and do) take f and H so that $P_H(f(\varphi(y_0))) = 1$. Consider then the open neighborhood

$$U_n := \left\{ x \in X : v(x) < v(\varphi(y_0)) + \frac{1}{n} \text{ and } P_H(f(x)) < 1 + \frac{1}{n} \right\}$$

of $\varphi(y_0)$ and take a continuous functions $g_n \in C_b(X)$ such that $g_n(\varphi(y_0)) = 1$, $0 \leq g_n \leq 1$ and $\operatorname{supp} g_n \subset U_n$. Since E is a $C_b(X)$-module and satisfies (M), the function $g_n P_H \circ f \otimes a$ belongs to E for arbitrary $a \in A$. Hence we have

$$u(y_0) P_G(\psi_{y_0}(a)) \leq \left(v(\varphi(y_0)) + \frac{1}{n} \right) \left(1 + \frac{1}{n} \right) P_H(a).$$

As n tends to infinity, we get

$$u(y_0) P_G(\psi_{y_0}(a)) \leq v(\varphi(y_0)) P_H(a),$$

which is the required inequality.

Sufficiency. Let $f \in E$, $G \in \mathcal{N}$ and $u \in U$ be given. By (1), there exist $v \in V$ and $H \in \mathcal{N}$ so that

$$(2) \qquad u(y) P_G(\psi_y(f(\varphi(y)))) \leq v(\varphi(y)) P_H(f(\varphi(y))), \quad \text{for all } y \in Y.$$

Therefore,

$$\begin{aligned} P_{G,u}(\psi C_\varphi(f)) &:= \sup\{u(y) P_G(\psi_y(f(\varphi(y)))),\ y \in Y\} \\ &\leq \sup\{v(\varphi(y)) P_H(f(\varphi(y))),\ y \in Y\} \\ &\leq P_{H,v}(f) < +\infty. \end{aligned}$$

This shows that $\psi C_\varphi(f) \in CU(Y, A)$ and, since f is arbitrary in E, that ψC_φ is continuous. □

It follows from Theorem 3 that, for any $C_b(X)$-module $E \subset CV(X, A)$ satisfying (M), if $\operatorname{coz}(E) = \operatorname{coz}(CV(X, A))$ and $\psi C_\varphi(CV(X, A)) \subset C(Y, A)$, then ψC_φ maps $CV(X, A)$ continuously into $CU(Y, A)$ if, and only if, the same holds for E.

Whenever $\psi C_\varphi(f)$ is continuous for some f, (1) implies that $\psi C_\varphi(f)$ belongs to $CU(Y,A)$. However, (1) does not imply the continuity of $\psi C_\varphi(f)$ although f,ψ and φ are all continuous. Such an example is obtained by taking $X = Y = \widehat{\mathbf{N}}$ the one point compactification of $\mathbf{N}$, $V = U$ the collection of all non-negative functions vanishing on X except on a finite set, φ the identity map and $A = C[0,1]$ with the norm of $L^1[0,1]$. Let $(g_n)_n$ be a null sequence in A such that $(g_n f_n)_n$ does not converge to 0 for some other null sequence $(f_n)_n$. Define ψ by $\psi(n) = L_{g_n} : f \mapsto f g_n$ for $n \in \mathbf{N}$ and $\psi(\infty) = 0$. Then ψ is continuous from X into $\mathcal{L}_\sigma(A)$ such that (1) is fulfilled, for $||\psi_y(h)|| \le ||\psi_y||_\infty ||h||$, $h \in A$. Nevertheless, for $f \in CV(X,A)$ defined by $f(n) = f_n$ for $n \in \mathbf{N}$ and $f(\infty) = 0$, $\psi C_\varphi(f)$ is not continuous at ∞, since $f_n g_n$ does not converge to 0.

Next, we will examine when ψC_φ is a continuous weighted composition operator ranging in a smaller subspace F of $CU(Y,A)$. We first look at the case $F = CU_0(Y,A)$. To this aim, let us set

$$\mathrm{Cst}\,(E) := \{K \subset X : \forall\, a \in A, \exists f \in E \text{ with } f = a \text{ identically on } K\}.$$

It is easily seen that every $v \in V$ is bounded on every $K \in \mathrm{Cst}(E)$.

