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CONSTRUCTING COMPLETE PROJECTIVELY FLAT CONNECTIONS

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ABSTRACT. On any open subset U of the Euclidean space $\mathbf{R}^n$ there is complete torsion-free connection whose geodesics are reparameterizations of the intersections of the straight lines of $\mathbf{R}^n$ with U. For any positive integer m, there is a complete projectively flat torsion free connection on the two-dimensional torus such that for any point p there is another point q so that any broken geodesic from p to q has at least m breaks. This example is also homogeneous with respect to a transitive Lie group action.

1. Introduction.

The purpose of this note is to tie up a couple of loose ends in the classical theory of linear connections. First, in [**6**, p. 395], Spivak raises the question of if, on a compact manifold with complete connection, any two points can be joined by a geodesic. The answer is "no" even when the connection is projectively flat and homogeneous:

Theorem 1. *Let T^2 be the two-dimensional torus. Then, for any positive integer m, there is a complete torsion free projectively flat connection, ∇, on T^2 such that for any point $p \in T^2$ there is a point $q \in T^2$ with the property that any broken ∇-geodesic between p and q has at least m breaks. Moreover if T^2 is viewed as a Lie group in the usual manner, this connection is invariant under translations by elements of T^2.*

Another natural question is: For a connected open subset, U, of the Euclidean space, $\mathbf{R}^n$, is the usual flat connection restricted to U projectively equivalent to complete torsion free connection on U? This is true and is a special case of a more general result about connections on incomplete Riemannian manifolds.

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Theorem 2. *Let (M,g) be a not necessarily complete Riemannian manifold. Then there is a complete torsion free connection on M that is projective with the metric connection on M. In particular, any connected open subset M of the Euclidean space, $\mathbf{R}^n$, has a complete torsion free connection ∇ such that the geodesics of ∇ are reparameterizations of straight line segments of $M \subseteq \mathbf{R}^n$.*

The main tool is Proposition 2.2, which gives an elementary method of constructing complete torsion free connections that are projective with a given torsion free connection.

1.1. Definitions, notation and preliminaries. All of our manifolds are smooth, i.e., C^∞, Hausdorff, paracompact and connected. The tangent bundle of M is denoted by $T(M)$. If $f{:}M \to N$ is a smooth map between manifolds, then the derivative map is $f_{*x}{:}T(M)_x \to T(M)_{f(x)}$.

We will use the term *connection* to stand for a linear connection on the tangent bundle, also called a Koszul connection, as defined in [**4**, Proposition 2.8, p. 123 and Proposition 7.5, p. 143] or [**6**, p. 241]. Let $c{:}(a,b) \to M$ be a smooth immersed curve. Then c is a is a ∇-*geodesic* if and only if $\nabla_{c'(t)}c'(t) = 0$. The curve is a ∇-*pregeodesic* if and only if there is a reparameterization of c that is a geodesic. This is equivalent to $\nabla_{c'(t)}c'(t) = \alpha(t)c'(t)$ for some smooth function $\alpha{:}(a,b) \to \mathbf{R}$. Given a pregeodesic $c{:}(a,b) \to M$, then an *affine parameterization* of c is a reparameterization $\sigma{:}(a_1,b_1) \to (a,b)$ so that $c \circ \sigma$ is a geodesic.

If $f{:}M \to N$ is a local diffeomorphism and ∇ is a connection on N, then the *pull back connection* is the connection $f^*\nabla$ defined on M by $f_*\big((f^*\nabla)_X Y\big) = \nabla_{f_*X} f_*Y$. The connection ∇ on M is *homogeneous* on M if and only if there is a transitive action on M by a Lie group, G, so that $\phi^*\nabla = \nabla$ for all $\phi \in G$.

Two connections $\overline{\overline{\nabla}}$ and ∇ on M are *projective* if and only if all geodesics of $\overline{\overline{\nabla}}$ are pregeodesics of ∇. This is an equivalence relation on the set of connections on M. If ∇_i is a connection on M_i for $i = 1,2$, then a map $f{:}M_1 \to M_2$ is a *projective map* if and only if it is a local diffeomorphism and maps ∇_1-geodesics to ∇_2-pregeodesics. This is equivalent to the connections ∇_1 and $f^*\nabla_2$ on M_1 being projective. The connection ∇ is *projectively flat* if and only if every point $p \in M$

has an open neighborhood U and projective map $f{:}U \to \mathbf{R}^n$ where $\mathbf{R}^n$ has its standard flat connection. Or, what is the same thing, for every geodesic c of M the image $f \circ c$ is a reparameterization of an interval in a line of $\mathbf{R}^n$. There is a well-known criterion, due to Hermann Weyl, for two connections to be projective. A proof can be found in [**6**, Corollary 19, p. 277].

