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DARBOUX INTEGRABILITY AND REVERSIBLE QUADRATIC VECTOR FIELDS

JAUME LLIBRE AND JOÃO CARLOS MEDRADO

ABSTRACT. In this paper we improve the Darboux theory of integrability for reversible polynomial vector fields in $\mathbf{R}^n$, and we classify the phase portraits of all $\varphi-$reversible quadratic polynomial vector fields of $\mathbf{R}^2$ such that the dimension of the set of fixed points of φ is equal to one.

1. Introduction and statement of the main results. The algebraic theory of integrability is a classical one. In 1878, Darboux [**11**] provided a link between algebraic geometry and the search of first integrals and showed how to construct the first integral of polynomial vector fields in $\mathbf{R}^2$ or $\mathbf{C}^2$ having sufficient invariant algebraic curves. The theory also received contributions from Poincaré [**24**], who mainly was interested in the rational first integrals.

Good extensions of the Darboux theory of integrability to polynomial systems in $\mathbf{R}^n$ or $\mathbf{C}^n$ are due to Jouanolou [**16**] and Weil [**29**], see also [**17**]. In [**4, 6–9**], the authors developed the Darboux theory of integrability essentially in $\mathbf{R}^2$ or $\mathbf{C}^2$ considering not only the invariant algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves. Recently, in [**13**] and [**18**] there are extensions of the Darboux theory of integrability to two-dimensional surfaces.

In this paper we present and prove properties of reversible polynomial vector fields. In Propositions 3 and 4 we prove that for $\varphi-$reversible polynomial vector fields, X, of degree greater than one and such that $\dim(\mathrm{Fix}(\varphi)) = k$, the involution φ is linear and conjugated to $\mathrm{diag}(+1, \dots, +1, -1, \dots, -1)$, where the number of -1 is equal to k. In Proposition 5 we prove that if $f = 0$ is an invariant curve of X, then $f \circ \varphi$ is also an invariant curve. The same occurs with the exponential

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factors and the first integrals. This result is useful to obtain first integral of vector fields using the Darboux theory of integrability, see Theorem 6, because if an invariant curve $f = 0$ or exponential factor is not symmetric, i.e., $f \circ \varphi \neq f$ or $F \circ \varphi \neq F$, then the system has another invariant algebraic curve or exponential factor, respectively.

Quadratic vector fields, i.e., quadratic polynomial vector fields, have been investigated intensively, and nearly 1000 papers have been published about these systems, see, for instance, [**25, 30, 31**]. But it is an open problem to know what are the integrable quadratic systems, see, for instance, [**19**]. Here, we characterize all $\varphi-$reversible quadratic vector fields such that the dimension of the set of fixed points of φ is equal to one, and we prove that they are integrable.

Teixeira [**28**] and Medrado [**21**], see also [**22**], studying $\varphi-$reversible vector fields X in $\mathbf{R}^n$ such that the $\dim(\mathrm{Fix}(\varphi)) = n-1$ used a change of variables and reduce the study of X to analyze vector fields defined on manifolds with boundary. In the proof of the next theorem we also use this technique.

Theorem A. *Let X be a reversible quadratic vector field with the dimension of the set of fixed points of the associated involution equal to one. Then, the phase portrait of X is topologically equivalent to one of the* 77 *phase portraits given in Figure* 1. *Moreover, each phase portrait of Figure* 1 *is realizable by some reversible quadratic reversible vector field with the dimension of the set of fixed points of the associated involution equal to one.*

The paper is organized as follows. In Section 2 we give some basic definitions that we will need to draw the phase portraits of reversible polynomial vector fields. In Section 3 we define the reversible vector fields and present their basic properties. In the same section we prove Propositions 3 and 4. In Section 4 we state the Darboux theory of integrability for real polynomial reversible vector fields and we prove Proposition 5. In Section 5 we prove that the $\varphi-$reversible polynomial vector fields in $\mathbf{R}^2$, such that the dimension of the set of fixed points of φ is equal to one, are integrable and we present their normal forms. In Section 6 we draw the phase portraits of reversible quadratic vector fields defined on half-plane. In Section 7 we draw the phase portraits of reversible quadratic vector fields and prove Theorem A.

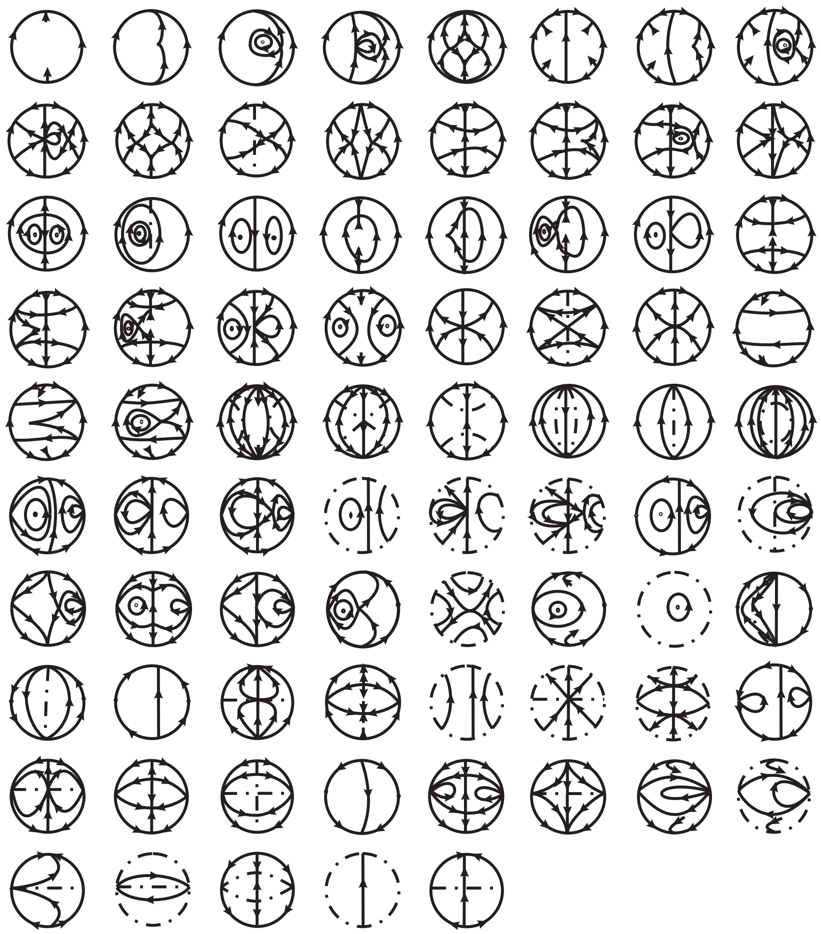

FIGURE 1. Phase portraits of $\varphi-$reversible quadratic vector fields in $\mathbf{R}^2$ such that $\dim(\mathrm{Fix}(\varphi)) = 1$.

2. Preliminary definitions. In this section we introduce some basic definitions and notations for the investigation of topological phase portraits of $\varphi-$reversible quadratic vector fields X.

2.1 Singular points. Let $X = (P(x,y), Q(x,y))$ be a planar real polynomial vector field of degree n. A point $q \in \mathbf{R}^2$ is a *singular point* of the vector field X if $P(q) = Q(q) = 0$.

If $D = P_x(q)Q_y(q) - P_y(q)Q_x(q)$ and $T = P_x(q) + Q_y(q)$, then a singular point q is *elementary nondegenerate* if $D \neq 0$. Then the singular point is isolated. Furthermore, q is a *saddle* if $D < 0$, a *node* if $T^2 \geq 4D > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a focus if $T^2 < 4D$ and $T \neq 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < D$; for more details see [**2**, p. 183]. A singular point q is *elementary degenerate* if $D = 0$ and $T \neq 0$, and then q is also isolated in the set of all singular points. The results on elementary degenerate singular points are summarized in the *elementary degenerate theorem* of the Appendix.

A singular point q is *nilpotent* if $D = T = 0$ and the Jacobian matrix at q is not the zero matrix and q is isolated in the set of all singular points. The results on nilpotent singular points are summarized in the nilpotent theorem of the Appendix.

If the Jacobian matrix at the singular point q is identically zero and q is isolated in the set of all singular points, we say that q is *linearly zero*. Then the study of its local phase portraits needs a particular treatment (directional blow-ups), see for more details [**23**] and [**26**]. If $q = (0,0)$ is linearly zero and the vector field X has some nonzero second degree term, then the local phase portraits are characterized in [**15**].

2.2 Poincaré compactification. We denote by $\mathcal{P}_2(\mathbf{R}^2)$ the set of all planar real vector fields of degree 2. For $X \in \mathcal{P}_2(\mathbf{R}^2)$ the *Poincaré compactified vector field* $p(X)$ corresponding to X is a vector field induced in $\mathbf{S}^2$ as follows, see for instance [**12**] and [**2**]. Let $\mathbf{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, called the *Poincaré sphere*, and $T_y\mathbf{S}^2$ be the tangent space to $\mathbf{S}^2$ at point y. Consider the central projections $f_+ : T_{(0,0,1)}\mathbf{S}^2 \to \mathbf{S}^2_+ = \{y \in \mathbf{S}^2 : y_3 > 0\}$ and $f_- : T_{(0,0,1)}\mathbf{S}^2 \to \mathbf{S}^2_- = \{y \in \mathbf{S}^2 : y_3 < 0\}$. These maps define two copies of X, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector fields $Df_+ \circ X$ and $Df_- \circ X$ in $\mathbf{S}^2$ except on its equator $\mathbf{S}^1 = \{y \in \mathbf{S}^2 : y_3 = 0\}$. Obviously $\mathbf{S}^1$ is identified to the infinity of $\mathbf{R}^2$. In order to extend X' to an analytic vector field in $\mathbf{S}^2$, including $\mathbf{S}^1$, it is necessary that X satisfies suitable hypotheses. In the case that $X \in \mathcal{P}_2(\mathbf{R}^2)$, the *Poincaré compactification* $p(X)$ is the only analytic extension of y_3X' to $\mathbf{S}^2$. The set of all compactified vector fields $p(X)$ with $X \in \mathcal{P}_2(\mathbf{R}^2)$ is denoted by $\mathcal{P}_2(\mathbf{S}^2)$. For the flow of the compactified vector field $p(X)$, the equator $\mathbf{S}^1$ is invariant. On $\mathbf{S}^2 \backslash \mathbf{S}^1$ there are two symmetric copies of X, and knowing the behavior of $p(X)$ around $\mathbf{S}^1$, we know the behavior of X near infinity. The projection of the closed northern hemisphere of $\mathbf{S}^2$ in $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbf{D}^2$.

As $\mathbf{S}^2$ is a differentiable manifold, for computing the expression of $p(X)$, we can consider the six local charts $U_i = \{y \in \mathbf{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbf{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i \to \mathbf{R}^2$ and $G_i : V_i \to \mathbf{R}^2$ defined as the inverses of the central projections from the tangent planes at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$, then z represents different things according to the local charts under consideration. Some straightforward calculations give for $p(X)$ the following expressions:

$$z_2^2\Delta(z)\left[Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right] \quad \text{in} \quad U_1,$$

$$z_2^2\Delta(z)\left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right] \quad \text{in} \quad U_2,$$

$$\Delta(z)[P(z_1, z_2), Q(z_1, z_2)] \quad \text{in} \quad U_3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-1/2}$. The expression for V_i is the same as that for U_i except for the multiplicative factor -1. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbf{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$. Thus we obtain a polynomial vector field of degree at most 3 in each local chart.

Since the unique singular point at infinity which cannot be contained into the charts $U_2 \cup V_2$ are the origins $(0, 0)$ of U_1 and V_1, when we study the infinity singular points on the charts $U_1 \cup V_1$, we only consider if the $(0,0)$ of these charts are or not singular points.

2.3 Topological equivalence. We say that polynomial vector fields X and Y in $\mathbf{R}^2$ are *topologically equivalent* if there exists a homeomorphism in $\mathbf{S}^2$ preserving the infinity $\mathbf{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$, preserving or reversing simultaneously the sense of all orbits.

A *separatrix* of $p(X)$ is an orbit which is a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point, finite or infinity. If a quadratic system has a polynomial first integral, then it has no limit cycles.

We denote by $\mathrm{Sep}\,(p(X))$ the set formed by all separatrices of $p(X)$. Neumann [**23**] proved that the set $\mathrm{Sep}\,(p(X))$ is closed. Each open connected component of $\mathbf{S}^2 \backslash \mathrm{Sep}\,(p(X))$ is called a *canonical region* of $p(X)$. A *separatrix configuration* is defined as a union of $\mathrm{Sep}\,(p(X))$ plus one representative solution chosen from each canonical region. We say that $\mathrm{Sep}\,(p(X))$ and $\mathrm{Sep}\,(p(Y))$ are *equivalent* if there exists a homeomorphism in $\mathbf{S}^2$ preserving the infinity $\mathbf{S}^1$ carrying orbits of $\mathrm{Sep}\,(p(X))$ into orbits of $\mathrm{Sep}\,(p(Y))$, preserving or reversing simultaneously the sense of all orbits.

The next theorem due to Neumann [**23**] states the characterization of two topologically equivalent Poincaré compactified vector fields. We shall need it later on for the analysis of the global phase portraits of the $\varphi-$reversible quadratic vector fields.

Theorem 1 (Neumann's theorem). *Suppose that $p(X)$ and $p(Y)$ are two continuous flows in $\mathbf{S}^2$ with isolated singular points. Then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

Neumann's theorem implies that in order to obtain the global phase portrait of a vector field $p(X)$ with isolated singular points, we essentially need to determine the α- and ω-limit sets of all separatrices of $p(X)$.

Neumann's theorem was obtained under the additional assumption that the flow has no limit separatrices by Markus [**20**] in 1954.

3. Reversible vector fields. Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be an involution, i.e., $\varphi \circ \varphi = \mathrm{Id}$. We say that X is a $\varphi-$*reversible vector field*, or only $\varphi-$*reversible*, if X satisfies

$$D\varphi(p)X(p) = -X \circ \varphi(p), \quad p \in \mathbf{R}^n.$$

We denote by $S \subset \mathbf{R}^n$ the set of fixed points of φ, or $S = \mathrm{Fix}\,(\varphi)$. If $p \in S$ and $X(p) = 0$, we say that p is a *symmetric singular point* of X; otherwise, it is an *asymmetric singular point*. Any periodic orbit of X crossing S is called a *symmetric periodic orbit*; otherwise, it is an *asymmetric periodic orbit*.

If p is a singular point of X, then $\varphi(p)$ is also a singular point of X, and since φ interchanges the stable and unstable manifolds, a symmetric singular point cannot be an attractor or a repellor. If γ is a periodic orbit of X, then $\varphi(\gamma)$ is also a periodic orbit.

Lemma 2. *Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be an involution, and let X be a $\varphi-$reversible vector field in $\mathbf{R}^n$. If γ is an asymmetric periodic orbit, then $\varphi(\gamma)$ is an asymmetric periodic orbit too; and if γ is a symmetric periodic orbit, then it is not a limit cycle.*

Proof. The proof follows directly from equation $D\varphi(p)X(p) = -X(\varphi(p))$ and from [**21**, Lemma 3.2]. □

Proposition 3. *Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial involution of degree q, and let X be a $\varphi-$reversible polynomial vector field of degree p in $\mathbf{R}^n$. If $p \neq 1$, then φ is a linear involution.*

Proof. As X is a $\varphi-$reversible vector field, then $D\varphi(p)X(p) = -X(\varphi(p))$. This equation implies that $q - 1 + p = pq$, or equivalently $q(p-1) = p - 1$. So, $q = 1$ provided that $p \neq 1$. □

Proposition 4. *Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a linear involution such that the vector subspace $Fix\,(\varphi)$ has dimension equal to k. Then the involution φ*

is conjugated to ψ *given by* $\psi = \operatorname{diag}(+1, \dots, +1, -1, \dots, -1)$*, where the number of elements* -1 *is equal to* k*.*

Proof. We observe that as φ is a linear involution, then $\det(\varphi) = \pm 1$. By Jordan's normal form theorem, there is a linear change of variables $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $\psi = h^{-1}\varphi h$ is formed by Jordan's blocks, and the elements of the principal diagonal of ψ are formed by not zero eigenvalues λ_i, $i = 1, \dots, n$. Now, we suppose that ψ has a $k \times k$ Jordan's block of nilpotent type associated to eigenvalue λ_{i_0}, $1 \le i_0 \le n$ which we denote by $C = (c_{ij})$, $i, j = 1, \dots, k$. We compute $C^2 = (d_{ij})$, $i, j = 1 \dots, k$, and we have that $d_{12} = 2\lambda_{i_0} \ne 0$. But, C is an involution, provided that ψ is an involution too, this implies that $d_{12} = 0$ and we have a contradiction. So, ψ has no nilpotent Jordan's blocks.

Now, if there is $1 \le i_0 \le n$ such that $\lambda_{i_0} = a + ib$, with $b \ne 0$, we have the associated Jordan' block:

$$C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Thus we have a contradiction because $C^2 = \mathrm{Id}$ if and only if $b = 0$.

In short, ψ is a diagonal matrix, and $\psi^2 = \mathrm{Id}$ implies $\lambda_i = \pm 1$, and the proof of the proposition is completed. □

4. Darboux theory of integrability for reversible polynomial vector fields. In this section we state the Darboux theory of integrability for *real* polynomial reversible vector fields. Of course, this theory can be extended in a natural way to complex polynomial vector fields, but here we do not consider these extensions. We consider the following polynomial vector fields in $\mathbf{R}^n$:

$$X = \sum_{i=1}^{n} P_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}, \quad (x_1, \dots, x_n) \in \mathbf{R}^n,$$

where P_i for $i = 1, \dots, n$, are polynomials of degree at most m. The integer $m = \max\{\deg P_1, \dots, \deg P_n\}$ is the *degree* of the vector field X.

The polynomial vector field X has a *first integral* in an open subset U of $\mathbf{R}^n$ if there exists a nonconstant analytic function $H : U \to \mathbf{R}^n$, which is constant on all solutions $(x_1(t), \dots, x_n(t))$ of X in U. Clearly H is a first integral of X in U if and only if $XH \equiv 0$ in U.

Let $f(x_1, \dots, x_n) \in \mathbf{C}[x_1, \dots, x_n]$. As usual, $\mathbf{C}[x_1, \dots, x_n]$ denotes the ring of all complex polynomials in the variables $x_1, \dots, x_n$. We say that $f = 0$ is an *invariant algebraic hypersurface of the vector field* X *on* $\mathbf{R}^n$, or simply an *invariant algebraic hypersurface on* $\mathbf{R}^n$, if there exists a polynomial $k \in \mathbf{C}[x_1, \dots, x_n]$ such that

$$Xf = \sum_{i=1}^{n} P_i \frac{\partial f}{\partial x_i} = kf, \quad \text{on} \quad \mathbf{C}^n,$$

the polynomial $k = k(x_1, \dots, x_n) \in \mathbf{C}[x_1, \dots, x_n]$ is called the *cofactor* of $f = 0$ in $\mathbf{C}^n$. We can prove easily that for a polynomial vector field X of degree m the cofactor of an invariant algebraic hypersurface is of degree at most $m - 1$.