Theorem 4. *Let $E \subset CV(X,A)$ be a $C_b(X)$-module satisfying (M) such that $\psi C_\varphi(E) \subset C(Y,A)$. Assume that, for every $v \in V$, $G \in \mathcal{N}$, $f \in E$ and $\varepsilon > 0$, $N(vP_G \circ f, \varepsilon) \in \mathrm{Cst}(E)$ and $f(N(vP_G \circ f, \varepsilon))$ is precompact in A. Then ψC_φ is continuous from E into $CU_0(Y,A)$ if, and only if,* (1) *in Theorem* 3 *holds and*

$$\varphi^{-1}(K) \cap \{y \in Y : u(y) P_G(\psi_y(a)) \ge \varepsilon\}$$

is relatively compact, for all $K \in \mathrm{Cst}(E)$, $G \in \mathcal{N}$, $u \in U$, $a \neq 0$ and $\varepsilon > 0$.

Proof. Necessity. (1) follows from Theorem 3. Now, assume that $K \in \mathrm{Cst}(E)$ and let $u \in U$, $G \in \mathcal{N}$, $a \in A \setminus \{0\}$ and $\varepsilon > 0$ are given. Choose $f \in E$ such that $f = a$ identically on K. As $\psi C_\varphi(f)$ belongs to $CU_0(Y,A)$, the set

$$S := \{y \in Y : u(y) P_G\left(\psi_y(f(\varphi(y)))\right) \ge \varepsilon\}$$

is compact and contains

$$\varphi^{-1}(K) \cap \{y \in Y : u(y)P_G(\psi_y(a)) \geq \varepsilon\}.$$

Hence the latter is relatively compact.

Sufficiency. Let $f \in E$, $G \in \mathcal{N}$, $u \in U$ and $\varepsilon > 0$ be arbitrary and consider again the set S defined as above. Let $H \in \mathcal{N}$ satisfy $H + H \subset G$. By (1), there are $I \in \mathcal{N}$ and $v \in V$ with

$$u(y)P_H\left(\psi_y(a)\right) \leq v(\varphi(y))P_I(a), \quad a \in A, \quad y \in Y_{E,\varphi}.$$

But $K := N(vP_I \circ f, (\varepsilon/2))$ belongs to $\mathrm{Cst}\,(E)$ and satisfies $\varphi(S) \subset K$. In order to show that $\psi C_\varphi(f)$ belongs to $CU_0(Y, A)$, it suffices to show that S is contained in some union of finitely many sets of the form

$$C_i := \left\{y \in Y : u(y)P_H(\psi_y(a_i)) \geq \frac{\varepsilon}{2}\right\}$$

for some $a_i \in A \setminus \{0\}$. But $f(\varphi(S))$ is contained in $f(K)$ which is precompact, then it is itself precompact in A. Thus there are $y_1, \dots, y_n \in S$ such that

$$f(\varphi(S)) \subset \bigcup_{i=1}^{n}\left(f(\varphi(y_i)) + \frac{\varepsilon}{2m}I\right),$$

with $m = \sup\{v(x), x \in K\}$. Then, for $y \in S$, there is some $i \in \{1, \dots, n\}$ so that

$$v(\varphi(y))P_I(f(\varphi(y)) - f(\varphi(y_i))) \leq \frac{\varepsilon}{2}.$$

By (1), We get

$$u(y)P_H\left(\psi_y(f(\varphi(y)) - \psi_y(f(\varphi(y_i)))\right) \leq \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}\varepsilon &\leq u(y)P_G\left(\psi_y(f(\varphi(y)))\right)\\ &\leq u(y)P_H[\psi_y(f(\varphi(y)) - \psi_y(f(\varphi(y_i))] + u(y)P_H\left(\psi_y(f(\varphi(y_i)))\right)\\ &\leq \frac{\varepsilon}{2} + u(y)P_H\left(\psi_y(f(\varphi(y_i)))\right),\end{aligned}$$

whereby

$$\frac{\varepsilon}{2} \leq u(y)P_H\left(\psi_y(f(\varphi(y_i)))\right),$$

and consequently

$$S \subset \bigcup_{i=1}^{n} \left\{ y \in Y : u(y)P_H(\psi_y(a_i)) \geq \frac{\varepsilon}{2} \right\}, \quad \text{with } a_i := f(\varphi(y_i)).$$

To prove the continuity of ψC_φ, just proceed as at the end of the proof of Theorem 3.