1.1. Proposition (H. Weyl). *Two connections $\overline{\nabla}$ and ∇ on a manifold are projective and have the same torsion tensor if and only if there is a smooth one form ω so that the connections are related by*

$$\nabla_X Y = \overline{\nabla}_X Y + \omega(X)Y + \omega(Y)X. \tag{1.1}$$

Therefore, if this relation holds and $\overline{\nabla}$ is torsion free, then so is ∇. □

Only the easy direction of this result will be used. That is, if $\overline{\nabla}$ is torsion free and ∇ is given by (1.1), then ∇ is torsion free and projective with $\overline{\nabla}$. Note in this case if $c{:}(a,b) \to M$ is a $\overline{\nabla}$-geodesic, then (1.1) implies $\nabla_{c'(t)}c'(t) = 2\omega(c'(t))c'(t)$, and therefore c is a ∇-pregeodesic. That ∇ is torsion free is equally as elementary.

The connection ∇ is *complete* if and only if every ∇-geodesic defined on a subinterval of $\mathbf{R}$ extends to a ∇-geodesic defined on all of $\mathbf{R}$. Letting $\exp^{\nabla}$ be the exponential of ∇, cf. [**4**, p. 140], then ∇ is easily seen to be complete if and only if the domain of $\exp^{\nabla}$ is all of $T(M)$. A curve $c{:}[0,b) \to M$ is an *inextendible $\overline{\nabla}$-geodesic ray* if and only if c is a $\overline{\nabla}$-geodesic and has no extension to $[0, b+\varepsilon)$ as a $\overline{\nabla}$-geodesic for any $\varepsilon > 0$. Therefore when $b = \infty$, so that $[0,\infty)$ is the domain of c, c is always inextendible.

1.2. Proposition. *Let $\overline{\nabla}$ be a torsion free connection on the manifold M, and let ∇ be torsion free and projective with $\overline{\nabla}$. Then ∇ is complete if and only if every inextendible $\overline{\nabla}$-geodesic ray $c{:}[0,b) \to M$ has an orientation preserving reparameterization $\sigma{:}[0,\infty) \to [0,b)$ such that $c \circ \sigma$ is a ∇-geodesic.*

Proof. First assume that the reparameterization condition holds, and we will show that ∇ is complete by showing the domain of the exponential map of ∇ is all of $T(M)$. Let $v \in T(M)$. As 0 is in the domain of $\exp^{\nabla}$, assume $v \neq 0$. Let $c{:}[0,b) \to M$ be the inextendible $\overline{\nabla}$-geodesic ray with $c'(0) = v$. By assumption, there is an orientation preserving reparameterization $\sigma{:}[0,\infty) \to [0,b)$ such that $\tilde{c} := c \circ \sigma$ is a ∇-geodesic. As the reparameterization is orientation preserving $\tilde{c}'(0) = \lambda c'(0) = v$ for some positive constant λ. Then $\hat{c}{:}[0,\infty) \to M$ given by $\hat{c}(t) := \tilde{c}(t/\lambda)$ is also a ∇-geodesic and $\hat{c}'(0) = v$. From the definition of $\exp^{\nabla}$ we have for all $t \geq 0$ that tv is in the domain of $\exp^{\nabla}$ and $\exp^{\nabla}(tv) = \hat{c}(t)$. In particular, letting $t = 1$ shows that v is in the domain of $\exp^{\nabla}$ and completes the proof that ∇ is complete.

Conversely, assume ∇ is complete and let $c{:}[0,b) \to M$ be an inextendible $\overline{\nabla}$-geodesic ray. Assume, toward a contradiction, there is an orientation preserving reparameterization $\sigma{:}[0,b_1) \to [0,b)$ with $b_1 < \infty$ and so that $\tilde{c} = c\circ\sigma$ is a ∇ geodesic. Then, as ∇ is complete, the curve $\tilde{c}$ extends to a ∇-geodesic $\hat{c}{:}[0,\infty) \to M$ and therefore is a proper extension of $\tilde{c}$. But then $\hat{c}$ can be reparameterized as a $\overline{\nabla}$-geodesic that extends c, contradicting that c was an inextendible $\overline{\nabla}$-geodesic ray and completing the proof. □

2. Constructing complete projectively equivalent connections on incomplete Riemannian manifolds. We first observe that, for some choices of the one form ω in Weyl's result 1.1, there is an explicit formula for reparameterizing a $\overline{\nabla}$-geodesic as a ∇-geodesic.