We allow that the invariant algebraic hypersurfaces (and later on the exponential factors) can be complex, because often the existence of a real first integral is forced by existence of these complex objects, for more details see [**4, 8**].

Let $f = 0$ be an invariant algebraic hypersurface of X in $\mathbf{R}^n$. Suppose that $f(x_1, \dots, x_n) \notin \mathbf{R}[x_1, \dots, x_n]$, if $f = 0$ is an invariant algebraic hypersurface of X in U, then the conjugate $\bar{f}(x_1, \dots, x_n)$ of the polynomial $f(x_1, \dots, x_n)$ (which means to conjugate all the coefficients of f) defines another invariant algebraic hypersurface $\bar{f} = 0$ of X in U.

We remark that, in the above definitions, in $\mathbf{R}^n$ with $n > 2$, then $f = 0$ is called an *invariant algebraic hypersurface*. If $n = 2$, then $f = 0$ is called an *invariant algebraic curve*. If $n = 3$, then $f = 0$ is called an *invariant algebraic surface*.

Since on an invariant algebraic hypersurface $f = 0$ the gradient ∇f in $f = 0$ is orthogonal to the polynomial vector $(P_1, \dots, P_n)$, it follows that the vector field X is tangent to the algebraic hypersurface $f = 0$. Therefore, the hypersurface $f = 0$ is formed by trajectories of the vector field X. This justifies the name of *invariant* by the flow of the vector field X in $\mathbf{R}^n$.

An *exponential factor* $F(x_1, \dots, x_n)$ of the polynomial vector field X of degree m in $\mathbf{R}^n$ is a function of the form $\exp(g/h)$ with g and h

polynomials of $\mathbf{C}[x_1, \dots, x_n]$ and satisfying $XF = KF$ in $\mathbf{C}^n$ for some $K \in \mathbf{C}_{m-1}[x_1, \dots, x_n]$, where $\mathbf{C}_{m-1}[x_1, \dots, x_n]$ denotes the set of all polynomials of $\mathbf{C}[x_1, \dots, x_n]$ of degree at most $m-1$. The notion of exponential factor is due to Christopher [**7**], and it controls the multiplicity of the invariant hypersurface $h = 0$, see [**10**].

Proposition 5. *Let X be a $\varphi-$reversible polynomial vector field of $\mathbf{R}^n$. Then the following statements hold.*

(a) *$f = 0$ is an invariant algebraic hypersurface of X with cofactor K if and only if $f_\varphi = f \circ \varphi = 0$ is also an invariant algebraic hypersurface with cofactor $K_\varphi = -K \circ \varphi$.*

(b) *$F = \exp(g/h)$ is an exponential factor of X with cofactor L, if and only if $F_\varphi = F \circ \varphi$ is also an exponential factor with cofactor $L_\varphi = -L \circ \varphi$.*

(c) *$H : \mathbf{R}^n \to \mathbf{R}$ is a first integral of X if and only if $H_\varphi = H \circ \varphi$ is also a first integral of X.*

Proof. Initially, we observe that, as $f = 0$ is an invariant algebraic hypersurface with cofactor K and X is φ-reversible, we have that

$$Xf = Kf \quad \text{and} \quad D\varphi X = -X \circ \varphi,$$

respectively. Then, we obtain

$$\begin{aligned} Xf_\varphi &= \nabla f_\varphi \cdot X = \nabla f \circ \varphi \cdot D\varphi \cdot X = \nabla f \circ \varphi(-X \circ \varphi) \\ &= -(Xf) \circ \varphi = -(K \circ \varphi)(f \circ \varphi) = K_\varphi f_\varphi. \end{aligned}$$

Thus, if $f = 0$ is an invariant algebraic hypersurface with cofactor K, then $f_\varphi = 0$ is also an invariant algebraic hypersurface with cofactor $K_\varphi = -K \circ \varphi$.

This implies that, if $f_\varphi = f \circ \varphi = 0$ is an invariant algebraic hypersurface with cofactor $K_\varphi = -K \circ \varphi$, then $f_\varphi \circ \varphi = 0$ is also an invariant algebraic hypersurface with cofactor $\tilde{K} = -K_\varphi \circ \varphi$. But, we observe that

$$f_\varphi \circ \varphi = f \circ \varphi \circ \varphi = f \quad \text{and} \quad \tilde{K} = -K_\varphi \circ \varphi = -(-K \circ \varphi)\varphi = K.$$

This proves statement (1).

Now, as $F = \exp(g/h)$ is an exponential factor with cofactor L, we have by Proposition 7 of [**8**] that $h = 0$ is an invariant algebraic hypersurface of X with cofactor K_h, i.e., $Xh = K_h h$, and g satisfies the equation $Xg = gK_h + hL$ where L is the cofactor of F.

We consider $g_\varphi = g \circ \varphi$ and $f_\varphi = f \circ \varphi$, we get

$$XF_\varphi = X \exp\left(\frac{g_\varphi}{h_\varphi}\right) = \exp\left(\frac{g_\varphi}{h_\varphi}\right) \frac{(Xg_\varphi)h_\varphi - g_\varphi(Xh_\varphi)}{(h_\varphi)^2}.$$

Now,

$$\begin{aligned}(Xg_\varphi)h_\varphi - g_\varphi(Xh_\varphi) &= -[g_\varphi(K_h \circ \varphi) + h_\varphi(L \circ \varphi)]h_\varphi - g_\varphi[-K_h \circ \varphi]h_\varphi \\ &= -(L \circ \varphi)(h_\varphi)^2.\end{aligned}$$

Consequently,

$$X \exp\left(\frac{g_\varphi}{h_\varphi}\right) = -L \circ \varphi \ \exp\left(\frac{g_\varphi}{h_\varphi}\right).$$

Thus, if F is an exponential factor with cofactor L, then F_φ is also an exponential factor with cofactor $L_\varphi = -L \circ \varphi$.

We now apply the argument used in the proof of above statement, with F replaced by F_φ and L replaced by L_φ to conclude the proof of statement (2).

Finally, H is a first integral if and only if $XH \equiv 0$, and we have that

$$XH_\varphi = \nabla H \circ \varphi \cdot X \circ \varphi = XH(\varphi) \equiv 0.$$

So, $XH \equiv 0$ if and only if $XH_\varphi \equiv 0$. This proves statement (3). □

The following result is a summary of the Darboux theory of integrability in $\mathbf{R}^n$, see for instance, [**16, 17, 29**].

Theorem 6. *Suppose that the polynomial vector field X defined in $\mathbf{R}^n$ of degree m admits p irreducible invariant algebraic hypersurfaces $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$; q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$; and r independent singular points $\mathbf{x}_k \in \mathbf{R}^n$ of X such that $f_i(\mathbf{x}_k) \neq 0$ for $i = 1, \dots, p$*

and $k = 1, \dots, r$. We note that the irreducible factors h_j are some f_i. Then the following statements hold.

(a) *There exist $\lambda_i, \mu_j \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the following real, multi-valued, function of Darbouxian type*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbf{R}$, is a first integral of the vector field X.

(b) *If $p + q + r \geq \binom{n+m-1}{m-1} + 1$, then there exist $\lambda_i, \mu_j \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$.*

(c) *There exist $\lambda_i, \mu_j \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\sigma$ for some $\sigma \in \mathbf{R}\backslash\{0\}$, if and only if the real, multi-valued, function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{\sigma t},$$

substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbf{R}$, is an invariant of the vector field X.

(d) *The vector field X has a rational first integral if and only if*

$$p + q + r \geq \binom{n+m-1}{m-1} + n.$$

Moreover, all trajectories are contained in invariant algebraic hypersurfaces.

For reversible vector fields we must take into account in the statements of Theorem 6 the existence of the symmetric invariant algebraic curves and exponential factors.

5. Normal forms for reversible quadratic vector fields. In this section we find the normal forms of all $\varphi-$reversible quadratic polynomial vector fields defined in $\mathbf{R}^2$ such that $\dim(\mathrm{Fix}\,(\varphi)) = 1$. By Propositions 3 and 4, we can consider that involution φ is given by $\varphi(x, y) = (x, -y)$. Let X be a $\varphi-$reversible quadratic (polynomial) vector field in $\mathbf{R}^2$. Then X has the following form:

$$X(x, y) = (y(a_0 + a_1 x), -b_0 + b_1 x + b_2 x^2 + b_3 y^2). \tag{1}$$

The next result provides the normal form for the family of $\varphi-$reversible quadratic vector fields.

Lemma 7. *Any $\varphi-$reversible quadratic vector field* (1) *can be written in one of the following normal forms*:

(a) $X_1(x,y) = (y(a_0 + a_1 x), -b_0 + x^2 + b_3 y^2)$.

(b) $X_2(x,y) = (y(a_0 + a_1 x), -b_0 + x + b_3 y^2)$.

(c) $X_3(x,y) = (y(a_0 + a_1 x), -b_0 + b_3 y^2)$.

Proof. If $b_2 \neq 0$, doing the change of variables $(u,v) = (x+(b_1/2b_2), y)$, and the rescaling of the time by $T = b_2 t$, system (1) becomes $(v((a_0 - ((a_1 b_1)/(2b_2)))(1/b_2) + a_1 u), -(b_0 - ((b_1^2)/(4b_2)))(1/b_2) + u^2 + (b_3/b_2)v^2)$. So, we obtain X_1 after changing (u,v) by (x,y) and rename their coefficients. If $b_2 = 0$ and $b_1 \neq 0$, then rescaling the time by $T = b_1 t$, we get X_2. If $b_2 = b_1 = 0$, we have X_3. □

Lemma 8. *Any $\varphi-$reversible quadratic vector field $X_1(x,y)$ can be written in one of the following normal forms*:

(1) *If $a_0 a_1 \neq 0$, then $X_1^{\pm}(x,y) = (y(1 \pm x), -b_0 + x^2 + b_3 y^2)$.*

(2) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 \neq 0$, then $X_{2,\pm}(x,y) = (y, \pm 1 + x^2 + b_3 y^2)$.*

(3) *If $a_0 \neq 0$, $a_1 = 0$ and $b_3 \neq 0$, then $X_{3,\pm}(x,y) = (y, -b_0 + x^2 \pm y^2)$.*

(4) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = b_3 = 0$, then $X_4(x,y) = (y, x^2)$.*

(5) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 \neq 0$, then $X_{5,\pm}^{\pm}(x,y) = (\pm xy, \pm 1 + x^2 + b_3 y^2)$.*

(6) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 = 0$, then $X_6^{\pm}(x,y) = (\pm xy, x^2 + b_3 y^2)$.*

(7) *If $a_0 = a_1 = 0$ and $b_0 b_3 \neq 0$, then $X_{7,\pm,\pm}(x,y) = (0, \pm 1 + x^2 \pm y^2)$.*

(8) *If $a_0 = a_1 = 0$ and $b_0 \neq 0$, $b_3 = 0$, then $X_{8,\pm}(x,y) = (0, \pm 1 + x^2)$.*

(9) *If $a_0 = a_1 = 0$ and $b_0 = 0$, $b_3 \neq 0$, then $X_{9,\pm}(x,y) = (0, x^2 \pm y^2)$.*

(10) *If $a_0 = a_1 = b_0 = b_3 = 0$, then $X_{10}(x,y) = (0, x^2)$.*

Proof. After the change of variables $(x, y, t) = (\alpha x_1, \beta y_1, \gamma T)$, the vector field $X_1(x, y)$ has the form

$$\tilde{X}_1(x_1, y_1) = \Big(y_1(a_0\gamma^2\alpha + a_1\gamma^2\alpha^2 x_1), -\frac{b_0}{\alpha^2} + x_1^2 + b_3\gamma^2\alpha^2 y_1^2\Big),$$

where $\beta = \alpha^2\gamma$. If $a_0a_1 \neq 0$, then we obtain the normal form $X_1^{\pm}(x, y) = (y_1(1 \pm x_1), -b_0 + x_1^2 + b_3 y_1^2)$, where $\alpha = \pm a_0/a_1$ and $\gamma^2 = \pm a_1/a_0^2$. In a similar way, we obtain the other normal forms. □

Lemma 9. *Any φ−reversible quadratic vector field $X_2(x, y)$ can be written in one of the following normal forms*:

(1) *If $a_0a_1 \neq 0$, then $X_{11}^{\pm}(x, y) = (y(\pm 1 + x), -b_0 + x + b_3y^2)$.*

(2) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 \neq 0$, then $X_{12}^{\pm}(x, y) = (\pm y, 1 + x + b_3y^2)$.*

(3) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = 0$, $b_3 \neq 0$, then $X_{13,\pm}(x, y) = (\pm y, x + y^2)$.*

(4) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = b_3 = 0$, then $X_{14}^{\pm}(x, y) = (\pm y, x)$.*

(5) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 \neq 0$, then $X_{15,\pm}(x, y) = (xy, \pm 1 + x + b_3y^2)$.*

(6) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 = 0$, then $X_{16}(x, y) = (xy, x + b_3y^2)$.*

(7) *If $a_0 = a_1 = 0$ and $b_0b_3 \neq 0$, then $X_{17,\pm}(x, y) = (0, 1 + x \pm y^2)$.*

(8) *If $a_0 = a_1 = 0$ and $b_0 \neq 0$, $b_3 = 0$, then $X_{18}(x, y) = (0, 1 + x)$.*

(9) *If $a_0 = a_1 = 0$ and $b_0 = 0$, $b_3 \neq 0$, then $X_{19}(x, y) = (0, x + y^2)$.*

(10) *If $a_0 = a_1 = b_0 = b_3 = 0$, then $X_{20}(x, y) = (0, x)$.*

Proof. For $X_2(x, y)$ we do the same change of variables as in the proof of Lemma 8, and we obtain

$$\tilde{X}_2(x_1, y_1) = (y_1(a_0\gamma^2 + a_1\alpha\gamma^2 x_1), -b_0/\alpha + x_1 + b_3\alpha\gamma^2 y_1^2),$$

where $\beta = \alpha\gamma$. If $a_0a_1 \neq 0$, then we have the normal form $X_{11}^{\pm}(x_1, y_1) = (y_1(\pm 1 + x_1), -b_0 + x_1 + b_3y_1^2)$, doing $\alpha = \pm a_0/a_1$ and $\gamma^2 = \pm 1/a_0$. Repeating these arguments we obtain the other normal forms. □

Lemma 10. *Any φ−reversible quadratic vector field $X_3(x, y)$ can be written in one of the following normal forms*:

(1) *If* $a_0 a_1 \neq 0$ *and* $b_0 \neq 0$*, then* $X_{21,\pm}(x,y) = (y(1+x), \pm 1 + b_3 y^2)$.

(2) *If* $a_0 a_1 \neq 0$ *and* $b_0 = 0$*, then* $X_{22}(x,y) = (y(1+x), b_3 y^2)$.

(3) *If* $a_0 \neq 0$*,* $a_1 = 0$ *and* $b_0 b_3 \neq 0$*, then* $X_{23,\pm}(x,y) = (y, 1 \pm y^2)$.

(4) *If* $a_0 \neq 0$*,* $a_1 = 0$ *and* $b_0 \neq 0$*,* $b_3 = 0$*, then* $X_{24}(x,y) = (y, 1)$.

(5) *If* $a_0 \neq 0$*,* $a_1 = 0$ *and* $b_0 = 0$*,* $b_3 \neq 0$*, then* $X_{25}(x,y) = (y, y^2)$.

(6) *If* $a_0 \neq 0$*,* $a_1 = 0$ *and* $b_0 = b_3 = 0$*, then* $X_{26}(x,y) = (y, 0)$.

(7) *If* $a_0 = 0$*,* $a_1 \neq 0$ *and* $b_0 \neq 0$*, then* $X_{27,\pm}(x,y) = (xy, \pm 1 + b_3 y^2)$.

(8) *If* $a_0 = 0$*,* $a_1 \neq 0$ *and* $b_0 = 0$*, then* $X_{28}(x,y) = (xy, b_3 y^2)$.

(9) *If* $a_0 = a_1 = 0$ *and* $b_0 b_3 \neq 0$*, then* $X_{29,\pm}(x,y) = (0, 1 \pm y^2)$.

(10) *If* $a_0 = a_1 = 0$ *and* $b_0 \neq 0$*,* $b_3 = 0$*, then* $X_{30}(x,y) = (0, 1)$.

(11) *If* $a_0 = a_1 = 0$ *and* $b_0 = 0$*,* $b_3 \neq 0$*, then* $X_{31}(x,y) = (0, y^2)$.

Proof. For $X_3(x,y)$ we do the same change of variables as above and we obtain

$$\tilde{X}_3(x_1,y_1) = \left(y_1 \left(a_0 \, \frac{\beta\gamma}{\alpha} + a_1 \beta\gamma x_1 \right), -b_0 \, \frac{\gamma}{\alpha} + b_3 \beta\gamma y_1^2 \right).$$

If $a_0 a_1 \neq 0$, then we have the normal form $X_{21,\pm}(x,y) = (y(1+x), \pm 1 + b_3 y^2)$, using $\alpha = a_0/a_1$ and $\beta\gamma = 1/a_1$. Following the same arguments we obtain the other normal forms. ◻

We have the following results:

Proposition 11. *Let* $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ *be the involution* $\varphi(x,y) = (x,-y)$*. If* X *is a* $\varphi-$*reversible quadratic vector field in* $\mathbf{R}^2$*, then* X *is integrable and has the following first integrals.*

(a) *For* X_1 *we have*:

(i) *If* $a_1 \neq 0$ *and* $b_3 \notin \{0, a_1/2, a_1\}$*, then*

$$H_1(x,y) = (a_1 x + a_0)^{-2b_3} \left(\left(x - \frac{a_0}{a_1 - 2b_3} \right)^2 - (a_1 - b_3) y^2 + K \right)^{a_1},$$

where $K = -((a_1 - b_3)(-b_0(a_1 - 2b_3)^2 + a_0^2))/(b_3(a_1 - 2b_3)^2)$.