In case $E \subset CV_0(X, A)$, we get the following

Theorem 5. *Let $E \subset CV_0(X, A)$ be a $C_b(X)$-module satisfying (M) such that $\psi C_\varphi(E) \subset C(Y, A)$. Then ψC_φ is continuous from E into $CU_0(Y, A)$ if, and only if,* (1) *holds and*

$$\varphi^{-1}(K) \cap \{y \in Y : u(y)P_G(\psi_y(a)) \geq \varepsilon\}$$

is relatively compact, for every compact $K \subset \operatorname{coz}(E)$, $G \in \mathcal{N}$, $u \in U$, $a \neq 0$ and $\varepsilon > 0$.

Proof. A similar proof as that of Theorem 4 works provided that, for every $v \in V$, $G \in \mathcal{N}$, $f \in E$ and $\varepsilon > 0$, $f(N(vP_G \circ f, \varepsilon))$ is precompact in A and $N(vP_G \circ f, \varepsilon) \in \operatorname{Cst}(E)$. But, $N(vP_G \circ f, \varepsilon)$ is compact, so its image by f is precompact. The second condition is a consequence of

Lemma 6. *Let $E \subset CV(X, A)$ be a $C_b(X)$-module satisfying (M). If $K \subset \operatorname{coz}(E)$ is a compact set and $C \subset X$ a closed set such that $K \cap C = \varnothing$, then for every $a \in A$, there exists $f \in E$ such that $f = a$ on K and $f = 0$ on C.*

Proof. For every $x \in K$, consider $G_x \in \mathcal{N}$ and $f_x \in E$ so that $P_{G_x}(f_x(x)) = 1$. Choose then $g_x \in C_b(X)$ with $g_x(x) = 1$, $0 \leq g_x \leq 1$ and $g_x = 0$ identically on C. Set $j_x := g_x P_{G_x} \circ f_x$ and $h_x := |j_x|^2 \Gamma(j_x^2)$, where, for a function g,

$$\Gamma(g)(t) := \begin{cases} |g(t)| & |g(t)| \leq 1 \\ 1/|g(t)| & \text{otherwise.} \end{cases}$$

Then $h_x(x) = 1$, $0 \le h_x \le 1$ and $h_x = 0$ on C. By a compactness argument, there exist $x_1, x_2, \dots, x_m$ in X such that

$$K \subset \bigcup_{i=1}^{m} \{t \in X : h_{x_i}(t) > 1/2\}.$$

Now, the function

$$h := \sum_{n=1}^{m} h_{x_n}$$

satisfies $h(t) > 1/2$ for every $t \in K$. Hence, for $a \in A$, the function $f := 2h\Gamma(2h) \otimes a$ belongs to E, for E is a $C_b(X)$-module and satisfies (M), and enjoys the required conditions. □

According to Theorem 5, for any $C_b(X)$-module $E \subset CV_0(X, A)$ satisfying (M), if $\operatorname{coz}(E) = \operatorname{coz}(CV_0(X, A))$ and $\psi C_\varphi(CV_0(X, A)) \subset C(Y, A)$, then ψC_φ maps continuously $CV_0(X, A)$ into $CU_0(Y, A)$ if, and only if, the same holds for E.

Remark 7. 1. A subspace E of $CV(X, A)$ satisfying the hypotheses of Theorem 4 need not be contained in $CV_0(X, A)$. For instance, take $X = \mathbf{R}$, A a topological vector space in which every bounded set is precompact (e.g., a semi-Montel space or a locally convex space with its weak topology) and $E = C_b(X, A)$. For every integer $n \ge 2$, define a continuous function w_n on X by

$$w_n(x) = \begin{cases} n(x - n + (1/n)) & n - (1/n) \le x \le n \\ n(n + (1/n) - x) & n \le x \le n + (1/n) \\ 0 & \text{otherwise.} \end{cases}$$

Put then

$$w(x) = \sum_{n=2}^{+\infty} w_n(x) \quad \text{and} \quad v(x) := \max(e^{-|x|}, w(x)).$$

Then $V = \{\lambda v, \lambda > 0\}$ is a Nachbin family on X such that $E \subset CV(X, A)$ satisfies all the hypotheses of Theorem 4. However, the non-zero constant functions belong to E but not to $CV_0(X, A)$.