2.1. Lemma. *Let $\overline{\nabla}$ be a smooth manifold, let $\overline{\nabla}$ be a connection on M and let $v{:}M \to (0,\infty)$ be a smooth positive function. Define a new connection by*

$$\nabla_X Y = \overline{\nabla}_X Y + \frac{1}{2v}\, dv(X)Y + \frac{1}{2v}\, dv(Y)X. \tag{2.1}$$

Let $c{:}(a,b) \to M$ be a $\overline{\nabla}$-geodesic and $\sigma{:}(\alpha,\beta) \to (a,b)$ be an orientation preserving reparameterization of c so that $\tilde{c} = c \circ \sigma$ is a ∇-geodesic.

Then the inverse of σ, $\sigma^{-1}:(a,b) \to (\alpha,\beta)$, *is given by*

$$\sigma^{-1}(t) = C_0 + C_1 \int_{t_0}^{t} v(c(\tau))\, d\tau \tag{2.2}$$

where $t_0 \in (a,b)$, $C_0, C_1 \in \mathbf{R}$ *and* $C_1 > 0$.

Proof. Let t be the natural coordinate on (a,b) and s the coordinate on (α,β) related to t by $t = \sigma(s)$. Our goal is to find $s = s(t) = \sigma^{-1}(t)$. Note $dt = \sigma'(s)\, ds$ so that $\sigma'(s) = dt/ds$. Therefore,

$$\tilde{c}'(s) = (c \circ \sigma)'(s) = \sigma'(s)c'(\sigma(s)) = \left. \frac{dt}{ds}\frac{dc}{dt} \right|_{t=\sigma(s)}.$$

Because of this, and because it makes applications of the chain rule easier to follow, we will denote $\tilde{c}'(s)$ as dc/ds and think of s as "the affine parameter for ∇ along c". We will abuse notation a bit and write $v(t) = v(c(t))$. As $\overline{\nabla}_{dc/dt} dc/dt = \overline{\nabla}_{c'(t)} c'(t) = 0$, we have using (2.1) that $\overline{\nabla}_{dc/ds} dc/dt = dt/ds \overline{\nabla}_{dc/dt} dc/dt = 0$, and $dv(dc/ds) = dv/ds$

$$\begin{aligned}
0 = \nabla_{dc/ds}\frac{dc}{ds} &= \overline{\nabla}_{dc/ds}\frac{dc}{ds} + \frac{1}{v}\left(\frac{dv}{ds}\right)\frac{dc}{ds} = \overline{\nabla}_{dc/ds}\left(\frac{dt}{ds}\frac{dc}{dt}\right) + \frac{d(\ln v)}{ds}\frac{dc}{ds} \\
&= \frac{d^2t}{ds^2}\frac{dc}{dt} + \frac{dt}{ds}\overline{\nabla}_{dc/ds}\frac{dc}{dt} + \frac{d(\ln v)}{ds}\frac{dc}{ds} = \frac{d^2t}{ds^2}\frac{dc}{dt} + \frac{d(\ln v)}{ds}\frac{dt}{ds}\frac{dc}{dt} \\
&= \left(\frac{dt}{ds}\right)\left(\left(\frac{dt}{ds}\right)^{-1}\frac{d^2t}{ds^2} + \frac{d(\ln v)}{ds}\right)\frac{dc}{dt} = \left(\frac{dt}{ds}\right)\left(\frac{d}{ds}\ln\left(v\frac{dt}{ds}\right)\right)\frac{dc}{dt}.
\end{aligned}$$

This shows that $\ln(v(dt/ds))$, and therefore also $v(dt/ds)$ is constant. As $v, (dt/ds) > 0$ (the reparameterization is orientation preserving implies $dt/ds = \sigma'(s) > 0$), there is a constant $C_1 > 0$ such that

$$v(t)\frac{dt}{ds} = \frac{1}{C_1}.$$

This differential equation can be integrated to give $s(t) = \sigma^{-1}(t)$ as a function of t, and the result is the required formula (2.2). □