(ii) *If* $a_1 \neq 0$ *and* $b_3 = a_1/2$, *then*

$$H_2(x,y) = -2x + \frac{4b_3^2(b_3y^2 - b_0) + a_0^2}{b_3(a_0 + 2b_3x)} + 2\,\frac{a_0 \ln(a_0 + 2b_3x)}{b_3}.$$

(iii) *If* $a_1 \neq 0$ *and* $b_3 = a_1$, *then*

$$H_3(x,y) = \frac{b_0b_3^2 - a_0^2 + 4a_0(a_0 + b_3x) + 2(a_0 + b_3x)^2}{(a_0 + b_3x)^2} \times \frac{\ln(a_0 + b_3x) - b_3^3y^2}{(a_0 + b_3x)^2}.$$

(iv) *If* $a_1 \neq 0$ *and* $b_3 = 0$, *then*

$$H_4(x,y) = y^2 - \frac{(2\ln(a_0 + a_1x)(a_0^2 - a_1^2b_0) + (a_1x - a_0)^2 - a_0^2)}{a_1^3}.$$

(v) *If* $a_1 = 0$ *and* $a_0b_3 \neq 0$, *then*

$$H_5(x,y) = \exp\left(-\frac{2b_3}{a_0}\,x\right)\left((2b_3x + a_0)^2 + 4b_3^3y^2 - 4b_0b_3^2 + a_0^2\right).$$

(vi) *If* $a_1 = b_3 = 0$ *and* $a_0 \neq 0$, *then*

$$H_6(x,y) = -2x^3 + 6b_0x + 3a_0y^2.$$

(b) *For* X_2 *we have*:

(i) *If* $a_1 \neq 0$ *and* $b_3 \notin \{0, a_1/2\}$, *then*

$$H_7(x,y) = (a_0 + a_1x)^{-2b_3}\left(x - \frac{a_1 - 2b_3}{2}\,y^2 + \frac{b_0(a_1 - 2b_3) + a_0}{2b_3}\right)^{a_1}.$$

(ii) *If* $a_1 \neq 0$ *and* $b_3 = a_1/2$, *then*

$$H_8(x,y) = 2\ln(a_0 + a_1x) + \frac{-a_1^2y^2 + 2a_1b_0 + 2a_0}{a_0 + a_1x}.$$

(iii) *If* $a_1 \neq 0$ *and* $b_3 = 0$, *then*

$$H_9(x,y) = \frac{1}{a_1^2}\left(-2a_1x + a_1^2y^2 + 2\ln(a_0 + a_1x)(b_0a_1 + a_0)\right).$$

(iv) *If* $a_1 = 0$ *and* $a_0b_3 \neq 0$, *then*

$$H_{10}(x,y) = \left(\frac{x}{b_3} + y^2 + \frac{a_0 - 2b_0b_3}{2b_3^2}\right)\exp\left(-\frac{2b_3}{a_0}\,x\right).$$

(v) *If* $a_1 = b_3 = 0$ *and* $a_0 \neq 0$, *then*

$$H_{11}(x,y) = -\frac{1}{a_0}\,(x^2 - 2b_0x - a_0y^2).$$

(c) *For* X_3 *we have*:

(i) *If* $a_1 \neq 0$ *and* $b_3 \notin \{0, a_1/2\}$, *then*

$$H_{12}(x,y) = (a_1x + a_0)^{-2b_3}\left(-b_0 + b_3y^2\right)^{a_1}.$$

(ii) *If* $a_1 \neq 0$ *and* $b_3 = a_1/2$, *then*

$$H_{13}(x,y) = \frac{-b_0 + b_3y^2}{b_3(a_0 + 2b_3x)}.$$

(iii) *If* $a_1 \neq 0$ *and* $b_3 = 0$, *then*

$$H_{14}(x,y) = \frac{1}{a_1}\left(a_1y^2 + 2b_0\ln(a_0 + a_1x)\right).$$

(iv) *If* $a_1 = 0$ *and* $a_0b_3 \neq 0$, *then*

$$H_{15}(x,y) = \frac{1}{b_3}\,(b_3y^2 - b_0)\exp\frac{2b_3}{a_0}.$$

(v) *If* $a_1 = b_3 = 0$ *and* $a_0 \neq 0$, *then*

$$H_{16}(x,y) = b_0x + a_0y^2.$$

(d) *If, for* X_i *with* $i = 1, 2, 3$, $a_0 = a_1 = 0$, *then* $H_{19} = x$.

Proof. The proposition follows easily from tedious computations from the equation $XH = 0$ and using Theorem 6. □

Lemma 12. *Let* $X(x, y) = (y(a_0 + a_1x), -b_0 + b_1x + b_2x^2 + b_3y^2)$ *be a* $\varphi-$*reversible quadratic vector field with* $\varphi(x, y) = (x, -y)$. *If* $a_1 \neq 0$, *then the straight line* $L := \{(x, y) \in \mathbf{R}^2 : f(x, y) = a_0 + a_1x = 0\}$ *is an invariant algebraic curve of* X. *If we denote by* $x_0 = -a_0/a_1$ *and* $\Delta = -b_0 + b_1x_0 + b_2x_0^2$, *then* $f = 0$ *has the following characterization*

(1) *Case* $b_3 \neq 0$.

(a) *If* $\Delta b_3 < 0$, *then the straight line* L *contains two singular points of* X, *denoted by* $A_+ = (x_0, \sqrt{-\Delta/b_3})$ *and* $A_- = (x_0, -\sqrt{-\Delta/b_3})$. *So,* L *is formed by three open trajectories of* X *without contact points* $\overline{\infty A_+}$, $\overline{A_+A_-}$ *and* $\overline{A_-\infty}$. *Moreover, the direction of the trajectories of* X *is the same in* $\overline{\infty A_+}$ *and in* $\overline{A_-\infty}$, *and opposite in* $\overline{A_+A_-}$.

(b) *If* $\Delta = 0$, *then the straight line* L *contains only one singular point of* X, *denoted by* $A_0 = (x_0, 0) = L \cap \{y = 0\}$. *So,* L *is formed by two open trajectories without contact points* $\overline{\infty A_0}$ *and* $\overline{A_0\infty}$. *Moreover, the directions of the trajectories of* X *on the segments are the same.*

(c) *If* $\Delta b_3 > 0$, *then the straight line* L *has no singular points of* X *and it contains a unique trajectory.*

(2) *Case* $b_3 = 0$. *All points of the straight line* L *are singular points of* X.

Proof. We start verifying that $Xf = a_1yf$, to conclude that L is an invariant algebraic curve. As $a_1 \neq 0$, we have that $X(x_0, y) = (0, \Delta + b_3y^2)$, thus $(x_0, y(t))$ is a solution of X with initial conditions in $f = 0$ where $y(t)$ is a solution of $\dot{y} = \Delta + b_3y^2$. When $y_0 = -\Delta/b_3 > 0$, we have two asymmetric singular points of X, $A_+ \in L$ and $A_- \in L$. If $y_0 = 0$, then X has one symmetric singular point $A_0 \in \{f = 0\}$. If $y_0 < 0$, then X has no singular points in $f = 0$. Finally, we observe that the directions of trajectories depends on the sign of $\dot{y}$, and then the lemma follows. □

6. Quadratic vectors fields in the half-plane. In this section we study a particular family of quadratic vector fields defined on the half-plane which will be very useful later on for studying the $\varphi-$reversible quadratic vector fields. Here, we deal with the family of quadratic vector fields

$$Y(u,v) = (a_0 + a_1u, -b_0 + b_1u + b_2u^2 + b_3v), \tag{2}$$

defined in $v \geq 0$.

To analyze the class of $\varphi-$reversible vector fields in $y \geq 0$, the following change of coordinates is useful (see, for instance, [**21**]). So, doing the change of variables $u = x$, $v = y^2/2$, to the vector field (1) in $y \geq 0$, we get $Y(u,v) = (a_0 + a_1u, -b_0 + b_1u + b_2u^2 + 2b_3v)$ in $v \geq 0$. Therefore, by the symmetry properties, Section 3, for the reversible vector fields knowing the phase portrait of Y, we can obtain the phase portrait of X. We comment that at a regular point of S the trajectory of X is always orthogonal to S. If $(u_0, 0)$ is a singular point of X, then the trajectory $v = (u - u_0)^\alpha + h.o.t.$ with $\alpha > 0$ of Y in $v \geq 0$ tangent to $v = 0$, becomes $y = 2^{-1+\alpha/2}(x - u_0)^{\alpha/2} + h.o.t.$ for X in $y \geq 0$.

Let $\theta : \mathbf{R}^2 \to \mathbf{R}$ be the projection $\theta(u,v) = v$. In this case $S = \theta^{-1}(0)$. We say that Y has an *internal (external) fold singularity* at $p \in S$ if $Y\theta(p) = 0$ and $Y^2\theta(p) > 0$ (< 0). We say that Y has a *cusp singularity*, $p \in S$, if $Y\theta(p) = Y^2\theta(p) = 0$ and $Y^3\theta(p) \neq 0$.

We note that a fold or cusp of Y is a singular point of X, and if Y has a singular point in $\{v > 0\}$, then X has two singular points, see Section 3. Figure 2 illustrates these comments.

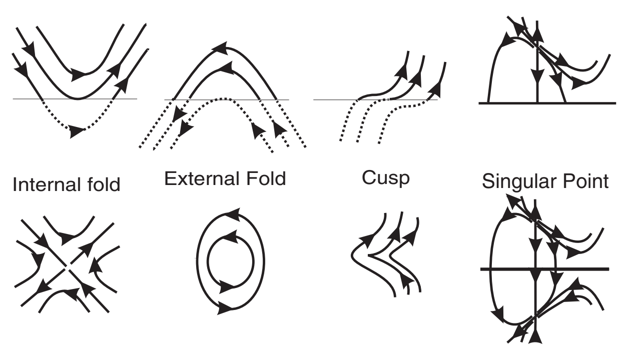

FIGURE 2. Relations between the singularities of X and Y.

Using the same arguments as in the proof of Proposition 7 and Lemmas 8, 9 and 10, we obtain the normal forms for the vector field Y associated to X, presented in the following results.

Proposition 13. *Any quadratic vector field* (2) *can be written in one of the following normal forms*:

(a) $Y_1(u,v) = (a_0 + a_1 u, -b_0 + u^2 + 2b_3 v)$.

(b) $Y_2(u,v) = (a_0 + a_1 u, -b_0 + u + 2b_3 v)$.

(c) $Y_3(u,v) = (a_0 + a_1 u, -b_0 + 2b_3 v)$.

Lemma 14. *Any $\varphi-$reversible quadratic vector field $Y_1(u,v)$ can be written in one of the following normal forms*:

(1) *If $a_0 a_1 \neq 0$, then $Y_1^{\pm}(u,v) = (1 \pm u, -b_0 + u^2 + 2b_3 v)$.*

(2) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 \neq 0$, then $Y_{2,\pm}(u,v) = (1, \pm 1 + u^2 + 2b_3 v)$.*

(3) *If $a_0 \neq 0$, $a_1 = 0$ and $b_3 \neq 0$, then $Y_{3,\pm}(u,v) = (1, -b_0 + u^2 \pm 2v)$.*

(4) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = b_3 = 0$, then $Y_4(u,v) = (1, u^2)$.*

(5) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 \neq 0$, then $Y_{5,\pm}^{\pm}(u,v) = (\pm u, \pm 1 + u^2 + 2b_3 v)$.*

(6) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 = 0$, then $Y_6^{\pm}(u,v) = (\pm u, u^2 + 2b_3 v)$.*

(7) *If $a_0 = a_1 = 0$ and $b_0 b_3 \neq 0$, then $Y_{7,\pm,\pm}(u,v) = (0, \pm 1 + u^2 \pm 2v)$.*

(8) *If $a_0 = a_1 = 0$ and $b_0 \neq 0$, $b_3 = 0$, then $Y_{8,\pm}(u,v) = (0, \pm 1 + u^2)$.*

(9) *If $a_0 = a_1 = 0$ and $b_0 = 0$, $b_3 \neq 0$, then $Y_{9,\pm}(u,v) = (0, u^2 \pm 2v)$.*

(10) *If $a_0 = a_1 = b_0 = b_3 = 0$, then $Y_{10}(u,v) = (0, u^2)$.*

Lemma 15. *Any $\varphi-$reversible quadratic vector field $Y_2(u,v)$ can be written in one of the following normal forms*:

(1) *If $a_0 a_1 \neq 0$, then $Y_{11}^{\pm}(u,v) = (\pm 1 + u, -b_0 + u + 2b_3 v)$.*

(2) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 \neq 0$, then $Y_{12}^{\pm}(u,v) = (\pm 1, 1 + u + 2b_3 v)$.*

(3) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = 0$, $b_3 \neq 0$, then $Y_{13}^{\pm}(u,v) = (\pm 1, u + 2v)$.*

(4) *If $a_0 \neq 0$, $a_1 = 0$ and $b_0 = b_3 = 0$, then $Y_{14}^{\pm}(u,v) = (\pm 1, u)$.*

(5) *If $a_0 = 0$, $a_1 \neq 0$ and $b_0 \neq 0$, then $Y_{15,\pm}(u,v) = (u, \pm 1 + u + 2b_3 v)$.*

(6) *If* $a_0 = 0$, $a_1 \neq 0$ *and* $b_0 = 0$, *then* $Y_{16}(u, v) = (u, u + 2b_3v)$.

(7) *If* $a_0 = a_1 = 0$ *and* $b_0b_3 \neq 0$, *then* $Y_{17,\pm}(u, v) = (0, 1 + u \pm 2v)$.

(8) *If* $a_0 = a_1 = 0$ *and* $b_0 \neq 0$, $b_3 = 0$, *then* $Y_{18}(u, v) = (0, 1 + u)$.

(9) *If* $a_0 = a_1 = 0$ *and* $b_0 = 0$, $b_3 \neq 0$, *then* $Y_{19}(u, v) = (0, u + 2v)$.

(10) *If* $a_0 = a_1 = b_0 = b_3 = 0$, *then* $Y_{20}(u, v) = (0, u)$.

Lemma 16. *Any* $\varphi-$*reversible quadratic vector field* $Y_3(u, v)$ *can be written in one of the following normal forms*:

(1) *If* $a_0a_1 \neq 0$ *and* $b_0 \neq 0$, *then* $Y_{21,\pm}(u, v) = (1 + u, \pm 1 + 2b_3v)$.

(2) *If* $a_0a_1 \neq 0$ *and* $b_0 = 0$, *then* $Y_{22}(u, v) = (1 + u, 2b_3v)$.

(3) *If* $a_0 \neq 0$, $a_1 = 0$ *and* $b_0b_3 \neq 0$, *then* $Y_{23,\pm}(u, v) = (1, 1 \pm 2v)$.

(4) *If* $a_0 \neq 0$, $a_1 = 0$ *and* $b_0 \neq 0$, $b_3 = 0$, *then* $Y_{24}(u, v) = (1, 1)$.

(5) *If* $a_0 \neq 0$, $a_1 = 0$ *and* $b_0 = 0$, $b_3 \neq 0$, *then* $Y_{25}(u, v) = (1, 2v)$.

(6) *If* $a_0 \neq 0$, $a_1 = 0$ *and* $b_0 = b_3 = 0$, *then* $Y_{26}(u, v) = (1, 0)$.

(7) *If* $a_0 = 0$, $a_1 \neq 0$ *and* $b_0 \neq 0$,*then* $Y_{27,\pm}(u, v) = (u, \pm 1 + 2b_3v)$.

(8) *If* $a_0 = 0$, $a_1 \neq 0$ *and* $b_0 = 0$, *then* $Y_{28}(u, v) = (u, 2b_3y)$.

(9) *If* $a_0 = a_1 = 0$ *and* $b_0b_3 \neq 0$, *then* $Y_{29,\pm}(u, v) = (0, 1 \pm 2v)$.

(10) *If* $a_0 = a_1 = 0$ *and* $b_0 \neq 0$, $b_3 = 0$, *then* $Y_{30}(u, v) = (0, 1)$.

(11) *If* $a_0 = a_1 = 0$ *and* $b_0 = 0$, $b_3 \neq 0$, *then* $Y_{31}(u, v) = (0, 2v)$.

6.1 Analysis of the family Y_1**.** In the sequel, if $a_1 \neq 0$ and $b_0 \geq 0$, we denote by $\delta = -b_0 + x_0^2$ (remember that $x_0 = -a_0/a_1$), $\delta_1 = \sqrt{b_0} + a_0/a_1$ and $\delta_2 = -\sqrt{b_0} + a_0/a_1$. We observe that $\delta = \delta_1\delta_2$ and $\delta_1 = \delta_2 = 0$ if and only if $a_0 = b_0 = 0$. As in Lemma 12, we denote by $A_0 = (x_0, 0) = L \cap \{v = 0\}$. If $b_0 > 0$, we denote by S_+ and S_- the points $(-\sqrt{b_0}, 0)$ and $(\sqrt{b_0}, 0)$, respectively. If $b_0 = 0$, then we denote by S_0 the point $(0, 0)$.

Lemma 17. *Assume for the vector field* Y_1 *that* $a_1 \neq 0$ *and* $b_0 > 0$. *If* $\delta < 0$ *then* $A_0 \in \overline{S_-S_+}$. *If* $\delta > 0$ *then* $A_0 \in \overline{\infty S_-}$ *or* $A_0 \in \overline{S_+\infty}$, *provided that* $\delta_2 > 0$ *or* $\delta_1 < 0$, *respectively. Finally,* $A_0 = S_+$ *or* $A_0 = S_-$, *if* $\delta_1 = 0$ *or* $\delta_2 = 0$, *respectively.*

Proof. As $a_1 \neq 0$, it follows that the position of point $A_0 = L \cap \{v = 0\}$ depends on $\operatorname{sgn} \delta$. So, if $\delta < 0$ then $x_0 \in (-\sqrt{b_0}, \sqrt{b_0})$. Now, if $\delta > 0$, then $|x_0| > \sqrt{b_0}$ and $\operatorname{sgn} x_0 = -\operatorname{sgn} \delta_1$. Finally, if $\delta_1 = 0$ then $x_0 > 0$ or if $\delta_2 = 0$, then $x_0 < 0$. □

Lemma 18. *Assume for the vector field Y_1 that $a_1 \neq 0$ and $b_0 = 0$. If δ_1 is positive or negative, then $A_0 \in \overline{S_0 \infty}$ or $A_0 \in \overline{\infty S_0}$, respectively. If $\delta_1 = 0$, then $A_0 = S_0$.*

Proof. As $a_1 \neq 0$, then $\delta_1 = a_0/a_1$. Thus, if $\delta_1 \neq 0$ the position of A_0 in relation to S_0 depends directly of $\operatorname{sgn} \delta_1$. So, if $\delta_1 = 0$, then $a_0 = 0$ and $A_0 = S_0$. □

Lemmas 19 and 20 characterize the isolated singular points of Y_1.