2. In Theorem 5 the second condition does not hold for compact sets not contained in $\operatorname{coz}(E)$. For such an example, take $X = [0, +\infty[$, $Y =]0, +\infty[$, $\varphi(y) = (1/y)$, $y \in Y$, $U = V = Z$, $A = \mathbf{C}$, ψ the constant function with value 1 and $E = \{f \in CV_0(X) : f(0) = 0\}$, where Z is the Nachbin family on X consisting of all the positive constant functions. Then ψC_φ is even an isometry from E into $CU_0(Y)$. However, for the compact $K = [0, 1]$, $\varphi^{-1}(K) \cap \{y \in Y : \psi_y(1) \geq 1\}$, being $[1, +\infty[$, is not relatively compact.

3. In Theorem 4 and Theorem 5, the second condition implies that $\psi C_\varphi(f)$ belongs to $CU_0(Y, A)$ whenever (1) holds. In general, we will say that F is $E\varphi$-solid if a function $g \in C(Y, A)$ belongs to F whenever for all $G \in \mathcal{N}$, $u \in U$, there exists $H \in \mathcal{N}$, $v \in V$, $f \in E$:

$$(uP_G \circ g)(y) \leq (vP_H \circ f)(\varphi(y)), \quad \text{for all } y \in Y_{E,\varphi}. \tag{3}$$

If $X = Y$, $\varphi = \mathrm{Id}_X$ and $A = \mathbf{K}$, we get the classical notion of being solid.

Examples of such solid spaces are given in the following

Examples. 1. Clearly, $CU(Y, A)$ is $E\varphi$-solid for every subspace E of $CV(X, A)$.

2. If, for every $v \in V$, $u \in U$ and every compact subset K of $N_v \cap \operatorname{coz}(E)$, $\varphi^{-1}(K) \cap N_u$ is relatively compact, then $CU_0(Y, A)$ is $E\varphi$-solid for every subspace E of $CV_0(X, A)$.

3. If $C_lV(X, A) = \{f \in CV(X, A)$; for all $v \in V$, $G \in \mathcal{N}$, there exists $v' \in V : vP_G(f) \leq v'\}$ and if $V \circ \varphi \leq U$, then $C_lU(Y, A)$ is $E\varphi$-solid for every $E \subset C_lV(X, A)$. Indeed, assume that $g \in C(Y, A)$ satisfies (3). Then

$$\begin{aligned} u(y)P_G(g(y)) &\leq (vP_H \circ f)(\varphi(y)) \\ &\leq v'(\varphi(y)), \quad (\text{for } P_H \circ f \in C_lV(X)) \\ &\leq u'(y), \quad (\text{since } V \circ \varphi \leq U), \end{aligned}$$

hence $P_G \circ g \in C_lU(Y)$ and then $g \in C_lU(Y, A)$.

4. If $C_AV(X, A) = \{f \in CV(X, A)$; for all $v \in V$, $f(N_v)$ is bounded in $A\}$ and, for every $u \in U$, there is some $v \in V$ such that $(v \circ \varphi)/u$ is bounded on N_u, then $C_AU(Y, A)$ is $E\varphi$-solid for every $E \subset C_AV(X, A)$.

If we combine 2 with 3, respectively 2 with 4, we get sufficient conditions for

$$C_lU_0(Y,A) := CU_0(Y,A) \cap C_lU(Y,A),$$

respectively

$$C_AU_0(Y,A) := C_AU(Y,A) \cap CU_0(Y,A)),$$

to be $E\varphi$-solid for every subspace E of $C_lV_0(X,A)$, respectively $C_AV_0(X,A)$. We refer to [**5**] and [**16**] for details on the spaces $C_AV(X,A)$ and $C_lV(X,A)$.

It is easy to show

Proposition 8. *Let $E \subset CV(X,A)$ be a $C_b(X)$-module satisfying (M) such that $\psi C_\varphi(E) \subset C(Y,A)$. If F is $E\varphi$-solid, then ψC_φ is continuous from E into F if, and only if,* (1) *holds.*

4. Bounded weighted composition operators. A linear map θ is said to be bounded, precompact or compact, if it maps some 0-neighborhood into a bounded, precompact or compact set respectively. It will be called locally bounded (respectively locally precompact, locally compact) if it maps every bounded set into a bounded (respectively precompact, compact) one. Whenever θ ranges in a space of continuous functions on Y, it will be said to be locally equicontinuous, respectively equicontinuous, on $Y_0 \subset Y$, if the image by θ of every bounded set, respectively of some 0-neighborhood, is equicontinuous at any point $y \in Y_0$.