2.2. Proposition. *Let* M *be a smooth manifold with smooth torsion free connection* $\overline{\nabla}$, *and let* $v:M \to (0,\infty)$ *be a smooth positive function.*

Then the connection ∇ defined by (2.1) is a torsion free connection projective with $\overline{\nabla}$, and ∇ is complete if and only if for each inextendible $\overline{\nabla}$-geodesic ray $c : [0, b) \to M$ the growth condition

$$\int_0^b v(c(t))\, dt = \infty. \tag{2.3}$$

holds.

Proof. That ∇ is projective to $\overline{\nabla}$ and torsion free follows from Proposition 1.1 using $\omega = (2v)^{-1}dv$. So all that is left to check is that ∇ is complete if and only if (2.3) holds along inextendible $\overline{\nabla}$-geodesic rays.

First assume that the growth condition (2.3) holds along inextendible $\overline{\nabla}$-geodesic rays. Let $c{:}[0, b) \to M$ be such a ray, and let $\sigma{:}[0, \beta) \to [0, b)$ be an orientation preserving reparameterization of c so that $\tilde{c} = c \circ \sigma$ is a ∇-geodesic. We claim that $\beta = \infty$. By Lemma 2.1 $\sigma^{-1}(t)$ is given by

$$\sigma^{-1}(t) = C_1 \int_0^t v(c(\tau))\, d\tau \tag{2.4}$$

with $C_1 > 0$. But then the growth condition (2.3) implies $\beta = C_1 \int_0^b v(c(\tau))\, d\tau = \infty$. As c was any inextendible $\overline{\nabla}$-geodesic ray, the completeness of ∇ follows from Proposition 1.2.

Conversely, assume ∇ is complete and let $c{:}[0, b) \to M$ be an inextendible $\overline{\nabla}$-geodesic ray. Then, by Proposition 1.2, there is an orientation preserving reparameterization $\sigma{:}[0, \infty) \to [0, b)$ so that $\tilde{c} = c \circ \sigma$ is a ∇-geodesic. Again, Lemma 2.1 implies that σ^{-1} is given by (2.4). Therefore, $C_1 \int_0^b v(c(\tau))\, d\tau = \lim_{t \uparrow b} \sigma^{-1}(t) = \infty$, which shows that the condition (2.3) holds along all inextendible $\overline{\nabla}$-geodesic rays. □

For a general connection, $\overline{\nabla}$, it is not clear how to choose a positive smooth function v so that the growth condition (2.3) holds along all inextendible $\overline{\nabla}$-geodesics rays. However, when ∇ is the metric connection of a Riemannian metric, the behavior of geodesics is closely related to the properties of the distance function of the metric and this can be exploited to find an appropriate v.

Proof of Theorem 2. If (M, g) is complete as a metric space, then the metric connection $\overline{\nabla}$ is complete, cf. [**7**, p. 462], and taking $\nabla = \overline{\nabla}$ completes the proof. Therefore, assume that M is incomplete. Let $\overline{M}$ be the completion of M as a metric space, and let $\partial M = \overline{M} \setminus M$ be the boundary of M in $\overline{M}$. For $x \in M$, let $\delta(x)$ be the distance of x from ∂M. A standard partition of unity argument shows that there is a smooth function v on M so that

$$v(x) \geq \max\{1, 1/\delta(x)\}$$

for all $x \in M$. Let $c{:}[0, b) \to M$ be an inextendible $\overline{\nabla}$-geodesic ray. There are two cases: $b = \infty$ and $b < \infty$. In the case $b = \infty$, then from the definition of v we have $v(c(t)) \geq 1$ and so $\int_0^b v(c(t))\,dt \geq \int_0^\infty 1\,dt = \infty$ and the condition (2.3) holds in this case.