Lemma 19 (Hyperbolic singular points of Y_1). *Assume for the vector field Y_1 that $a_1 b_3 \neq 0$ and $\delta/b_3 \leq 0$. Then Y_1 has a unique hyperbolic singular point in L, $A = (-a_0/a_1, -\delta/(2b_3))$. If $\operatorname{sgn} b_3 = -\operatorname{sgn} a_1$, then A is a saddle. If $\operatorname{sgn} b_3 = \operatorname{sgn} a_1$ and negative, respectively positive, then A is an attractor, respectively repellor. Moreover, if $a_1 = 2b_3$, the singular point is a degenerate node. Note that $A = A_0$ when $\delta = 0$.*

Proof. If $p \in \mathbf{R}^2$ satisfies $Y_1(p) = 0$, then $p = A$ with $\delta/b_3 \leq 0$. Thus, Y_1 has isolated singular points if $a_1 b_3 \neq 0$. If $Y_1(A) = 0$ and $\delta/b_3 < 0$, then A is singular point of Y_1 in $v > 0$. If $\delta = 0$ we have that $A_0 \in L \cup \{v = 0\}$ or A_0 is a singular point in $v = 0$. If $a_0 \neq 0$, then $(DY_1)_A$ has eigenvalues a_1 and $2b_3$ and with eigenvectors $(-a_1(a_1 - 2b_3), 2a_0)$ and $(0, 1)$, respectively. If $a_0 = 0$, then $(DY_1)_A$ has the same eigenvalues a_1 and $2b_3$, but with eigenvectors $(1, 0)$ and $(0, 1)$, respectively. Therefore, the proof is done. □

Lemma 20 (Fold and cusp of Y_1). *Assume for the vector field Y_1 that $b_0 \geq 0$.*

(1) *If $b_0 > 0$ and $\delta \neq 0$, then the trajectories of Y_1 are tangent to $\{v = 0\}$ only at two points, S_- and S_+. Moreover, the trajectories of Y_1 which intersect $\overline{\infty S_-} \cup \overline{S_+ \infty}$, respectively $\overline{S_- S_+}$, are increasing,*

respectively decreasing. If $b_0 = 0$*, then all trajectories that intersect* $\{v = 0\}$ *are increasing, except the trajectory that is tangent at the point* $(0,0)$.

(2) *If* $b_0 > 0$*, then* Y_1 *has two-fold singularities. If* $\operatorname{sgn}\delta = -1$ *and* $\operatorname{sgn} a_1 = 1$*, respectively* -1*, the singularities* S_- *and* S_+ *are internal, respectively external, folds. If either* $\operatorname{sgn}\delta = 1$ *or* $a_1 = 0$*, the singularities* S_- *and* S_+ *are internal and external, respectively external and internal, folds, provided that* $\operatorname{sgn} a_0 = -1$*, respectively* (1).

(3) *If* $\delta = 0$ *and* $a_0 a_1 \neq 0$*, then* Y_1 *has one internal, respectively external, fold,* $S_1 = (a_0/a_1, 0)$*, provided* $\operatorname{sgn} a_1$ *is positive, respectively negative.*

(4) *If* $b_0 = 0$ *and* $a_0 \neq 0$*, then* Y_1 *has one cusp singularity at* S_0.

Proof. The tangencies between the orbits of Y_1 and S are given by the solutions of the equation:

$$Y_1\theta(u,0) = -b_0 + u^2 = 0. \tag{3}$$

If $b_0 > 0$, this equation has two solutions, and this implies Y_1 has two singularities, S_- and S_+. If $a_1 \neq 0$, then we have that ${Y_1}^2\theta(S_+) = a_1\sqrt{b_0}\,\delta_1$ and ${Y_1}^2\theta(S_-) = -a_1\sqrt{b_0}\,\delta_2$. If $a_1 = 0$, then ${Y_1}^2\theta(S_+) = 2a_0\sqrt{b_0}$ and ${Y_1}^2\theta(S_-) = -2a_0\sqrt{b_0}$. But, $\operatorname{sgn}\delta = \operatorname{sgn}\delta_1 \operatorname{sgn}\delta_2$, and this implies that if $\operatorname{sgn}\delta = -1$, respectively 1, $\operatorname{sgn}{Y_1}^2\theta(S_+) = -\operatorname{sgn}{Y_1}^2\theta(S_-) = \operatorname{sgn} a_1$, respectively $\operatorname{sgn}{Y_1}^2\theta(S_+) = \operatorname{sgn} Y_1^2\theta(S_-) = \operatorname{sgn} a_1$. If $a_1 = 0$, then $\operatorname{sgn}{Y_1}^2\theta(S_+) = -\operatorname{sgn} Y_1^2\theta(S_-) = \operatorname{sgn} a_0$, and the proof of statement (1) follows from the definition of internal and external singularities. If $b_0 = 0$ and $a_0 \neq 0$, Y_1 has a unique symmetric singularity S_0. In this case $Y_1\theta(S_0) = {Y_1}^2\theta(S_0) = 0$ and $Y_1^3\theta(S_0) = 2a_0^2 \neq 0$, so S_0 is a cusp symmetric singularity. Finally, to end the proof of this lemma, it is sufficient to observe that: if $\delta = 0$ and $a_0a_1b_3 \neq 0$, equation (3) has two solutions, $S_- = A_0$ (see Lemma 19) and S_+. For the point S_+ we have $F^2\theta(S_+) = 4a_0^2/a_1 \neq 0$. $\square$

Lemma 21 characterizes the vector fields Y_1 which have curves of singularities.

Lemma 21 (Non-hyperbolic singular points of Y_1). (1) *If* $a_1 \neq 0$ *and* $b_3 = \delta = 0$*, then* Y_1 *has the invariant straight line* $\beta : u = -a_0/a_1$,

$v \geq 0$, filled by singularities of Y_1 and, for each singularity $p \in \beta$, the $DY_1(p)$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = a_1$ and with eigenvectors $\omega_1 = (0, 1)$, $\omega_2 = (-a_1^2, 2a_0)$, respectively.

(2) If $a_0 = a_1 = b_0 = b_3 = 0$, then Y_1 has the invariant straight line $\beta : u = 0$, $v \geq 0$, filled by singularities of Y_1 and, for each singularity $p \in \beta$, the $DY_1(p)$ has the eigenvalue $\lambda = 0$ with multiplicity two, and with eigenvector $\omega = (0, 1)$.

(3) If $a_0 = a_1 = b_3 = 0$ and $b_0 > 0$, then Y_1 has two invariants straight lines $\beta_1 : u = -\sqrt{b_0}$, $v \geq 0$, $\beta_2 : u = \sqrt{b_0}$, $v \geq 0$, filled by singularities of Y_1 and, for each singularity $p \in \beta_1 \cup \beta_2$, the $DY_1(p)$ has the eigenvalue $\lambda = 0$ with multiplicity two and eigenvector $\omega = (1, 0)$.

(4) If $a_0 = a_1 = 0$ and $\operatorname{sgn} b_3 = -1$ or $a_0 = a_1 = 0$ and $\operatorname{sgn} b_3 = \operatorname{sgn} b_0 = 1$, then the singularities of Y_1 are in the curve $\beta : v = (b_0 - u^2)/(2b_3)$, $v \geq 0$. In this case, for each singularity $p = (u_0, v_0) \in \beta$, the $DY_1(p)$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2b_3$ and with eigenvectors $\omega_1 = (-b_3, u_0)$, $\omega_2 = (0, 1)$, respectively.

(5) If $a_0 = a_1 = b_0 = 0$ and $\operatorname{sgn} b_3 = 1$, then Y_1 has a unique singularity, $p = (0, 0)$ and the $DY_1(p)$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2b_3$ and with eigenvectors $\omega_1 = (-b_3, 0)$ and $\omega_2 = (0, 1)$.

Proof. The proof of the lemma, follows from the computing of solutions of $Y_1(u, v) = (a_0 + a_1 u, -b_0 + u^2 + 2b_3 v) = 0$ and the eigenvalues of $DY_1(p)$ with $p \in \{Y_1(u, v) = 0\}$. □

The next lemma describes the vectors fields Y_1 with no singularities.

Lemma 22 (Y_1 without singularities). *Assume for vector field Y_1 that $b_0 < 0$. If either $a_1 b_3 \neq 0$ and $\delta / b_3 > 0$ or $a_1 \delta \neq 0$ and $b_3 = 0$, or $a_0 = a_1 = b_3 = 0$ or $a_1 = 0$ and $a_0 \neq 0$, then Y_1 has no singularities.*

Proof. As $b_0 < 0$, then by Lemma 20, Y_1 has no folds or cusps and $Y_1(p) = 0$ implies that a_0, a_1, b_0 and b_3 satisfy one of the following conditions:

(i) if $a_1 \neq 0$, then $\operatorname{sgn} \delta = -\operatorname{sgn} b_3 \neq 0$ or $\delta = b_3 = 0$.

(ii) If $a_1 = 0$, then $a_0 = 0$ and $b_3 \neq 0$.

The proof of the lemma follows when the conditions (i) and (ii) do not hold. ◻

The next lemma characterizes the connections between singularities.

Lemma 23 (Non-connection orbits between the singular points of Y_1). *If either $a_1 = 0$, $a_0 \neq 0$ and $b_0 > 0$ or $\delta < 0$ $b_3 < 0$ and $a_1 \neq 0$, then there is no connection between the singular points of Y_1.*

Proof. In the first case, without loss of generality, we assume that $a_0 > 0$. By Lemma 20 (1), Y_1 has two singular points, an external fold, S_- and an internal fold, S_+. Let γ be a solution of Y_1 such that $\gamma(0) = S_+$. As $Y_1\theta(u,0) = -b_0 + u^2$ is negative for $-\sqrt{b_0} < u < \sqrt{b_0}$, this implies that γ decreases in this interval and increases out of this. Thus, let $Q \neq S_+ \in \overline{\infty S_-}$ be the other point that γ intersects $\{v = 0\}$. By continuity, all solutions of Y_1 passing by a point of $\overline{S_-S_+}$ cross $\{v = 0\}$ in $\overline{QS_-}$. The solutions of Y_1, crossing $\overline{\infty Q} \cup \overline{S_+\infty}$, have only this point in common with $\{v = 0\}$.

In the second case, without loss of generality, we assume that $a_1 > 0$. By Lemma 20 (1), Y_1 has two internal folds S_- and S_+. As $\delta < 0$ and $b_3 < 0$, we have that L cross $\overline{S_-S_+}$ in the point $(-a_0/a_1, 0)$ and by Lemma 17, L has no singular point of Y_1 and L is invariant by Y_1. This complete the proof of the lemma. ◻

Lemma 24. *Assume that $a_1 \neq b_3, 2b_3$ and $a_1 b_3 < 0$. Then A is a hyperbolic saddle of Y_1, and the parabola $v = h(u)$ with*

$$(4) \qquad v = h(u) = \frac{1}{2(a_1 - b_3)}\left(u - \frac{a_0}{a_1 - 2b_3}\right)^2 + \frac{b_0(a_1 - 2b_3)^2 - a_0^2}{2b_3(a_1 - 2b_3)^2};$$

contains A and two separatrices of A. If $b_0 - (a_0/(a_1 - 2b_3))^2 = 0$, then $v = h(u)$ has a quadratic contact with $\{v = 0\}$ at the point S_+, respectively S_-, if $\operatorname{sgn} a_0/(a_1 - 2b_3)$ is positive, respectively negative.

Proof. The function $v = h(u)$ is a first integral of Y_1. So $v - h(u) = 0$ is invariant. The discriminant of the equation $h(u) = 0$ is given by $D = b_3(a_1 - b_3)(-b_0(a_1 - 2b_3)^2 + a_0^2)$, and $D = 0$ implies that

$b_0 = (a_0/(a_1 - 2b_3))^2$ because $b_3 \neq 0$ and $a_1 - b_3 \neq 0$ and we have that $h''(u) = 1/(a_1 - b_3)$. So, $h(u) = h'(u) = 0$ implies that $u = a_0/(a_1 - 2b_3) = \sqrt{b_0}$ or $u = a_0/(a_1 - 2b_3) = -\sqrt{b_0}$ according to whether $\operatorname{sgn} a_0/(a_1 - 2b_3)$ is 1 or -1, respectively. □

Lemma 25 (Phase portraits of $Y_1^{\pm}$). *The vector field Y_1^+, respectively Y_1-, is topologically equivalent to Figure* 3, *respectively Figure* 4.

Proof. If $b_3 \notin \{0, 1/2, 1\}$, then by Proposition 11, the function H_1, after the change of variables $(u, v) = (x, y^2/2)$, becomes the first integral $\tilde{H}_1(u, v)$ of Y_1^+. Now we isolate the variable v in the equation $\tilde{H}_1(u, v) - k = 0$ to obtain $v = h_k(u)$.

$$h_k(u) = \frac{1}{2(1-b_3)}\left(\left(u - \frac{1}{1-2b_3}\right)^2 - k(1+u)^{2b_3} + \frac{K}{(1-2b_3)^2}\right),$$

where $K = (1 - b_3)(-b_0(1 - 2b_3)^2 + 1)/b_3$. So we have that

$$\lim_{u\to-1} h_k(u) = -\frac{-b_0+1}{2b_3} - \frac{k}{2(1-b_3)} \lim_{u\to-1}(1+1u)^{2b_3}.$$

If $b_3 = 1/2$, then as above using H_2, from Proposition 11, we get

$$h_k(u) = (2(b_0 - 1) + k + (2 + k)u - 4\ln(1 + u)(1 + u) + 2u^2).$$

So we have that $\lim_{u\to-1} h_k(u) = b_0 - 1$. If $b_3 = 0$, then as above using H_4 from Proposition 11, we have

$$h_k(u) = (k - 2u + 2(-b_0 + 1)\ln(1 + u) + u^2).$$

It follows that $\lim_{u\to-1} h_k(u) = -\operatorname{sgn} -b_0 + 1\infty$. If $b_3 = 1$, then using H_3 from Proposition 11, we get

$$h_k(u) = -(1 + u)k + 2(1 + u)^2 \ln(1 + u) + 4(1 + u) + b_0 - 1.$$

So we have that $\lim_{u\to-1} h_k(u) = (b_0 - 1)/2$.

Assume that $b_0 < 0$. By Lemma 20, the vector field Y_1^+ has no folds and cusps. So all trajectories of Y_1^+ are transversal to $v = 0$. By

Lemma 19, if $b_3 < 0$, then Y_1^+ has a unique hyperbolic saddle at A and the separatrices are given by the straight line L and the parabola (4). If $b_3 \geq 0$, then Y_1^+ has no singular points and it is topologically equivalent to the vertical vector field $(0, 1)$.

Assume that $b_0 = 0$. By Lemma 20, the vector field Y_1^+ has a cusp at S_0. By Lemma 19, if $b_3 < 0$, then Y_1^+ has a unique hyperbolic saddle at A and the separatrices are given by the straight line L and the parabola (4). If $b_3 \geq 0$, then Y_1^+ has no singular points.

Assume that $0 < b_0 < 1$. By Lemma 20, the vector field Y_1^+ has two folds, one external and the other internal at S_- and S_+, respectively. By Lemma 19, if $b_3 < 0$, then Y_1^+ has a unique hyperbolic saddle at A. If $b_3 \geq 0$, then Y_1^+ has no singular points, only the two folds. Moreover, for any b_3, we have the separatrices L and the trajectory $v = h_k(u)$ with $h_k(\sqrt{b_0}) = 0$ which has a quadratic contact at S_+.

Assume that $b_0 = 1$. By Lemma 20, the vector field Y_1^+ has one internal fold at S_+. By Lemma 19, if $b_3 < 0$, then Y_1^+ has a unique hyperbolic saddle at S_-. If $b_3 = 0$, then $Y_1^+(u, v) = (1 + u)(1, -1 + u)$. So L is filled with singular points and Y_1^+ is topologically equivalent to the vector field $(1, -1 + u)$. If $0 < b_3 \leq 1/2$, then Y_1^+ has a repellor at S_- and the trajectories starting at A are tangent to the straight line L. If $b_3 > 1/2$, then Y_1^+ has a repellor at S_- and the trajectories starting at S_- are tangent to $v = h_k(u)$ with $h_k(\sqrt{b_0}\,) = 0$. Moreover, for any $b_3 > 0$ we have that L and the trajectory $v = h_k(u)$ with $h_k(\sqrt{b_0}\,) = 0$ are separatrices of Y_1^+.

Assume that $b_0 > 1$. By Lemma 20, the vector field Y_1^+ has two internal folds at S_- and S_+. By Lemma 19, if $b_3 \leq 0$, then Y_1^+ has no singular points, only two folds. If $0 < b_3 \leq 1/2$, then Y_1^+ has a repellor at A and the trajectories starting at A are tangent to the straight line L. If $b_3 > 1/2$, then Y_1^+ has a repellor at S_- and the trajectories starting at S_- are tangent to $v = h_k(u)$ with $h_k(\sqrt{b_0}\,) = 0$. Moreover, for any $b_3 > 0$ we have that L and the trajectory $v = h_k(u)$ with $h_k(\pm\sqrt{b_0}\,) = 0$ are separatrices of Y_1^+.