In this section we deal with bounded, (locally) equicontinuous and (locally) precompact weighted composition operators. For this purpose, we need the following lemma generalizing Lemma 10 of [**15**] to the non-locally convex setting.

Lemma 9. *Let L be a subset of $CV(X,A)$ such that $C_b^+(X)L \subset L$ and L satisfies (M). Then, for every $G \in \mathcal{N}$, $v \in V$ and $x \in coz(L)$, the equality*

$$\frac{1}{v(x)} = \sup\{P_G(f(x)),\ f \in B_{G,v} \cap L\}$$

holds, with $1/0 = +\infty$.

Proof. The same as that of Lemma 10 of [**15**] with gauges of members of $\mathcal{N}$ instead of continuous semi-norms. □

Henceforth, when there is no risk of confusion, we will use the same notation $B_{G,v}$ to mean $B_{G,v} \cap E$.

Theorem 10. *Assume that $E \subset CV(X,A)$ is a $C_b(X)$-module satisfying (M) and that $\psi C_\varphi(E) \subset C(Y,A)$. Then ψC_φ is bounded from E into $CU(Y,A)$ if, and only if, the following condition holds. There exists $H \in \mathcal{N}$, $v \in V$: for all $G \in \mathcal{N}$, $u \in U$, there exists $\lambda > 0$:*

$$u(y)P_G(\psi_y(a)) \leq \lambda v(\varphi(y))P_H(a), \quad \textit{for all } a \in A, \quad y \in Y_{E,\varphi}. \tag{4}$$

Moreover, if ψC_φ is bounded, then so is also ψ_y for every $y \in Y_{E,\varphi}$.

Proof. Necessity. Since $\psi C_\varphi : E \to CU(Y,A)$ is bounded, there exist $H \in \mathcal{N}$ and $v \in V$ such that, for every $G \in \mathcal{N}$ and $u \in U$, there is some $\lambda > 0$ enjoying

$$P_{G,u}(\psi C_\varphi(f)) \leq \lambda, \quad f \in B_{H,v}.$$

In particular,

$$u(y)P_G(\psi_y(P_H \circ f \otimes a)(\varphi(y))) \leq \lambda, \quad y \in Y,\ f \in B_{G,v},\ a \in H$$

or

$$u(y)P_H(f(\varphi(y)))P_G(\psi_y(a)) \leq \lambda, \quad y \in Y, \quad f \in B_{G,v}, \quad a \in H.$$

By Lemma 9 and a classical argument, we get

$$u(y)P_G(\psi_y(a)) \leq \lambda v(\varphi(y))P_H(a), \quad y \in Y_{E,\varphi}, \quad a \in A. \tag{5}$$

Sufficiency. Assume that, for every $G \in \mathcal{N}$ and $u \in U$, (4) holds. Then, for $f \in E$ and $y \in Y$, we have

$$u(y)P_G(\psi_y(f(\varphi(y)))) \leq \lambda v(\varphi(y))P_H(f(\varphi(y)), \quad y \in Y.$$

In particular, for $f \in B_{H,v}$, we get

$$u(y)P_G(\psi_y(f(\varphi(y)))) \leq \lambda, \quad y \in Y_{E,\varphi},$$

giving $P_{G,u}(\psi C_\varphi(f)) \le \lambda$, $f \in B_{H,v}$.

Now, assume that $\psi C_\varphi(B_{G,v})$ is bounded in A, and let $y_0 \in Y_{E,\varphi}$ and $f_0 \in B_{G,v}$ be such that $f_0(\varphi(y_0)) \neq 0$. Let $H \in \mathcal{N}$ enjoy $P_H(f_0(\varphi(y_0))) = 1$ and $\alpha > P_{H,v}(f_0)$. Then $(1/\alpha)P_H \circ f_0 \otimes a$ belongs to $B_{G,v}$ for all $a \in G$. Given $I \in \mathcal{N}$ and let $u \in U$ be such that $u(y_0) \neq 0$. There exists $\lambda > 0$ so that:

$$\left[u\psi C_\varphi\left(\frac{1}{\alpha}\, P_H \circ f_0 \otimes a\right)\right](Y) \subset \frac{\lambda}{\alpha}\, I, \quad a \in G.$$

In particular, $u(y_0)\psi_{y_0}(a) \in \lambda I$. Since $a \in G$ and $I \in \mathcal{N}$ are arbitrary, $\psi_{y_0}(G)$ is bounded in A. □