In the second case, where $b < \infty$, the length of the velocity vector $c'(t)$ is constant, and thus there is a constant $C > 0$ so that, for all $t_1, t_2 \in [0, b)$, the distance $d(c(t_1), c(t_2))$ between $c(t_1)$ and $c(t_2)$ satisfies

$$d(c(t_1), c(t_2)) \leq C|t_2 - t_1|.$$

Therefore, in the completion $\overline{M}$, the limit $p = \lim_{t \uparrow b} c(t)$ will exist and, from the definition of δ as the distance from the boundary ∂M, the estimate $\delta(c(t)) \leq d(c(t), p) \leq C|b - t|$ holds. This yields

$$\int_0^b v(c(t))\,dt \geq \int_0^b \frac{dt}{\delta(c(t))} \geq \int_0^b \frac{dt}{C|b - t|} = \infty.$$

Thus, (2.3) holds in all cases, and therefore ∇ is complete by Proposition 2.2. $\square$

Remark 2.3. In a complete Riemannian manifold, any two points can be joined by a geodesic. For complete connections this is no longer true and Hicks [**3**] has constructed an example of a complete connection on a manifold, M, so that for any positive integer m there are two points of M that not only cannot be connected by a

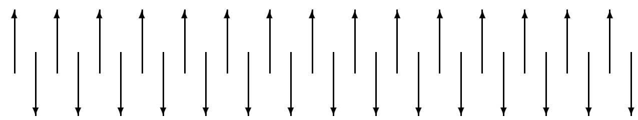

FIGURE 1. Let $U \subset \mathbf{R}^2$ be the compliment of the pictured rays. Then there is a complete torsion free connection on U whose geodesics are the restriction of the line segments of $\mathbf{R}^2$ to U.

geodesic, but any broken geodesic between the points must have at least m breaks. For open sets U in $\mathbf{R}^2$ the behavior of geodesics is easy to visualize and, using Theorem 2, it is trivial to generate such examples that are also projectively flat. For example, set $K := \cup_{k=-\infty}^{\infty}\{2k\} \times [-1,\infty) \cup \cup_{k=-\infty}^{\infty}\{2k+1\} \times (-\infty, 1]$, which is a union of rays parallel to the y-axis, and let $U = \mathbf{R}^2 \setminus K$, see Figure 1. Use Theorem 1 to put a complete projectively flat connection on U that has line segments as its geodesics and polygonal paths as its broken geodesics. With this connection, U has the property that any broken geodesic between the points $(1/2, 0)$ and $(m+1/2, 0)$ must have at least $m+1$ corners. $\square$

3. Homogeneous examples. Before specializing to two dimensions for the proof of Theorem 1, we do the preliminary calculations in arbitrary dimensions. This leads to higher dimensional examples.

Let $\overline{\nabla}$ be the standard flat connection on $\mathbf{R}^n$, and let $U := \mathbf{R}^n \setminus \{0\}$ be $\mathbf{R}^n$ with the origin deleted. Then any nonsingular linear map $A{:}\mathbf{R}^n \to \mathbf{R}^n$ preserves the connection $\overline{\nabla}$, and therefore the general linear group $\mathbf{GL}(n, \mathbf{R})$ has a transitive action on U that preserves $\overline{\nabla}$. Let $\mathbf{O}(n)$ be the orthogonal group of the standard inner product, $\langle\,,\rangle$, on $\mathbf{R}^n$, and let $\mathbf{R}^+$ be the multiplicative group of positive real numbers. Let G be the product group $G = \mathbf{O}(n) \times \mathbf{R}^+$. View G as a subgroup of $\mathbf{GL}(n, \mathbf{R})$ by letting it act on $\mathbf{R}^n$ by $(P, c)x = cPx$. This action of G is transitive on U and preserves the connection $\overline{\nabla}$. Let $v{:}U \to (0, \infty)$ be the function $v(x) = 1/\|x\|$. Then, if $g = (P, c) \in G$, the pull back of v by g is $(g^*v)(x) = v(gx) = \|cPx\|^{-1} = c^{-1}\|x\|^{-1} = c^{-1}v(x)$ as $P \in \mathbf{O}(n)$ so that $\|Px\| = \|x\|$. The pull back of the one form dv/v is

$$g^*\left(\frac{dv}{v}\right) = \frac{g^*dv}{g^*v} = \frac{d(g^*v)}{g^*v} = \frac{d(c^{-1}v)}{c^{-1}v} = \frac{dv}{v},$$

and so dv/v is invariant under the action of G. Therefore, if we define a connection ∇ on U by

(3.1)
$$\nabla_X Y = \overline{\nabla}_X Y + \overline{\nabla}_X Y + \frac{1}{2v}\big(dv(X)Y + dv(Y)X\big) \quad \text{with } v(x) = \frac{1}{\|x\|},$$

then ∇ will be invariant under the action of the group G. The inextendible $\overline{\nabla}$-geodesic rays in U are the curves $c{:}[0,b) \to U$ given by $c(t) = x_0 + tx_1$ where $x_1 \neq 0$ and either $b = \infty$ or $c(b) := \lim_{t\uparrow b} c(t) = 0$. In either case it is easy to check that $\int_0^b v(c(t))\,dt = \infty$, and therefore by Proposition 2.2 the connection ∇ is complete and projectively flat on U.