The vector field Y_1^- can be studied using the same arguments as for Y_1^+. In Figures 3 and 4 we draw the phase portraits in $v \geq 0$ of Y_1^+ and Y_1^-, respectively. $\square$

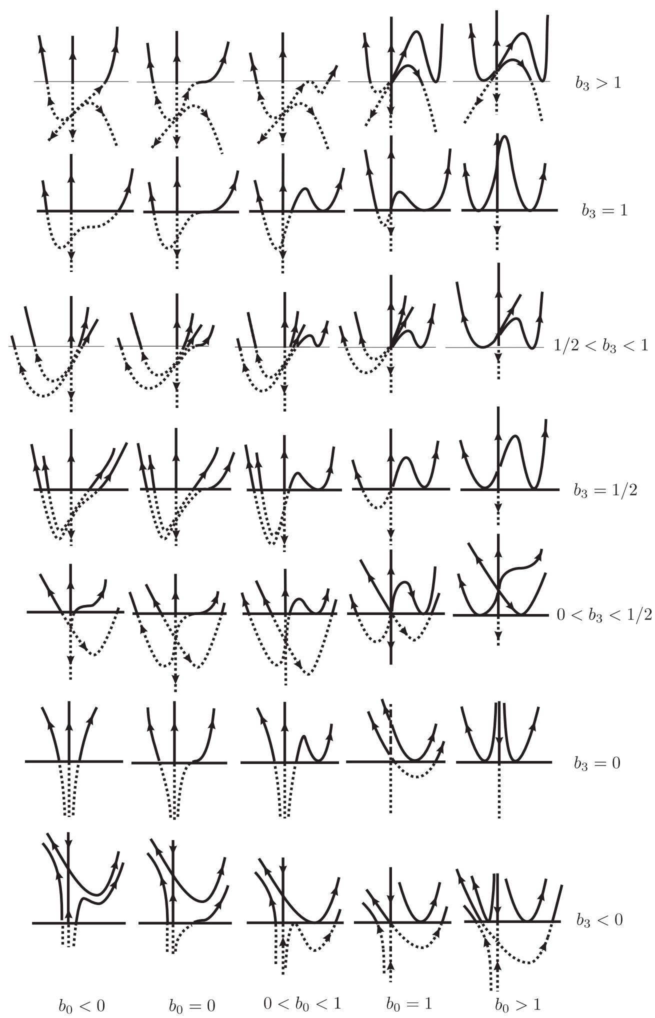

FIGURE 3. Phase portraits of Y_1^+. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Lemma 26 (Phase portraits of $Y_{2,\pm}, Y_{3,\pm}$ and Y_4). *The vector fields* $Y_{2,+}$*, respectively* $Y_{2,-}$*, and* $Y_{3,\pm}$ *such that* $b_0 > 0$*, respectively* $b_0 < 0$*, are topologically equivalent to Figure* 5 (a)*, respectively* (c)*. The vector fields* Y_4 *is topologically equivalent to Figure* 5 (b)*.*

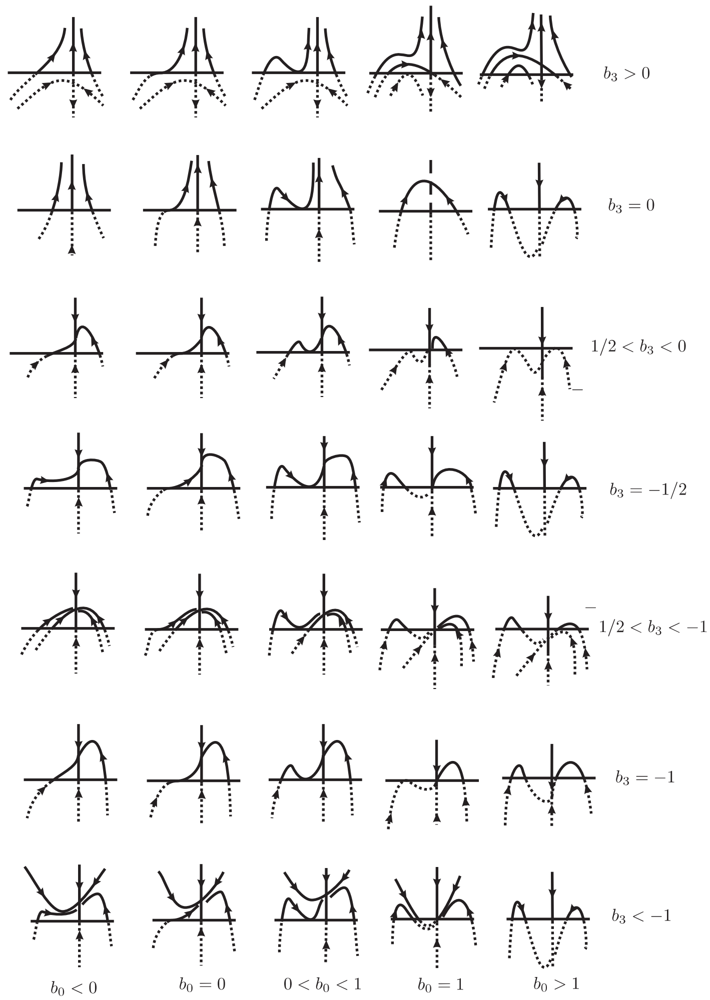

FIGURE 4. Phase portraits of Y_1^-. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Proof. Applying Proposition 11 to vector fields $Y_{2,\pm}$ with $b_3 \neq 0$ and to $Y_{3,\pm}$, we have that the function H_5, after the change of variables $(u, v) = (x, y^2/2)$, becomes a first integral $\tilde{H}_5(u, v)$ of $Y_{2,\pm}$ with $b_3 \neq 0$ and to $Y_{3,\pm}$. Now we isolate the variable v in the equation

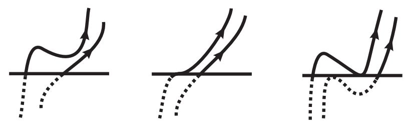

FIGURE 5. Phase portraits of $Y_{2,\pm}, Y_{3,\pm}$ and Y_4.

$\tilde{H}_5(u, v) - k = 0$ to obtain $v = h_k(u)$, where

$$h_k(u) = \frac{1}{8b_3^3}(-(2b_3u + 1)^2 + 4b_0b_3^2 - 1 + k \exp 2b_3u).$$

For the vector fields $Y_{2,\pm}$ with $b_3 = 0$ and Y_4, we use the function H_6 and in a similar way we get that $h_k(u) = k + 2u^3 - 6b_0u$.

By Lemma 22, $Y_{2,\pm}, Y_{3,\pm}$ and Y_4 have no singular points. By Lemma 20, $Y_{2,-}$ and $Y_{3,\pm}$ with $b_0 < 0$ has two folds, an external at S_- and an internal at S_+, see Figure 5 (c). The vector fields $Y_{2,+}$ and $Y_{3,\pm}$ with $b_0 > 0$ has no folds or cusps, and they are topologically equivalent to the vertical field, see Figure 5 (a). The vector fields Y_4 and $Y_{3,\pm}$ with $b_0 = 0$, have one cusp at $(0, 0)$, see Figure 5 (b). We obtain the separatrices $v = h_k(u)$ following the same arguments of the proof of Lemma 25. □

Lemma 27 (Phase portraits of $Y_{5,\pm}^{\pm}$ and $Y_6^{\pm}$). *The vector fields $Y_{5,+}^{\pm}$, respectively $Y_{5,-}^{\pm}$, are topologically equivalent to Figure* 3, *respectively Figure* 4, *for columns $b_0 < 0$ and $b_0 > 1$, respectively. The vector fields Y_6^+, respectively Y_6^-, are topologically equivalent to Figure* 6 (a), *respectively Figure* 6 (b).

Proof. In this proof we use the same arguments as in the proof of Lemma 25.

Assume that $b_0 = -1$. By Lemma 20, $Y_{5,\pm}^+$ has no folds and cusps. Hence all trajectories of $Y_{5,\pm}^+$ are transversal to $v = 0$. By Lemma 19, if $b_3 < 0$, then $Y_{5,\pm}^+$ has a unique hyperbolic saddle at A and the separatrices are given by the straight line L and the parabola (4). If $b_3 \geq 0$, then $Y_{5,\pm}^+$ has no singular points and it is topologically equivalent to vertical vector field $(0, 1)$.

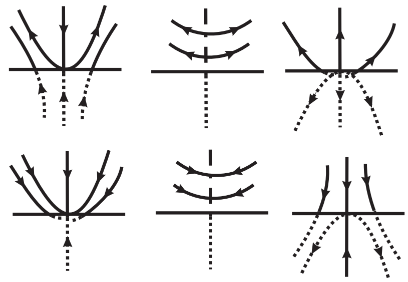

FIGURE 6. Phase portraits of $Y_6^{\pm}$. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Assume that $b_0 = 0$. By Lemma 19, if $b_3 < 0$, then Y_6^+ has a unique hyperbolic saddle at A and the separatrices are given by the straight line L and the parabola (4). If $b_3 > 0$, then Y_6^+ has no singular points. If $b_3 = 0$, then $Y_6^+(u,v) = u(1,u)$ has the line $u = 0$ filled with singular points.

Assume that $b_0 = 1$. By Lemma 20, the vector field Y_1^+ has two internal folds at S_- and S_+. By Lemma 19, if $b_3 \leq 0$, then Y_1^+ has no singular points, only two folds. If $0 < b_3 \leq 1/2$, then Y_1^+ has a repellor at A and the trajectories starting at A are tangent to straight line L. If $b_3 > 1/2$, then Y_1^+ has a repellor at S_- and the trajectories starting at S_- are tangent to $v = h_k(u)$ with $h_k(\sqrt{b_0}) = 0$. Moreover, for any $b_3 > 0$ we have that L and the trajectory $v = h_k(u)$ with $h_k(\pm\sqrt{b_0}) = 0$ are separatrices of Y_1^+.

The vector field $Y_{5,\pm}^-$ can be studied using the same arguments as for $Y_{5,\pm}^+$. In Figures 3 and 4, for $b_0 < 0$ and $b_0 > 1$, we draw the phase portraits in $v \geq 0$ of $Y_{5,+}^{\pm}$ and $Y_{5,-}^{\pm}$, respectively. In Figure 6 (a), respectively 6 (b), we draw the vector fields that are topologically equivalent to Y_6^+, respectively Y_6^-. ◻

Lemma 28 (Phase portraits of $Y_{7,\pm,\pm}$, $Y_{8,\pm}$, $Y_{9,\pm}$ and Y_{10}). *The vector fields $Y_{7,+,+}$, $Y_{7,+,-}$, $Y_{7,-,+}$, $Y_{7,-,-}$, $Y_{8,+}$, $Y_{8,-}$, $Y_{9,+}$, $Y_{9,-}$ and Y_{10} are topologically equivalent to Figure* 7 (a), (b), (c), (d), (e), (f), (g), (h) *and* (i), *respectively.*

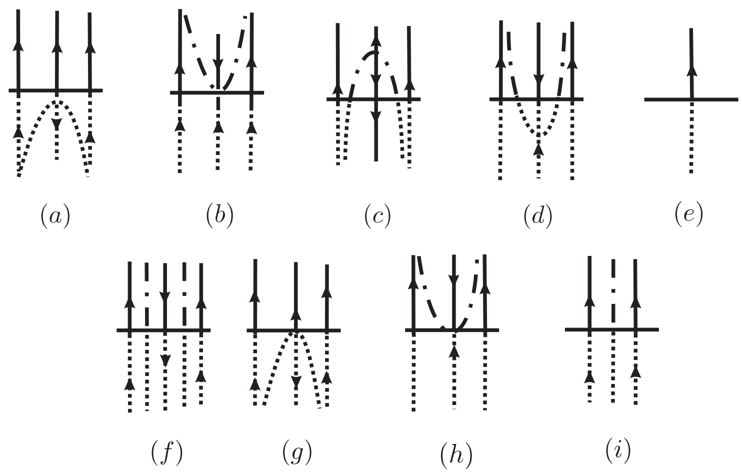

FIGURE 7. Phase portraits of $Y_{7,\pm,\pm}$, $Y_{8,\pm}$, $Y_{9,\pm}$ and Y_{10}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Proof. Except in their singular points, the vector fields $Y_{7,\pm,\pm}$, $Y_{8,\pm}$, $Y_{9,\pm}$ and Y_{10} are topologically equivalent to the vertical field. The vector fields $Y_{7,+,\pm}$, $Y_{7,-,\pm}$ and $Y_{9,\pm}$ have the parabolas $v = \mp(1+u^2)/2 \geq 0$, $v = \mp(-1+u^2)/2 \geq 0$ and $v = \mp u^2/2 \geq 0$, respectively, filled with with singular points. The vector fields $Y_{8,+}$ have no singular points. The vector field $Y_{8,-}$ has the two lines $u^2 = 1$ in $v \geq 0$ filled with singular points. Finally, Y_{10} has the line $u = 0$, filled with singular points. $\square$

6.2 Analysis of the family Y_2. In this subsection we analyze the vector field $Y_2(u,v) = (a_0 + a_1 u, -b_0 + u + 2b_3 v)$ and we draw the phase portraits of Y_2 in $\{v \geq 0\}$.

Lemma 29. *Assume for the vector field Y_2 that $a_1 \neq 0$. Then $L = \{u = -a_0/a_1\}$ is an invariant straight line of Y_2.*

Proof. As $a_1 \neq 0$, we have that $Y_2(-a_0/a_1, v) = (0, -b_0 - a_0/a_1 + 2b_3 v)$ and the proof follows. $\square$

Lemma 30 (Hyperbolic singular points of Y_2). *Assume for the vector field Y_2 that $a_1b_3 \neq 0$ and $(b_0 + a_0/a_1)/b_3 \geq 0$. Then Y_2 has a unique hyperbolic singular point in L, $A = (-a_0/a_1, (b_0 + a_0/a_1)/(2b_3))$. If $\operatorname{sgn} b_3 = -\operatorname{sgn} a_1$, then A is a hyperbolic saddle. If $\operatorname{sgn} b_3 = \operatorname{sgn} a_1$ and negative, respectively positive, then A is an attractor, respectively repellor. Moreover, if $a_1 = 2b_3$, the singular point is a degenerate node.*

Proof. If $p \in \mathbf{R}^2$ satisfies $Y_2(p) = 0$, then $p = A$ with $(b_0+a_0/a_1)/b_3 \geq 0$. Thus, Y_2 has isolated singular points if $a_1b_3 \neq 0$. The linear part $(DY_2)_A$ has eigenvalues a_1 and $2b_3$ with eigenvectors $(a_1 - 2b_3, 1)$ and $(0, 1)$, respectively. Therefore, the proof is done. □

Lemma 31 (Fold of Y_2). *Assume for Y_2 that $a_1b_0 + a_0 \neq 0$. If $a_1b_0 + a_0 > 0$, respectively < 0, then Y_2 has an internal, respectively external, fold at $S_1 = (b_0, 0)$.*

Proof. The tangencies between the orbits of Y_2 and S are given by the solutions of the equation $Y_2\theta(u, 0) = -b_0 + u = 0$. So we have that $Y_2^2\theta(b_0, 0) = a_1b_0 + a_0$, and S_1 is an internal, respectively external, fold if $\operatorname{sgn} b_0 + a_0/a_1 > 0$, respectively < 0. We observe that if $b_0 + a_0/a_1 = 0$, then S_1 is a singular point of Y_2. □

Lemma 32 (Non-hyperbolic singular points of Y_2). *Assume for Y_2 that $a_1b_0 + a_0 = 0$. If either $a_0 = a_1 = 0$, or $a_1 \neq 0$ and $b_3 = 0$, then Y_2 has a straight line filled with singular points.*

Proof. If $a_1 \neq 0$ and $b_3 = 0$, then $Y_2(-a_0/a_1, v) = 0$ for all $v \geq 0$. Hence, the straight line $(-a_0/a_1, v)$ is filled with singular points of Y_2. If $a_0 = a_1 = 0$ and $b_3 \neq 0$, respectively $b_3 = 0$, then $Y_2(b_0 - 2b_3v, v) = 0$, respectively $Y_2(b_0, v) = 0$, for all $v \geq 0$, and this ends the proof. □

Lemma 33 (Phase portraits of $Y_{11}^{\pm}$, $Y_{15,\pm}$ and Y_{16}). *The vector field Y_{11}^{+}, respectively Y_{11}^{-}, is topologically equivalent to Figure 8, respectively Figure 9. The vector fields $Y_{15,+}$, respectively $Y_{15,-}$, and Y_{16} are topologically equivalent to column $b_0 < -1$, respectively $b_0 > -1$, of Figure 8.*

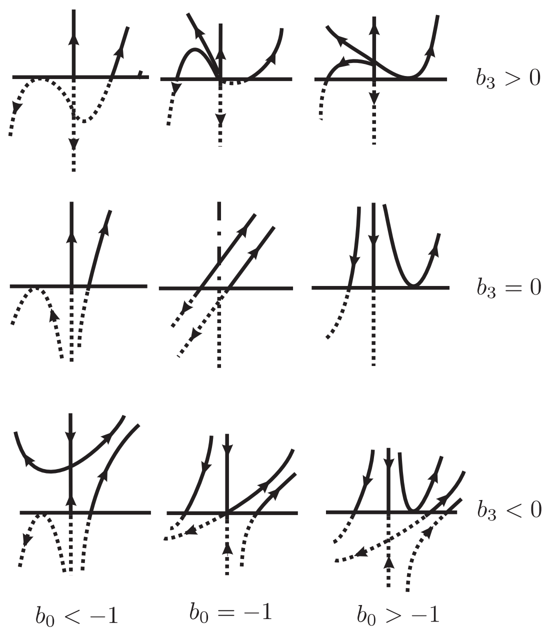

FIGURE 8. Phase portraits of Y_{11}^{+}, $Y_{15,\pm}$ and Y_{16}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Proof. We prove this lemma in a similar way to the proof of Lemma 25.

If $b_3 \notin \{0, 1/2\}$, then by Proposition 11, the function H_7, after the change of variables $(u, v) = (x, y^2/2)$, becomes the first integral $\tilde{H}_7(u, v)$ of Y_{11}^{+}. Now we isolate the variable v in the equation $\tilde{H}_7(u, v) - k = 0$ to obtain

$$v = h_k(u) = \frac{1}{2(1-b_3)} \left(\frac{1 + 2b_3 u + b_0 - 2b_0 b_3 + 1}{2b_3} - 2^{2b_3} k \right).$$

So $\lim_{u \to -1} h_k(u) = (b_0+1)/(2b_3) - (k)/(2(1-b_3)) \lim_{u \to -1} (1+1u)^{2b_3}$. If $b_3 = 1/2$, then using the same arguments for H_8, we get $v = h_k(u) = ((1+u)(k - 2\ln(1+u)) - 2(b_0 + a_0))/2$. So $\lim_{u \to -1} h_k(u) = b_0 + 1$. If $b_3 = 0$, we have $v = h_k(u) = (k + 2u - 2(1+b_0)\ln(1+u))/2$. Therefore, $\lim_{u \to -1} h_k(u) = -\mathrm{sgn}\, b_0 + 1\infty$.

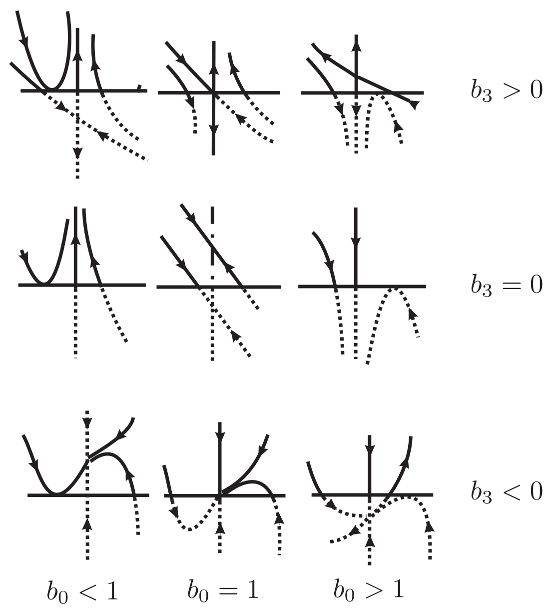

FIGURE 9. Phase portraits of Y_{11}^{-}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Assume that $b_0 < -1$. By Lemma 31, Y_{11}^{+} has one external fold at $(b_0, 0)$. By Lemma 30, if $b_3 < 0$, then Y_{11}^{+} has a unique hyperbolic saddle at A. The separatrices are L and the trajectory $v = h_k(u)$ with $h_k(-1) = (b_0 + 1)/(2b_3)$. If $b_3 \geq 0$, then Y_{11}^{+} has no singular points.

Assume that $b_0 = -1$. By Lemma 31, Y_{11}^{+} has no folds. By Lemma 30, if $b_3 < 0$, then Y_{11}^{+} has a unique hyperbolic saddle at A. The separatrices are L and the trajectory $v = h_k(u)$ with $h_k(-1) = 0$. If $b_3 \geq 0$, then Y_1^{+} has no singular points.