A similar proof yields

Theorem 11. *Assume that $E \subset CV_0(X, A)$ is a $C_b(X)$-module satisfying (M) and that $\psi C_\varphi(E) \subset C(Y, A)$. Then ψC_φ is bounded from E into $CU_0(Y, A)$ if, and only if,* (4) *holds and*

$$\varphi^{-1}(K) \cap \{y \in Y : u(y)P_G(\psi_y(a)) \ge \varepsilon\}$$

is relatively compact for every compact $K \subset \operatorname{coz}(E)$, $u \in U$, $G \in \mathcal{N}$, $a \neq 0$ and $\varepsilon > 0$.

We now examine the equicontinuity of ψC_φ.

Theorem 12. *Assume that $E \subset CV(X, A)$ is a $C_b(X)$-module satisfying (M) and that $\psi C_\varphi(E) \subset C(Y, A)$. Then ψC_φ is locally equicontinuous on $Y_{E,\varphi,\psi}$ if, and only if, the following conditions hold*:

1. *φ is locally constant on $Y_{E,\varphi,\psi}$.*
2. *ψ is continuous from Y into $\mathcal{L}_\beta(A)$.*

Proof. Necessity. 1. Assume that φ is constant on no neighborhood of some $y_0 \in Y_{E,\varphi,\psi}$ and choose $f_0 \in E$ with $\psi_{y_0}(f_0(\varphi(y_0))) \neq 0$. If $\mathcal{V}$ is the collection of all neighborhoods of y_0, then every $\Omega \in \mathcal{V}$ contains some y_Ω with $\varphi(y_0) \neq \varphi(y_\Omega)$. Consider $f_\Omega \in C_b(X)$ such that $0 \le f_\Omega \le 1$, $f_\Omega(\varphi(y_\Omega)) = 0$ and $f_\Omega(\varphi(y_0)) = 1$. The set

$\{g_\Omega := f_\Omega f_0, \Omega \in \mathcal{V}\}$ is bounded in E and then its image by ψC_φ is equicontinuous at y_0. Therefore, for every $G \in \mathcal{N}$, there exists $\Omega_0 \in \mathcal{V}$ such that

$$\psi_y(g_\Omega(\varphi(y))) - \psi_{y_0}(g_\Omega(\varphi(y_0))) \in G, \quad \text{for all } y \in \Omega_0, \quad \Omega \in \mathcal{V}.$$

Hence, for every $\Omega \subset \Omega_0$ and $y = y_\Omega$, we get $\psi_{y_0}(f_0(\varphi(y_0))) \in G$. Since G is arbitrary, $\psi_{y_0}(g_\Omega(\varphi(y_0))) = 0$ which is a contradiction.

2. Let $y_0 \in Y_{E,\varphi,\psi}$, B a bounded set in A and $H \in \mathcal{N}$ be given. By 1 there exists a neighborhood Ω_0 of y_0 on which φ is constant with value, say, x_0. Choose $f_0 \in E$ so that $\psi_{y_0}(f_0(x_0)) \neq 0$ and $H \in \mathcal{N}$ with $P_H(\psi_{y_0}(f_0(\varphi(y_0)))) = 1$. Since the set

$$K := \{P_H \circ f_0 \otimes b, b \in B\}$$

is bounded in E, $\psi C_\varphi(K)$ is equicontinuous at y_0. Hence there is some y_0-neighborhood Ω contained in Ω_0 such that

$$[\psi_y(P_H(f_0(\varphi(y)))b) - \psi_{y_0}(P_H(f_0(\varphi(y_0)))b)] \in G, \quad y \in \Omega, \quad b \in B.$$

This yields $\psi_y - \psi_{y_0} \in N(B, G)$ for every $y \in \Omega$, showing that ψ is β-continuous at y_0. Since y_0 is arbitrary in $Y_{E,\varphi,\psi}$, ψ is β-continuous on $Y_{E,\varphi,\psi}$.