Assume that $b_0 > -1$. By Lemma 31, Y_{11}^{+} has one internal fold at $(b_0, 0)$. By Lemma 30, if $b_3 > 0$, then Y_{11}^{+} has a unique repellor at A. If $b_3 \leq 0$, then Y_{11}^{+} has no singular points, only the internal fold at S_1. The separatrices are the straight line L and the trajectory $v = h_k(u)$ with $h_k(b_0) = 0$ which has a quadratic contact at S_1.

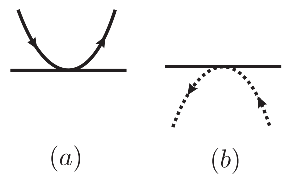

FIGURE 10. Phase portraits of $Y_{12}^{\pm}$, $Y_{13}^{\pm}$ and $Y_{14}^{\pm}$.

The vector field Y_{11}^- can be studied using the same arguments as for Y_{11}^+. In Figures 8 and 9, we draw the phase portraits in $v \geq 0$ of Y_{11}^+ and Y_{11}^-, respectively. The vector fields $Y_{15,+}$, respectively $Y_{15,-}$, and Y_{16} are topologically equivalent to column $b_0 < -1$, respectively $b_0 > -1$, of Figure 8. □

Lemma 34 (Phase portraits of $Y_{12}^{\pm}$, $Y_{13}^{\pm}$ and $Y_{14}^{\pm}$.) *The vector fields* Y_{12}^+, Y_{13}^+ *and* Y_{14}^+, *respectively* Y_{12}^-, Y_{13}^- *and* Y_{14}^-, *are topologically equivalent to Figure* 10 (a), *respectively Figure* 10 (b).

Proof. Using similar arguments as in the proof of Lemma 26, we obtain from H_{10} and H_{11}, for Y_{12}^+ with $b_3 \neq 0$, respectively $b_3 = 0$, that $h_k(u) = (2b_3u - 2b_3 + 1 - 2b_3^2k\ \exp(2b_3u))/(4b_3^2)$, respectively $h_k(u) = (u^2 - 2u + k)/2$. For Y_{13}^+, $v = h_k(u) = (2b_3u + 1 - 2b_3^2k\ \exp(2b_3u))/(4b_3^2)$. For Y_{14}^+, $v = h_k(u) = u(u+k)/2$.

The vector fields Y_{12}^+, Y_{13}^+ and Y_{14}^+ have no singular points and, by Lemma 31, Y_{12}^+ has a unique internal fold at $(-1, 0)$, and Y_{13}^+ and Y_{14}^+ at $(0,0)$. The separatrix for Y_{12}^+, respectively Y_{13}^+ and Y_{14}^+, is the trajectory $v = h_k(u)$ with $h_k(-1) = 0$, respectively $h_k(0) = 0$, which has a quadratic contact at S_1.

For the vector fields Y_{12}^-, Y_{13}^- and Y_{14}^-, applying the same arguments of this proof we draw their phase portraits in $v \geq 0$, in Figure 10 (b). □

Lemma 35 (Phase portraits of $Y_{17,\pm}$, Y_{18}, Y_{19} and Y_{20}). *The vector fields* $Y_{17,+}$ *and* Y_{19} *are topologically equivalent to Figure* 11 (a). *The vector field* $Y_{17,-}$ *is topologically equivalent to Figure* 11 (b). *The vector fields* Y_{18} *and* Y_{20} *are topologically equivalent to Figure* 11 (c).

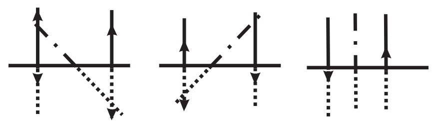

FIGURE 11. Phase portraits of $Y_{17,\pm}$, Y_{18}, Y_{19} and Y_{20}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

Proof. Except in their singular points the vector fields $Y_{17,\pm}$, Y_{18}, Y_{19} and Y_{20} are topologically equivalent to the vertical field. These vector fields have one straight line filled with singular points. □

6.3 Analysis of the family Y_3**.** In this subsection we analyze the vector field $Y_3(u,v) = (a_0 + a_1 u, -b_0 + 2b_3 v)$ and we draw the phase portraits of Y_3 in $\{v \geq 0\}$.

Lemma 36. *Assume for the vector field* Y_3 *that* $a_1 \neq 0$*. Then* $L = \{u = -a_0/a_1\}$ *is a invariant straight line of* Y_3*.*

Proof. As $a_1 \neq 0$, we have that $Y_3(-a_0/a_1, v) = (0, -b_0 + 2b_3 v)$, and the proof is done. □

Lemma 37 (Hyperbolic singular points of Y_3). *Assume for the vector field* Y_3 *that* $a_1 b_3 \neq 0$ *and* $b_0/b_3 \geq 0$*. Then* Y_3 *has a unique hyperbolic singular point in* L *at* $A = (-a_0/a_1, b_0/(2b_3))$*. If* $\operatorname{sgn} b_3 = -\operatorname{sgn} a_1$*, then* A *is a hyperbolic saddle. If* $\operatorname{sgn} b_3 = \operatorname{sgn} a_1$ *and negative, respectively positive, then* A *is an attractor, respectively repellor. Moreover, if* $a_1 = 2b_3$*, the singular point is a degenerate node.*

Proof. If $p \in \mathbf{R}^2$ satisfies $Y_3(p) = 0$, then $p = A$ with $b_0/b_3 \geq 0$. Thus, Y_3 has isolated singular points if $a_1 b_3 \neq 0$. The matrix $(DY_3)_A$ has eigenvalues a_1 and $2b_3$, with eigenvectors $(1,0)$ and $(0,1)$, respectively. So, this completes the proof. □

Lemma 38 (Orbits of Y_3 are transversal to $\{v = 0\}$). *Assume for the vector field* Y_3 *that* $b_0 \neq 0$*. Then the orbits of* Y_3 *are transversal to* $\{v = 0\}$*.*

Proof. The tangencies between the orbits of Y_3 and $S = \{v = 0\}$ are given by the solutions of the equation $Y_3\theta(u,0) = -b_0 \neq 0$. Thus, the orbits of Y_3 are transversal to S. $\square$

Lemma 39 (Non-hyperbolic singular points of Y_3). *If either $a_0 = a_1 = 0$ and $b_3 \neq 0$, or $a_1 \neq 0$ and $b_0 = b_3 = 0$, then Y_3 has a straight line filled with singular points.*

Proof. If $a_1 \neq 0$ and $b_0 = b_3 = 0$, then $Y_3(-a_0/a_1, v) = 0$ for all $v \geq 0$. Hence, $u = -a_0/a_1$ is the line of singular points of Y_3. If $a_0 = a_1 = 0$, then $Y_3(u, b_0/(2b_3)) = 0$ for all $v \geq 0$. So the proof is done. $\square$

Lemma 40 (Phase portraits of $Y_{21,\pm}$, Y_{22}, $Y_{27,\pm}$ and Y_{28}). *The vector fields $Y_{21,+}$ and $Y_{27,+}$, $Y_{21,-}$ and $Y_{27,-}$, and Y_{22} and Y_{28}, are topologically equivalent to column* (a), (b) *and* (c), *respectively, of Figure* 12.

Proof. We prove this lemma using the same arguments as in the proof of Lemma 25. Using H_{12}, H_{13} and H_{14}, we have that

$$h_k(u) = \frac{1}{2b_3}(b_0 + k(u+1)^{2b_3})$$

and

$$\lim_{u\to -1} h_k(u) = \frac{b_0}{2b_3} - \frac{k}{2b_3} \lim_{u\to -1}(1+1u)^{2b_3},$$

$$h_k(u) = b_0 + kb_3(1+u) \qquad \text{and} \qquad \lim_{u\to -1} h_k(u) = b_0,$$

and

$$h_k(u) = k - 2b_0 \ln(1+u) \quad \text{and} \quad \lim_{u\to -1} h_k(u) = -\operatorname{sgn} b_0 \infty.$$

We observe that $Y_{27,+}$, respectively $Y_{27,-}$, is topologically equivalent to $Y_{21,+}$, respectively $Y_{21,-}$, because the difference between them is a

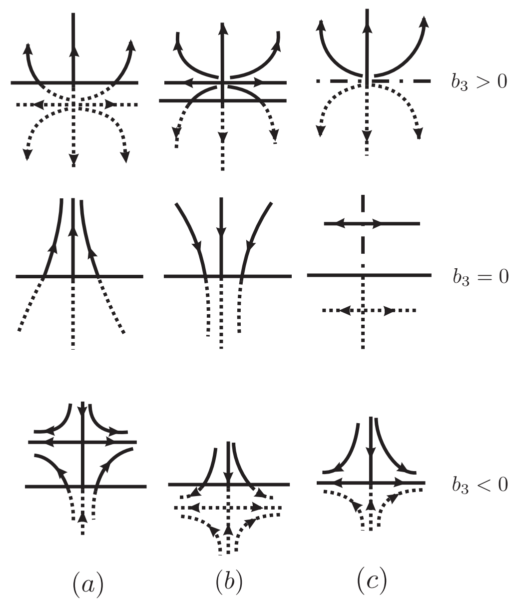

FIGURE 12. Phase portraits of $Y_{21,\pm}$, Y_{22}, Y_{27} and Y_{28}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

translated of their invariant straight line. The same occurs with Y_{22} and Y_{28}.

By Lemma 38, the orbits of vector fields $Y_{21,\pm}$ and Y_{22} are transversal to $v = 0$. By Lemma 37, if $b_3 < 0$, then $Y_{21,+}$ has a unique hyperbolic saddle at A and the separatrices for $Y_{21,+}$ are L and the trajectory $v = h_k(u)$ with $h_k(-1) = b_0/(2b_3)$. If $b_3 \geq 0$, then $Y_{21,+}$ has no singular points. By Lemma 37, if $b_3 < 0$, then Y_{22} has a unique hyperbolic saddle at A and the separatrices for Y_{22} are L and the trajectory $v = h_k(u)$ with $h_k(-1) = 0$. If $b_3 \geq 0$, then Y_{22} has no singular points. By Lemma 37, if $b_3 > 0$, then $Y_{21,-}$ has a unique repellor at A. If $b_3 \leq 0$, then Y_{11}^+ has no singular points and the proof is done. □

The proof of Lemmas 41 and 42 will be omitted here because they are similar to the proofs of Lemmas 34 and 35.

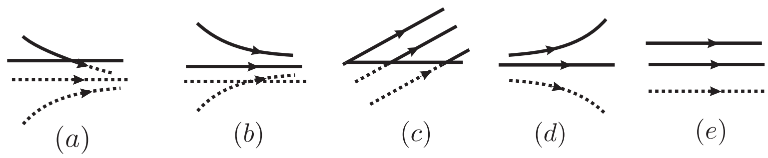

FIGURE 13. Phase portraits of $Y_{23,-}$, $Y_{23,+}$, Y_{24}, Y_{25} and Y_{26}.

Lemma 41 (Phase portraits of $Y_{23,\pm}$, Y_{24}, Y_{25} and Y_{26}). *The vector fields* $Y_{23,-}$, $Y_{23,+}$, Y_{24}, Y_{25} *and* Y_{26} *are topologically equivalent to Figure* 13 (a), (b), (c), (d) *and* (e)*, respectively.*

Lemma 42 (Phase portraits of $Y_{29,\pm}$, Y_{30} and Y_{31}). *The vector fields* $Y_{29,-}$, $Y_{29,+}$, Y_{30} *and* Y_{31} *are topologically equivalent to Figure* 14 (a), (b), (c) *and* (d)*, respectively.*

7. Phase portraits of $\varphi-$reversible quadratic vector fields. In this section, in order to prove Theorem A, we use the normal forms of Proposition 7 for φ-reversible quadratic vector fields. We remember that, for drawing the phase portraits of φ-reversible quadratic vector fields X in $\mathbf{R}^2$, we use the phase portrait of the associated vector field Y defined in $v \geq 0$ and the symmetry properties of X. So, using the phase portraits given in Lemmas 17–42 and the symmetry properties of reversible vector fields, we shall prove Lemmas 44–52. In these lemmas, using Neumann's theorem, see Theorem 1, we show

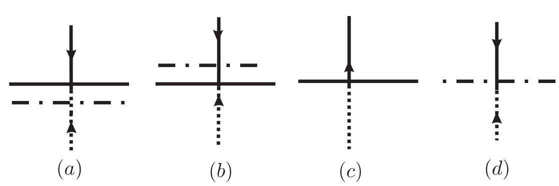

FIGURE 14. Phase portraits of $Y_{29,-}$, $Y_{29,+}$, Y_{30} and Y_{31}. The dotted lines in $v \geq 0$ denote lines filled with singular points.

all the phase portraits of φ-reversible quadratic vector fields in the Poincaré disc, drawing their separatrices and sometimes one orbit for every canonical region.

Lemma 43. *If $a_1 - b_3 \leq 0$, then the vector field X_1 has a unique infinite singular point at $I_1 = (0,0)$ in U_2. If $a_1 - b_3 > 0$, then X_1 has three infinite singular points in U_2, the I_1, $I_2 = (\sqrt{a_1 - b_3}, 0)$ and $I_3 = (-\sqrt{a_1 - b_3}, 0)$. The $(0,0)$ of U_1 is never a singular point for X_1.*

Proof. In the local charts U_1 and U_2, the compactified vector field associated to X_1 is given by $Z_1(z_1, z_2) = (1 - (a_1 - b_3)z_1^2 - b_0 z_2^2 - a_0 z_1^2 z_2, -a_1 z_1 z_2 - a_0 z_1 z_2^2)$ and $Z_2(z_1, z_2) = (a_0 z_2 + (a_1 - b_3)z_1 - z_1^3 + b_0 z_1 z_2^2, -b_3 z_2 + b_0 z_2^3 - z_1^2 z_2)$, respectively. The point $(0,0)$ is not a singular point for the vector field Z_1 because $Z_1(0,0) = (1,0)$. So, we only consider the infinite singular points in U_2. If $a_1 - b_3 \leq 0$, then Z_2 has a unique infinite singular point, the I_1 in U_2. If $a_1 - b_3 > 0$, then Z_2 has three infinite singular points the I_1, I_2 and I_3. □

Lemma 44 (Phase portraits of $X_1^{\pm}$, $X_{5,\pm}^{+}$ and X_6^{+}). *The phase portraits of the vector field X_1^+, respectively X_1^-, is topologically equivalent to Figure* 15*, respectively Figure* 16*. The phase portraits of the vector field X_5^+, respectively X_5^-, is topologically equivalent to column $b_0 < 0$, respectively $b_0 > 0$, of Figure* 15*, respectively Figure* 16*. The phase portraits of the vector field $X_6^{\pm}$ is topologically equivalent to Figure* 17*.*

Proof. For drawing the phase portraits in the Poincaré disc, we use Lemmas 25 and 27 and the following characterization of the infinite singular points.

For the vector field X_1^+, we have that

$$Z_2 = (z_2 + (1 - b_3)z_1 - z_1^3 + b_0 z_1 z_2^2, -b_3 z_2 + b_0 z_2^3 - z_1 z_2^2). \tag{5}$$

The I_1 is a singular point of Z_2, and $(DZ_2)_{I_1}$ has eigenvalues $1 - b_3$ and $-b_3$ with eigenvectors $((1,0)$ and $(-1,1)$. If $1 - b_3 > 0$, then I_2 and I_3 are singular points of Z_2. Their linear parts $(DZ_2)_{I_2}$ and $(DZ_2)_{I_3}$ have the same eigenvalues, $-2(1 - b_3)$ and -1 with eigenvectors $(1,0)$ and $(1, 1 - 2b_3)$.

If $b_3 < 0$, then I_1 is a repellor and I_2, I_3 are attractors.

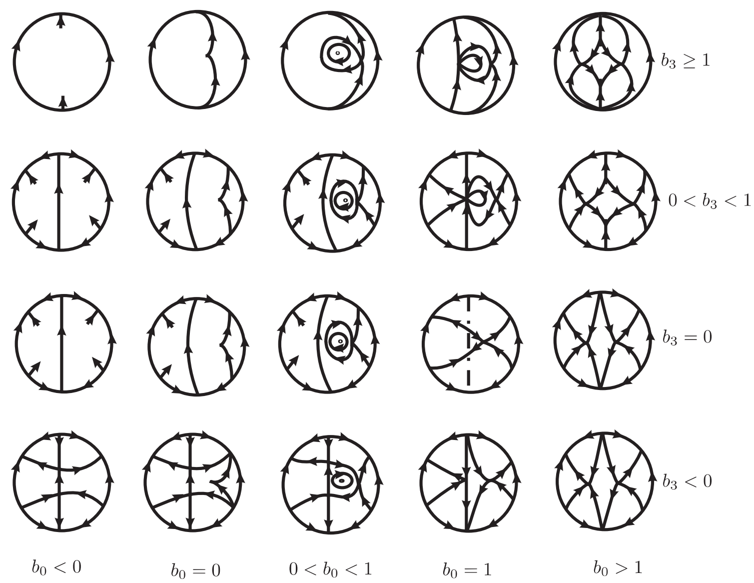

FIGURE 15. Phase portraits of X_1^+ and X_5^+. The dotted lines denote lines filled with singular points.

If $b_3 = 0$ then I_2, I_3 are attractors and for I_1, we have that:

(1) If $b_0 \neq 1$, then (5), after a linear change of variables, has the form $Z_2(z_1, z_2) = ((-b_0+1)z_1^3+2z_1^2z_2-z_1z_2^2, z_2-z_2^3+2z_1z_2^2+(-b_0+1)z_1z_2^2)$, and in order to apply the elementary degenerate theorem, we have the function $g(z_1) = (-b_0+1)z_1^3+\cdots$. Hence, as $m = 3$, if $b_0 > 1$, respectively $b_0 < 1$, then I_1 is a topological unstable node, respectively saddle.

(2) If $b_0 = 1$, then $Z_2(z_1, z_2) = (z_1 + z_2 - z_1^3 + z_1z_2^2, -z_1^2z_2 + z_2^3)$. The singular points of Z_2 are the straight line $z_1 = -z_2$, and for each point of this straight line, the Jacobian matrix associated to Z_2 has eigenvalues 1 and 0 with eigenvectors $(1, 0)$ and $(-1, 1)$.

If $0 < b_3 < 1$, then I_1 is a hyperbolic saddle and I_2 and I_3 are attractors.

If $b_3 = 1$, then (5) has the form $Z_2(z_1, z_2) = (a_0z_2 - z_1^3 + b_0z_1z_2^2, -z_2 - z_1^2z_2 + b_0z_2^3)$. The Jacobian matrix associated to Z_2 at I_1 has eigenvalues

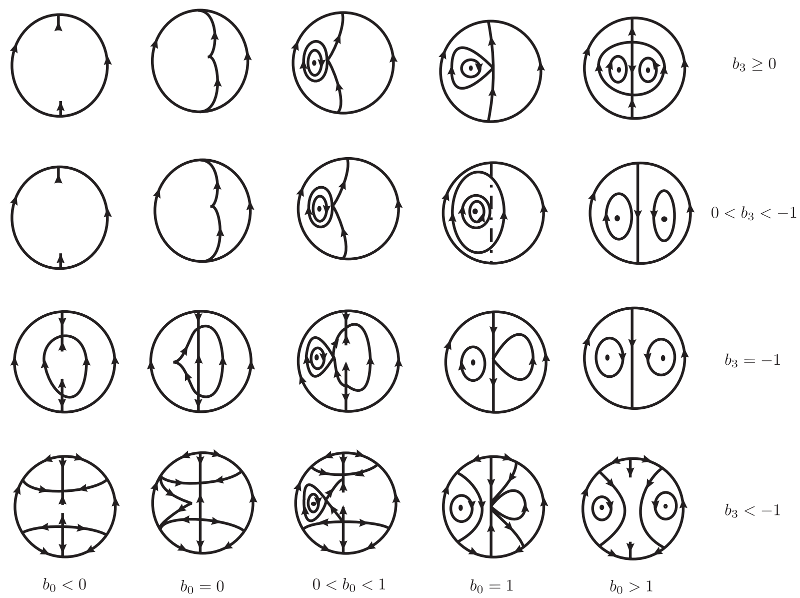

FIGURE 16. Phase portraits of X_1^- and X_5^-. The dotted lines denote lines filled with singular points.

0 and -1 with eigenvectors $(1, 0)$ and $(-1, 1)$. After a linear change of coordinates and applying the elementary degenerate theorem, we have that the function $g(x) = z_2^3$. So I_1 is a topological stable node.

If $b_3 > 1$, then I_1 is a hyperbolic node.

Using similar arguments we study the infinite singular points of X_1^-. If $b_3 < -1$, then I_1 is a repellor and I_2 and I_3 are hyperbolic saddles. If $b_3 = -1$, then I_1 is a topological saddle. If $-1 < b_3 < 0$, then I_1 is a hyperbolic saddle. If $b_3 = 0$ and $b_0 < 1$, respectively $b_0 > 1$, then I_1 is a topological saddle, respectively stable node. If $b_3 = 0$ and $b_0 = 1$, then I_1 belongs to the straight line of singular points $x = a_0$. If $b_3 > 0$, then I_1 is an attractor.

The vector field X_5^+, respectively X_5^-, is topologically equivalent to X_1^+, respectively X_1^-, for $b_0 < 0$, respectively $b_0 > 1$.

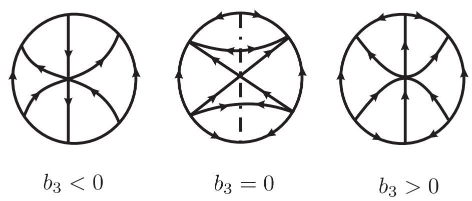

FIGURE 17. Phase portraits of $X_6^{\pm}$. The dotted lines denote lines filled with singular points.

From Lemma 27, X_6^+ is topologically equivalent to X_6^-. The description of infinite singular points is the same as for X_1, except when $b_3 = 1$. For $b_3 = 1$, we have that $Z_2(z_1, z_2) = (-z_1^3 + b_0 z_1 z_2^2, -z_2 - z_1^2 z_2 + b_0 z_2^3)$, and $(DZ_2)_{I_1}$ has eigenvalues 0 and -1 with eigenvectors $(1, 0)$ and $(0, 1)$. So, by the elementary degenerate theorem, it follows that I_1 is a topological stable node. □

Lemma 45 (Phase portraits of $X_{2,\pm}$, $X_{3,\pm}$ and X_4). *If $b_3 < 0$ or $b_3 \geq 0$, then the phase portrait of the vector field $X_{2,+}$, respectively $X_{2,-}$, is topologically equivalent to Figure* 18 (a) *or* (d)*, respectively* (c) *or* (f)*. If either $b_0 < 0$ or $b_0 = 0$ or $b_0 > 0$, then the phase portrait*

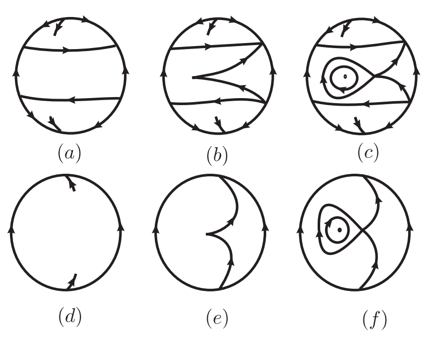

FIGURE 18. Phase portraits of $X_{2,\pm}$, $X_{3,\pm}$ and X_4.

of the vector field $X_{3,+}$*, respectively* $X_{3,-}$*, is topologically equivalent to Figure* 18 (d) *or* (e) *or* (f)*, respectively* (a) *or* (b) *or* (c)*. The vector field* X_4 *is topologically equivalent to Figure* 18 (e)*.*

Proof. We use Lemma 26 and the same arguments as in the proof of Lemma 44. The point I_1 is a singular point of Z_2, and $(DZ_2)_{I_1}$ has eigenvalue $-b_3$ with multiplicity 2, having eigenvector $(1,0)$. If $b_3 < 0$ the linear parts $(DZ_2)_{I_2}$ and $(DZ_2)_{I_3}$ have the same eigenvalues, $2b_3$ and 0, with eigenvectors $(1,0)$ and $(1,2b_3)$.

If $b_3 < 0$, then I_1 is a repellor, and to study I_2 and I_3, we apply the elementary degenerate theorem. Thus,

$$g(z_1) = \frac{\sqrt{-b_3}}{2b_3^2} z_1^2 + \cdots \quad \text{for} \quad I_2,$$

and

$$g(z_1) = -\frac{\sqrt{-b_3}}{2b_3^2} z_1^2 + \cdots \quad \text{for} \quad I_3.$$

Hence, I_2 and I_3 are saddle-nodes.

If $b_3 = 0$, applying the nilpotent theorem we obtain that $f(z_1) = -z_1^5(1+\cdots)$, $\Phi(z_1) = -4z_1^2(1+\cdots)$ and $b^2 + 4a(\beta+1) = 4$. Thus, I_1 is a topological stable node.

If $b_3 > 0$, then I_1 is an attractor. □

Lemma 46 (Phase portraits of $X_{7,\pm,\pm}$, $X_{8,\pm}$, $X_{9,\pm}$ and X_{10}). *The phase portraits of the vector fields* $X_{7,-,-}$*,* $X_{9,-}$*,* $X_{7,+,-}$*,* $X_{8,-}$*,* X_{10}*,* $X_{8,+}$*,* $X_{7,-,+}$*,* $X_{9,+}$ *and* $X_{7,+,+}$ *are topologically equivalent to Figure* 19 (a), (b), (c), (d), (e), (f), (g), (h) *and* (i)*, respectively.*

Proof. We use Lemma 28 and the same arguments as in the proof of Lemma 44. First, we consider the vector field $X_{7,\pm,+}$ and, by Lemma 43, we have that

$$Z_2 = (-b_3 z_1 - z_1^3 + b_0 z_1 z_2^2, -b_3 z_2 + b_0 z_2^3 - z_1^2 z_2), \tag{6}$$

The point I_1 is a singular point of Z_2, and $(DZ_2)_{I_1}$ has eigenvalue $-b_3$ with multiplicity 2 with eigenvectors $(1,0)$ and $(0,1)$. If $b_3 < 0$,

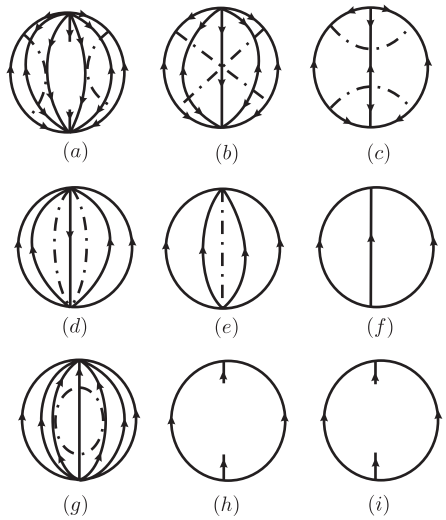

FIGURE 19. Phase portraits of $X_{7,\pm,\pm}$, $X_{8,\pm}$, $X_{9,\pm}$ and X_{10}. The dotted lines denote lines filled with singular points.

then $(DZ_2)_{I_2}$ and $(DZ_2)_{I_3}$ have the same eigenvalues, $2b_3$ and 0 with eigenvectors $(1,0)$ and $(0,1)$, respectively.

The vector field $X_{7,\pm,+}$ has a unique infinite singular point, the I_1, and it is an attractor.

The vector field $X_{7,\pm,-}$ has three infinite singular points, a repellor, in I_1 and, applying the elementary degenerate theorem, we have topological stable node at that I_2 and I_3 because $g(z_1) = z_1^5/8(1+\cdots)$.

Associated to $X_{8,+}$ we obtain from (6), $Z_2 = (z_1^2 + z_2^2)(-z_1, -z_2)$. So, Z_2 is topologically equivalent to (z_1, z_2) and has a unique infinite singular point, the topological stable node at I_1.

Associated to $X_{8,-}$ we obtain from (6), $Z_2 = (-z_1^2 + z_2^2)(z_1, z_2)$. So Z_2 is topologically equivalent to (z_1, z_2) and has the infinite singular point I_1. Moreover, the straight lines $z_2 = \pm z_1$ are filled by singular points of Z_2.

For the vector field $X_{9,+}$, we get $Z_2(z_1, z_2) = (1 + z_1^2)(-z_1, -z_2)$. So I_1 is a topological stable node.

For the vector field $X_{9,-}$, we obtain $Z_2(z_1, z_2) = (1 - z_1^2)(z_1, z_2)$ that has the infinite singular point I_1. The vector field Z_2 has two straight lines $z_1 = \pm 1$ filled with singular points.

For X_{10} we get $Z_2(z_1, z_2) = z_1^2(-z_1, -z_2)$; it is topologically equivalent to the vector field $(-z_1, -z_2)$ and has the straight line $z_1 = 0$ filled with singular points. □

Lemma 47 (Phase portraits of $X_{11}^{\pm}$, $X_{15,\pm}$ and X_{16}). *The phase portraits of the vector fields $X_{11}^{\pm}$ are topologically equivalent to $X_{11,+}$, and these are topologically equivalent to Figure* 20. *The vector fields $X_{15,+}$, respectively $X_{15,-}$, and X_{16} are topologically equivalent to X_{11}^{+} for $b_0 < -1$, respectively $b_0 > -1$.*

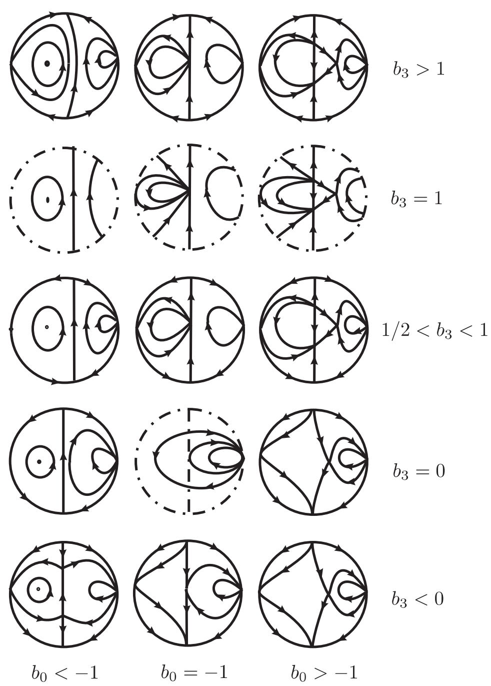

FIGURE 20. Phase portraits of $X_{11}^{\pm}$, $X_{15,\pm}$ and X_{16}. The dotted lines denote lines filled with singular points.

Proof. From Lemma 33, we get that the phase portraits of Y_{11}^+ is topologically equivalent to $Y_{11,-}$, and the vector fields $Y_{15,+}$, respectively $Y_{15,-}$, and Y_{16} are topologically equivalent to Y_{11}^+ for $b_0 < -1$, respectively $b_0 > -1$.

The associated compactified vector field to X_{11}^+ in the local chart U_1 is

$$Z_1(z_1, z_2) = (z_2 - (1-b_3)z_1^2 - b_0 z_2^2 - z_1^2 z_2, -z_1 z_2 - z_1 z_2^2).$$

If $b_3 \neq 1$, then Z_1 has a unique infinite singular point in U_1 at $(0,0)$. Using the nilpotent theorem, since $F(z_1) = (1-b_3)z_1^2 + \cdots$, $f(z_1) = -(1-b_3)z_1^3(1+\cdots)$, $\Phi(z_1) = -(3-2b_3)z_1(1+\cdots)$ and $b^2+4a(\beta+1) = (1-2b_3)^2 \geq 0$, we have that if $b_3 < 1$, respectively $b_3 > 1$, then it is a singularity whose neighborhood is the union of a hyperbolic and an elliptic sector (index $+1$), see Figure 31 (e), respectively topological saddle. If $b_3 = 1$, then $Z_1(z_1, z_2) = z_2(1 - b_0 z_2 - z_1^2, -z_1 - z_1 z_2)$. Hence, the equator is filled with infinite singular points.

The associated compactified vector field to X_{11}^+ in the local chart U_2 is

$$Z_2(z_1, z_2) = ((1-b_3)z_1 + z_2 - z_1^2 z_2 + b_0 z_1 z_2^2, -b_3 z_2 + b_0 z_2^3 - z_1 z_2^2).$$

If $b_3 \neq 1$, then Z_2 has a unique infinite singular point in U_2 at $(0,0)$. The linear part $(DZ_2)_{(0,0)}$ has eigenvalues $1-b_3$ and $-b_3$, with eigenvectors $(1,0)$ and $(-1,1)$. Thus, if either $b_3 < 0$ or $0 < b_3 < 1$ or $b_3 > 1$, then X_{11}^+ has a unique infinite singular point at $(0,0) \in U_2$, and it is either a repellor, or hyperbolic saddle or attractor, respectively. If $b_3 = 1$, then $Z_2(z_1, z_2) = z_2(1 - z_1^2 + b_0 z_1 z_2, -1 + b_0 z_2^2 - z_1 z_2)$. So the equator is filled with infinite singular points. If $b_3 = 0$ and $1 + b_0 > 0$, respectively $1 + b_0 < 0$, then, using the elementary degenerate theorem, where $f(z_1) = 0$, $g(z_1) = (\pm 1 + b_0)z_1^3(1+\cdots)$, we have that $(0,0) \in U_2$ is a topological node, respectively saddle.

The infinite singular points in U_1 and U_2 of $X_{15,\pm}$ and X_{16} are the same of X_{11}^+. $\square$

Lemma 48 (Phase portraits of $X_{12}^\pm$, $X_{13}^\pm$ and $X_{14}^\pm$). *The phase portraits of the vector field X_{12}^+, respectively X_{12}^-, are topologically equivalent to Figure* 21 (a), *respectively Figure* 21 (b). *The vector fields X_{13}^+ and X_{14}^+, respectively X_{13}^- and X_{14}^-, are topologically equivalent X_{12}^+, respectively X_{12}^-.*

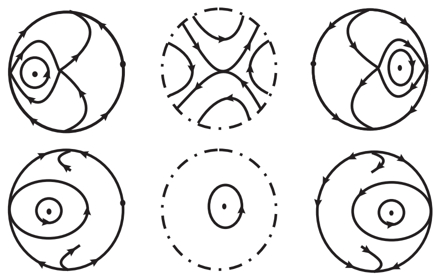

FIGURE 21. Phase portraits of $X_{12}^{\pm}$, $X_{13}^{\pm}$, and $X_{14}^{\pm}$. The dotted lines denote lines filled with singular points.

Proof. By Lemma 34, it is sufficient that we analyze $X_{12}^{\pm}$. The associated compactified vector field to $X_{12}^{\pm}$ in the local chart U_1 is $Z_1(z_1,z_2) = (z_2 + b_3z_1^2 - b_0z_2^2 \mp z_1^2z_2, \mp z_1z_2^2)$. So, if $b_3 \neq 0$, then Z_1 has a unique infinite singular point in U_1 at $(0,0)$. Using the nilpotent theorem where $F(z_1) = -b_3z_1^2 + \cdots$, $f(z_1) = \mp b_3^2z_1^5(1+\cdots)$, $\Phi(z_1) = 2b_3z_1(1+\cdots)$ and $b^2 + 4a(\beta+1) = (1-2b_3)^2 \geq 0$, we get that the $(0,0) \in U_1$ is an infinite singular point of X_{12}^{+}, whose neighborhood is the union of a hyperbolic and an elliptic sector (index $+1$), see Figure 31 (e). The vector field X_{12}^{-} has an infinite singular point at $(0,0) \in U_1$ and it is a topological saddle. If $b_3 = 0$, then $Z_1(z_1,z_2) = z_2(1 - b_0z_2 \mp z_1^2, \mp z_1z_2)$. So, the equator is filled with infinite singular points.

The associated compactified vector field to $X_{12}\pm$ in the local chart U_2 is

$$Z_1(z_1,z_2) = (-b_3z_1 \pm z_2 - z_1^2z_2 + b_0z_1z_2^2, -b_3z_2 + b_0z_2^3 - z_1z_2^2).$$

If $b_3 \neq 0$, then Z_2 has a unique infinite singular point in U_2 at $(0,0)$ and $(DZ_2)_{(0,0)}$ has eigenvalue $-b_3$ with multiplicity 2, having eigenvector $(1,0)$. So, if $b_3 < 0$, respectively $b_3 > 0$, then $X_{12}^{\pm}$ has a unique infinite singular point at $(0,0) \in U_2$ and it is a repellor, respectively attractor. If $b_3 = 0$, then $Z_2(z_1,z_2) = z_2(\pm 1 - z_1^2 + b_0z_1z_2, b_0z_2^2 - z_1z_2)$. So, the equator is filled with infinite singular points. □

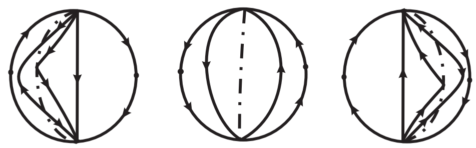

FIGURE 22. Phase portraits of $X_{17,\pm}$, X_{18}, X_{19} and X_{20}. The dotted lines denote lines filled with singular points.

Lemma 49 (Phase portraits of $X_{17,\pm}$, X_{18}, X_{19} and X_{20}). *The phase portraits of the vector fields $X_{17,+}$ and X_{19}, X_{18} and X_{20}, and $X_{17,-}$ are topologically equivalent to Figure* 22 (a), (b) *and* (c)*, respectively.*

Proof. The associated compactified vector field to $X_{17,\pm}$ in the local chart U_1 is $Z_1(z_1, z_2) = (z_2 \pm z_1^2 + z_2^2)(1, 0)$. Then Z_1 is topologically equivalent to the vector field $(1, 0)$ and has the curve $z_2 \pm z_1^2 + z_2^2 = 0$ filled with singular points.