Sufficiency. Given a bounded set $\mathbf{B} \subset E$, $y_0 \in Y_{E,\varphi,\psi}$ and $G \in \mathcal{N}$. By our assumption, there is some neighborhood Ω_0 of y_0 so that φ is constant on Ω_0. Choose $v \in V$ with $v(\varphi(y_0)) \neq 0$. Since the set $B := \{v(x)f(x),\ f \in \mathbf{B},\ x \in X\}$ is bounded in A and ψ is β-continuous at y_0, there is some other neighborhood Ω of y_0 such that $\Omega \subset \Omega_0$ and

$$\psi_y - \psi_{y_0} \in N(B, v(\varphi(y_0))G), \quad y \in \Omega.$$

This is

$$\psi_y(v(x)f(x)) - \psi_{y_0}(v(x)f(x)) \in v(\varphi(y_0))G, \quad y \in \Omega, \quad x \in X,$$

yielding

$$\psi C_\varphi(f)(y) - \psi C_\varphi(f)(y_0) \in G, \quad y \in \Omega,$$

whereby $\psi C_\varphi(\mathbf{B})$ is equicontinuous at y_0 and then on $Y_{E,\varphi,\psi}$.

A trivial consequence of Theorem 12 is

Corollary 13. *Assume that E is a $C_b(X)$-module satisfying (M). If φ is not constant on any open set (in particular, if X has no isolated point and φ is one to one), then ψC_φ is locally equicontinuous from E into $C(Y, A)$ if, and only if, it is identically zero.*

In case of multiplication operators, we get the following corollary improving Proposition 11 of [**15**].

Corollary 14. *Let E be a $C_b(X)$-module satisfying (M) and $\psi : X \to \mathcal{L}(A)$ a map. If $M_\psi : E \to C(X, A)$ is locally equicontinuous, then $\operatorname{coz}(M_\psi(E))$ is a discrete space.*

Proof. Take in Theorem 12 $Y = X$ and $\varphi = \mathrm{Id}_X$. Then M_ψ is nothing but ψC_φ and then Id_X is locally constant on $Y_{E,\varphi,\psi}$. This means that $\operatorname{coz}(M_\psi(E)) = Y_{E,\varphi,\psi}$ is discrete. □

Theorem 15. *Assume that $E \subset CV(X, A)$ is a $C_b(X)$-module satisfying (M) and that $\psi C_\varphi(E) \subset C(Y, A)$. Then ψC_φ is equicontinuous on $Y_{E,\varphi,\psi}$ if, and only if, the following two conditions hold*:

1. *φ is locally constant on $Y_{E,\varphi,\psi}$.*

2. *There exists $G \in \mathcal{N}$ such that, for every $y_0 \in Y_{E,\varphi,\psi}$ and $H \in \mathcal{N}$, there is some neighborhood Ω of y_0 so that $\psi_y - \psi_{y_0} \in N(G, H)$ for every $y \in \Omega$.*

Under conditions 1 and 2 every point $y_0 \in Y_{E,\varphi,\psi}$ admits a neighborhood whose image by ψ is equicontinuous on A.

Proof. Assume that there exist $v \in V$ and $G \in \mathcal{N}$ so that $\psi C_\varphi(B_{G,v})$ is equicontinuous on $Y_{E,\varphi,\psi}$. By Theorem 12, there exists a neighborhood Ω_0 of y_0 on which φ is constant with value, say, x_0.

Necessity. 1. follows from Theorem 12 since an equicontinuous map is already locally equicontinuous.

2. Let $y_0 \in Y_{E,\varphi,\psi}$ and $H \in \mathcal{N}$ be given. By property (M), we may choose $f_0 \in B_{G,v}$ so that $P_G(f_0(x_0)) \neq 0$. Since the set

$$K := \{P_G \circ f_0 \otimes a,\ a \in G\}$$

is contained in $B_{G,v}$, $\psi C_\varphi(K)$ is equicontinuous on Y. Hence there is some y_0-neighborhood Ω contained in Ω_0 such that

$$[\psi_y(P_G(f_0(\varphi(y)))a) - \psi_{y_0}(P_G(f_0(\varphi(y_0)))a)] \in P_G(f_0(x_0))H, \\ y \in \Omega, \quad a \in G.$$

This leads to $\psi_y - \psi_{y_0} \in N(G,H)$ for every $y \in \Omega$.