The associated compactified vector field to $X_{17,\pm}$ in the local chart U_2 is $Z_2(z_1, z_2) = (\pm 1 - z_2^2 - z_1 z_2)(-z_1, -z_2)$. It follows that Z_2 is topologically equivalent to the vector field $(-z_1, -z_2)$, and it has the curve $\pm 1 - z_2^2 - z_1 z_2 = 0$ filled with singular points.

The associated compactified vector field to X_{18} in the local chart U_1 is $Z_1(z_1, z_2) = (z_2 + z_2^2)(1, 0)$. So Z_1 is topologically equivalent to the vector field $(1, 0)$ and has the curve $z_2(1 + z_2) = 0$ filled with singular points. The associated compactified vector field to X_{18} in local chart U_2 is $Z_2(z_1, z_2) = (-z_2^2 - z_1 z_2)(-z_1, -z_2)$. Hence, Z_2 is topologically equivalent to vector field $(-z_1, -z_2)$ and has the curve $-z_2^2 - z_1 z_2 = 0$ filled with singular points.

The associated compactified vector field to X_{19} in the local chart U_1 is $Z_1(z_1, z_2) = (z_2 + z_1^2)(1, 0)$. The vector field Z_1 is topologically equivalent to the vector field $(1, 0)$ and has the curve $z_2 + z_1^2 = 0$ filled with singular points. The associated compactified vector field to X_{19} in the local chart U_2 is $Z_2(z_1, z_2) = (1 - z_1 z_2)(-z_1, -z_2)$. It follows that Z_2 is topologically equivalent to vector field $(-z_1, -z_2)$, and it has the curve $z_1 z_2 = 1$ filled with singular points.

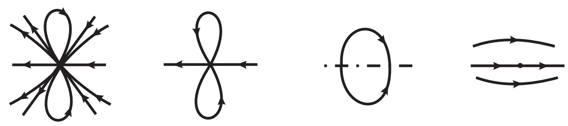

FIGURE 23. Infinite singular points in U_1 for the vector field $X_{21,+}$. The dotted lines denote lines filled with singular points.

The associated compactified vector field to X_{20} in the local chart U_1 is $Z_1(z_1, z_2) = z_2(1, 0)$. Thus, Z_1 is topologically equivalent to the vector field $(1, 0)$, and it has the curve $z_2 = 0$ filled with singular points. The associated compactified vector field to X_{20} in the local chart U_2 is $Z_2(z_1, z_2) = z_1 z_2(-z_1, -z_2)$. The vector field Z_2 is topologically equivalent to the vector field $(-z_1, -z_2)$, and it has the curves $z_1 z_2 = 0$ filled with singular points. □

Lemma 50 (Phase portraits of $X_{21,\pm}$, X_{22}, $X_{27,\pm}$ and X_{28}). *The phase portraits of the vector fields $X_{21,+}$, X_{22}, or $X_{21,-}$, are topologically equivalent to Figure* 26 (a), (b), *or* (c), *respectively. The vector fields $X_{27,+}$, respectively $X_{27,-}$, is topologically equivalent to $X_{21,+}$, respectively $X_{21,-}$, and X_{28} to X_{22}.*

Proof. The associated compactified vector field to $X_{21,\pm}$ in the local chart U_1 is

$$Z_1(z_1, z_2) = (\mp z_2^2 + (b_3 - 1)z_1^2 - z_1^2 z_2, -z_1 z_2^2 - z_1 z_2).$$

If $b_3 = 1$, respectively $b_3 \neq 1$, then Z_1 has the equator filled with infinite singular points, respectively a unique infinite singular point at $(0, 0)$. In order to analyze the infinite singular point at $(0, 0)$ of Z_1 for $b_3 \neq 1$, we use directional blow-up and polar blow-up for obtaining Figures 23 and 24 for $X_{21,+}$ and $X_{21,-}$, respectively. We observe that for $X_{27,+}$, respectively $X_{27,-}$, the infinite singular points in U_1 have the same characterization of $X_{21,+}$, respectively $X_{21,-}$. Using the same arguments for X_{22} and X_{28}, we obtain Figure 25.

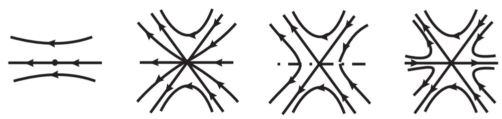

FIGURE 24. Infinite singular points in U_1 for the vector field $X_{21,-}$. The dotted lines denote lines filled with singular points.

The associated compactified vector field to $X_{21,\pm}$ in the local chart U_2 is

$$Z_2(z_1, z_2) = ((1 - b_3)z_1 + z_2 \pm z_1 z_2^2, -b_3 z_2 \pm z_2^3).$$

If $b_3 = 1$, respectively $b_3 \neq 1$, then Z_2 has the equator filled with infinite singular points, respectively a unique infinite singular point at $(0, 0)$. The matrix $D(Z_2)_{(0,0)}$ has eigenvalues $1-b_3$ and $-b_3$, with eigenvectors $(1, 0)$ and $(-a_0, 1)$. If $b_3 < 0$, then $(0, 0) \in U_2$ is a repellor. If $b_3 = 0$, then by the elementary degenerate theorem, we have that the infinite singular point $(0, 0) \in U_2$ of $X_{21,-}$ and $X_{27,-}$, respectively $X_{21,+}$ and $X_{27,+}$, is a topological saddle, respectively node. For X_{22} and X_{28}, $\mathbf{S}^1$ is filled with infinite singular points and Z_2 is topologically equivalent to a vertical vector field. If $0 < b_3 < 1$, then the infinite singular point $(0, 0) \in U_2$ is a hyperbolic saddle. If $b_3 = 1$, then $\mathbf{S}^1$ is filled with infinite singular points and Z_2 is topologically equivalent to a vertical vector field. If $b_3 > 1$, then the infinite singular point $(0, 0) \in U_2$ is an attractor. So, we use Lemma 40 and we draw the phase portraits of these vector fields. □

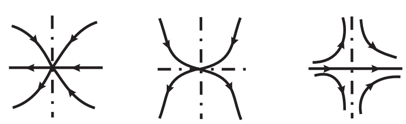

FIGURE 25. Infinite singular points in U_1 for the vector field X_{22}. The dotted lines denote lines filled with singular points.

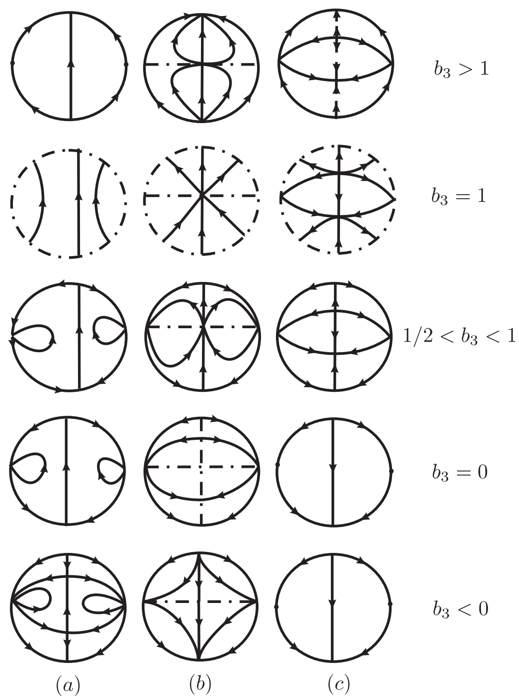

FIGURE 26. Phase portraits of $X_{21,\pm}$, X_{22}, $X_{27,\pm}$ and X_{28}. The dotted lines denote lines filled with singular points.

Lemma 51 (Phase portraits of $X_{23,-}$, $X_{23,+}$ X_{24}, X_{25} and X_{26}). *The phase portraits of the vector fields* $X_{23,-}$, $X_{23,+}$, X_{24}, X_{25} *and* X_{26} *are topologically equivalent to Figure* 29 (a), (b), (c), (d) *and* (e), *respectively.*

Proof. The associated compactified vector field to $X_{23,\pm}$ in the local chart U_1 is $Z_1(z_1, z_2) = (-z_2^2 \pm z_1^2 - z_1^2 z_2, -z_1 z_2^2)$. For the vector fields X_{24}, X_{25} and X_{26}, we have that $Z_1(z_1, z_2) = z_2(-z_2 - z_1^2 z_2, -z_1)$, $Z_1(z_1, z_2) = z_1(z_1 - z_1 z_2, -z_2^2)$ and $Z_1(z_1, z_2) = z_1 z_2(-z_1, -z_2)$, respectively. For these vector fields we use directional blow-up and polar blow-up, and we obtain Figure 27 (a), (b), (c), (d) and (e), where

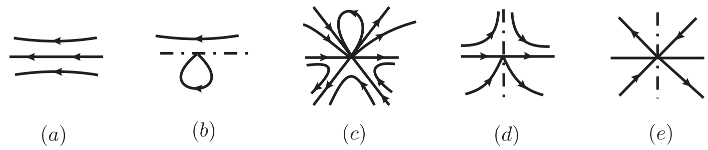

FIGURE 27. Infinite singular points in U_1 for the vector field X_{23}. The dotted lines denote lines filled with singular points.

we draw the local phase portrait at the infinite singular point of Z_1 associated to $X_{23,-}$, $X_{23,+}$, X_{24}, X_{25} and X_{26}, respectively.

The associated compactified vector field to $X_{23,\pm}$ in the local chart U_2 is $Z_2(z_1, z_2) = (-b_3 z_1 + z_2 + z_1 z_2^2, -b_3 z_2 + z_2^3)$. The matrix $D(Z_2)_{(0,0)}$ has eigenvalue $-b_3$ of the multiplicity 2, having eigenvectors $(1, 0)$ and $(-1, 1)$. If $b_3 < 0$, then $(0, 0) \in U_2$ is a repellor. If $b_3 = 0$, then $Z_2(z_1, z_2) = z_2(1 + z_1 z_2, z_2^2)$. So, $\mathbf{S}^1$ is filled with infinite singular points and Z_2 is topologically equivalent to a horizontal vector field. If $b_3 > 0$, then the infinite singular point $(0, 0) \in U_2$ is an attractor. So, we use Lemma 41 and we draw the phase portraits of these vector fields. $\square$

Lemma 52 (Phase portraits of $X_{29,-}$, $X_{29,+}$, X_{30} and X_{31}). *The phase portraits of the vector fields $X_{29,-}$, $X_{29,+}$, X_{30} and X_{31} are topologically equivalent to Figure* 29 (a), (b), (c) *and* (d).

Proof. The associated compactified vector field to $X_{29,\pm}$ in the local chart U_1 is $Z_1(z_1, z_2) = (-z_2^2 \pm z_1^2)(1, 0)$. For the vector fields X_{30}

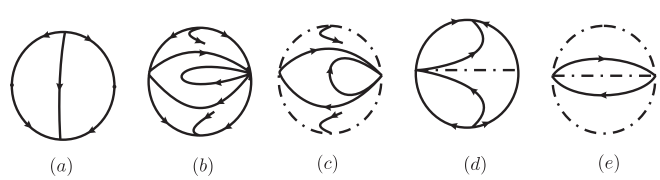

FIGURE 28. Phase portraits of $X_{23,-}$, $X_{23,+}$, X_{24}, X_{25} and X_{26}. The dotted lines denote lines filled with singular points.

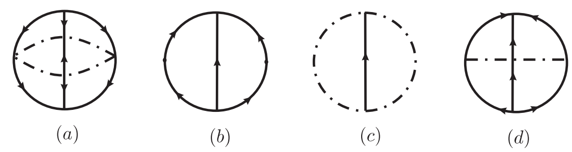

FIGURE 29. Phase portraits of $X_{29,-}$, $X_{29,+}$, X_{30} and X_{31}. The dotted lines denote lines filled with singular points.

and X_{31}, we have that $Z_1(z_1, z_2) = z_2^2(1, 0)$ and $Z_1(z_1, z_2) = z_1^2(1, 0)$, respectively. These vector fields are topologically equivalent to a horizontal field.

The associated compactified vector field to $X_{29,\pm}$ in the local chart U_2 is $Z_2(z_1, z_2) = (\mp z_1 + z_1 z_2^2, \mp z_2 + z_2^3)$. So the vector field $X_{29,-}$, respectively $X_{29,+}$, has an infinite singular point at $(0,0) \in U_2$ and it is a repellor, respectively an attractor. For the vector field X_{30}, respectively X_{31}, we have that $Z_2(z_1, z_2) = z_2^2(z_1, z_2)$, respectively $Z_2(z_1, z_2) = (-z_1, -z_2))$. So we use Lemma 42 and we draw the phase portraits of these vector fields. □

Appendix

The next theorem corresponds to Theorem 65 of [**2**].

Theorem 53 (Elementary Degenerate Theorem). *Let* $(0,0)$ *be an isolated singularity of the system* $(\dot{x}, \dot{y}) = (X(x,y), y + Y(x,y))$, *where* X *and* Y *are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in* x *and* y. *Let* $y = f(x)$ *be the solution of the equation* $y + Y(x,y) = 0$ *in the neighborhood of* $(0,0)$, *and assume that the series expansions of the function* $g(x) = X(x, f(x))$ *has the form* $g(x) = a_m x^m + \cdots$, *where* $m \geq 2$, $a_m \neq 0$. *Then*

(1) *If* m *is odd and* $a_m > 0$, *then* $(0,0)$ *is a topological node.*

(2) *If* m *is odd and* $a_m < 0$, *then* $(0,0)$ *is a topological saddle, two of whose separatrices tend to* $(0,0)$ *in the directions* 0 *and* π, *the other two in the directions* $\pi/2$ *and* $3\pi/2$.

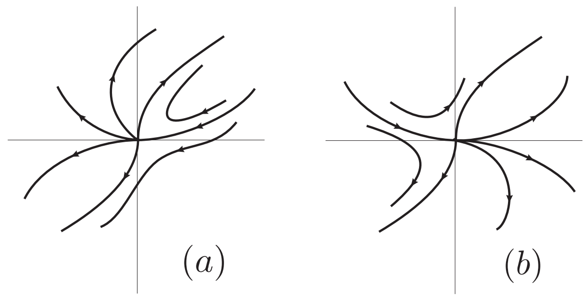

FIGURE 30. The elementary degenerate saddle-nodes (the orientation of the orbits can be reversed).

(3) *If* m *is even, then* $(0,0)$ *is a saddle-node, i.e., a singularity whose neighborhood is the union of one parabolic and two hyperbolic sectors, two of whose separatrices tend to* $(0,0)$ *in the directions* $\pi/2$ *and* $3\pi/2$*, and the other in the direction* 0 *or* π *according to* $a_m < 0$ (*Figure* 30 (a)) *or* $a_m > 0$ (*Figure* 30 (b)).

The corresponding topological indices of these singular points are $+1, -1, 0$, *so they may serve to distinguish the three types.*

For the proof of the following theorem, see [**1**], or Theorems 66 and 67 of [**2**].

Theorem 54 (Nilpotent Theorem). *Let* $(0,0)$ *be an isolated singularity of the system* $(\dot{x}, \dot{y}) = (y + X(x,y), Y(x,y))$, *where* X *and* Y *are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in* x *and* y. *Let* $y = F(x)$ *be the solution of the equation* $y + X(x,y) = 0$ *in the neighborhood of* $(0,0)$, *and assume that the series expansions for the functions* $f(x) = Y(x, f(x)) = ax^{\alpha}(1+\cdots)$ *and* $\Phi(x) = ((\partial X)/(\partial x) + (\partial Y)/(\partial y))(x, F(x)) = bx^{\beta}(1+\cdots)$, *where* $a \neq 0$, $\alpha \geq 2$ *and* $\beta \geq 1$. *Then*

(1) *If* α *is even, and*

(a) $\alpha > 2\beta + 1$, *then the origin is a saddle-node* (*index* 0), *see Figure* 31 (a).

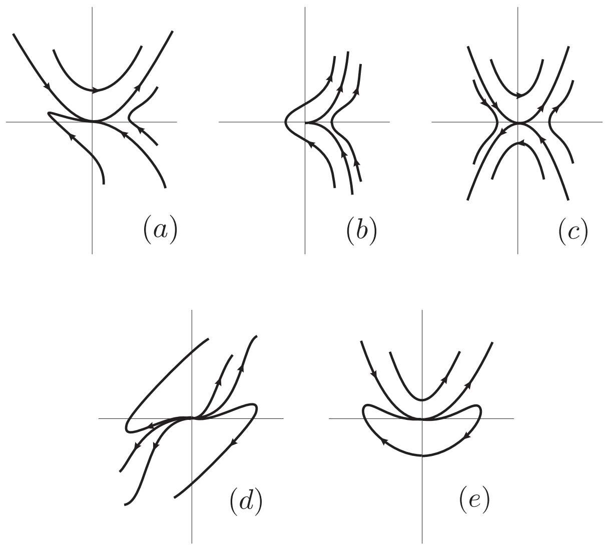

FIGURE 31. The local behavior near a nilpotent singularity (the orientation of the orbits can be reversed).

(b) *Either* $\alpha < 2\beta + 1$ *or* $\Phi(x) \equiv 0$, *then the origin is a singularity whose neighborhood is the union of two hyperbolic sectors* (*index* 0), *see Figure* 31 (b).

(2) *If* α *is odd and* $a > 0$, *then the origin is a saddle* (*index* -1), *see Figure* 31 (c).

(3) *If* α *is odd,* $a < 0$, *and*

(a) *either* $\alpha > 2\beta + 1$ *and* β *even; or* $\alpha = 2\beta + 1$, β *even and* $b^2 + 4a(\beta + 1) \geq 0$, *then the origin is a node* (*index* $+1$), *see Figure* 31 (d). *The node is stable if* $b < 0$, *or unstable if* $b > 0$.

(b) *Either* $\alpha > 2\beta + 1$ *and* β *odd, or* $\alpha = 2\beta + 1$, β *odd and* $b^2 + 4a(\beta + 1) \geq 0$, *then the origin is the union of a hyperbolic and an elliptic sector* (*index* $+1$), *see Figure* 31 (e).

(c) *Either* $\alpha = 2\beta + 1$ *and* $b^2 + 4a(\beta + 1) < 0$, *or* $\alpha < 2\beta + 1$ (*or* $\Phi(x) \equiv 0$), *then the origin is either a focus, or a center, respectively* (*index* $+1$).

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 – Bellaterra, Barcelona, Spain
E-mail address: `jllibre@mat.uab.es`

Instituto de Matemática e Estatística, Universidade Federal de Goiás, 74011-970 – Goiânia, Goiás, Brazil
E-mail address: `medrado@mat.ufg.br`