For the remainder, let $y_0 \in Y_{E,\varphi,\psi}$ and $H \in \mathcal{N}$. Choose again $f_0 \in B_{G,v}$ so that $\alpha := P_G(f(x_0)) \neq 0$. Since $\psi C_\varphi(B_{G,v})$ is equicontinuous at y_0, for every $I \in \mathcal{N}$ satisfying $I + I \subset H$, there exists a neighborhood $\Omega \subset \Omega_0$ of y_0 such that

$$\psi C_\varphi(f)(y) - \psi C_\varphi(f)(y_0) \in \alpha I, \quad y \in \Omega, \quad f \in B_{G,v}.$$

This is

$$\psi_y(f(x_0)) - \psi_{y_0}((f(x_0)) \in \alpha I, \quad y \in \Omega, \quad f \in B_{G,v}.$$

For an arbitrary $a \in G$, $P_G \circ f_0 \otimes a$ still belongs to $B_{G,v}$. Hence

$$P_G(f_0(x_0))[\psi_y(a) - \psi_{y_0}(a)] \in \alpha I, \quad y \in \Omega,$$

or $\psi_y(a) - \psi_{y_0}(a) \in I$, $y \in \Omega$. Now, the continuity of ψ_{y_0} yields a 0-neighborhood $J \in \mathcal{N}$ such that $\psi_{y_0}(J) \subset I$. Finally, for $a \in R := J \cap G$,

$$\psi_y(a) = [\psi_y(a) - \psi_{y_0}(a)] + \psi_{y_0}(a) \in I + I \subset H, \quad y \in \Omega,$$

showing that $\{\psi_y, y \in \Omega\}$ is equicontinuous at 0 and then everywhere on A. $\square$

In [**1**], Bierstedt showed that the precompact sets are equicontinuous in $CV(X)$ whenever X is a $V_{\mathbf{R}}$-space. Bierstedt's result was extended in [**18**] to the space

$$CV_p(X,A) := \{f \in CV(X,A) : (vf)(X) \text{ is precompact in } A,\ \forall v \in V\},$$

where A is a locally convex space. Later, this result was extended in [**15**] to $CV(X,A)$, again with A locally convex. Actually, this result holds for $CV(X,A)$ for arbitrary topological vector space A. Indeed, let δ_x denote the evaluation $f \mapsto f(x)$ at the point x and Δ the evaluation map $x \mapsto \delta_x$ defined from X into $\mathcal{L}(CV(X,A),A)$. Then we have

Proposition 16. 1. *The evaluation map* Δ *is continuous from* X *into* $\mathcal{L}_c(CV(X,A),A)$ *if, and only if, every precompact subset of* $CV(X,A)$ *is equicontinuous.*

2. *If* X *is a* $V_{\mathbf{R}}$*-space, then every precompact subset of* $CV(X,E)$ *is equicontinuous.*

Proof. 1. is straightforward. Next, in view of 1 and our assumption on X, it suffices to show that Δ is continuous on each $N_{v,1} := \{x \in X : v(x) \geq 1\}$. Let then $v \in V$ and $x \in N_{v,1}$ be given. If Λ is a neighborhood of δ_x in $\mathcal{L}_c(CV(X,A),A)$, then there exist $G \in \mathcal{N}$ and a precompact set $C \subset CV(X,E)$ such that $\delta_x + N(C,G) \subset \Lambda$. But there exist $h_i \in C$, $i \in \{1,2,\dots n\}$, so that $C \subset \cup_{i=1}^n (h_i + H)$, where $H+H+H \subset G$. Consider a neighborhood Ω of x with $h_i(t)-h_i(x) \in H$ for every $i = 1,2\dots n$ and $t \in \Omega$. Now, if $t \in \Omega \cap N_{v,1}$ and $h \in C$, then $h = h_i + f$ for some $i \in \{1,2\dots,n\}$ and some $f \in B_{H,v}$. Hence

$$\begin{aligned}\delta_t(h) - \delta_x(h) &= h(t) - h(x)\\ &= h_i(t) - h_i(x) + \frac{1}{v(t)}\,(v(t)f(t)) - \frac{1}{v(x)}\,(v(x)f(x))\\ &\in H + \frac{1}{v(t)}\,H + \frac{1}{v(x)}\,H \subset G.\end{aligned}$$

Since h is arbitrary in C, $\Delta(t) - \Delta(x) \in N(C,G)$ and thus Δ is continuous on $N_{v,1}$ and 2 is proved. □

According to Corollary 13 and Proposition 16, for a $C_b(X)$-module E satisfying (M), if Y is a $U_{\mathbf{R}}$-space and φ is constant on no open set, then ψC_φ is locally precompact from E into $CU(Y,A)$ if, and only if, it is identically zero.

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