To get compact examples let $\lambda > 1$ and let Γ be the cyclic subgroup of G given by $\Gamma := \{(I, \lambda^k) : k \in \mathbf{Z}\}$ where $\mathbf{Z}$ is the integers. The action of Γ on U is fixed point free and properly discontinuous, and therefore if M is defined to be the quotient space $M := \Gamma\backslash U$, then M is a smooth manifold, cf. [**1**, Theorem 8.3, p. 97], and it is not hard to see that M is diffeomorphic to $S^{n-1} \times S^1$. Let $\pi{:}U \to M$ be the natural projection. Then π is a covering map and Γ is the group of deck transformations. As the connection ∇ is invariant under these transformations, it follows there is a unique connection ∇^M on M so that $\pi^*\nabla^M = \nabla$. The ∇^M-geodesics on M are $\pi \circ c$ where c is a ∇-geodesic on U. As the ∇-geodesics in U are complete, it follows that the ∇^M geodesics in M are complete. Also this implies that π is a projective map, and therefore ∇^M is projectively flat on M.

For any $g = (P, c) \in G$ and $a = (I, \lambda^k) \in \Gamma$ we have $ag = ga$. As for $x \in U$ the image $\pi(x)$ is the orbit $\pi(x) = \Gamma x$ we see for $g \in \Gamma$ that $\pi(gx) = \Gamma gx = g\Gamma x = g\pi(x)$. Therefore, there is a well-defined action of G on M given by $g\pi(x) = \pi(gx)$. This action is transitive on M as G is transitive on U.

We now claim that, if $x \in U$ and $y = -\alpha x$ for $\alpha > 0$, then there is no geodesic from $\pi(x)$ to $\pi(y)$ in M. Assume, toward a contradiction, that there is a geodesic $c{:}[a,b] \to M$ with $c(a) = \pi(x)$ and $c(b) = \pi(y)$. Then there is a unique geodesic $\hat{c}{:}[a,b] \to U$ with $\hat{c}(a) = x$ and $\pi \circ \hat{c} = c$. Therefore, $\pi(\hat{c}(b)) = c(b) = \pi(y)$ which implies that $\hat{c}(b) = ay$ for some $a \in \Gamma$. From the definition of Γ, this implies that, for some $k \in \mathbf{Z}$, $\hat{c}(b) = \lambda^k y = -\lambda^k \alpha x$. But as ∇ is projective with the flat metric $\overline{\nabla}$ the geodesics segments of ∇ are reparameterizations of straight line segments in U. But then $\hat{c}$ is a reparameterization of a straight line segment of U form $\hat{c}(a) = x$ to $\hat{c}(b) = -\lambda^k \alpha x$, which is impossible as $\lambda^k \alpha > 0$, so that any line segment connecting these points must pass through the origin, which is not in U. This contradiction verifies our claim that there is no geodesic of M from $\pi(x)$ to $\pi(y)$. Letting α vary over the positive real numbers, we get uncountable many points $\pi(y)$ that cannot be connected to $\pi(x)$ by a geodesic. As every point $p \in M$ is of the form $p = \pi(x)$ this can be summarized as:

3.1. Proposition. *Let $M = \Gamma \backslash U$ and ∇^M be the manifold and connection just constructed. Then M is diffeomorphic to $S^{n-1} \times S^1$ and the connection ∇^M on M is complete, projectively flat and homogeneous with respect to the group action of G on M. For any $p \in M$ there are uncountable many points q that cannot be connected to p by a ∇^M-geodesic.* $\square$

3.1. *Proof of Theorem* 1. In the case that $n = 2$ it is possible to be more explicit. On $U = \mathbf{R}^2 \backslash \{0\}$ there are several sets of coordinates that will be convenient to use. First the standard Euclidean coordinates x and y. With respect to these coordinates the standard flat connection $\overline{\nabla}$ is given by

$$\overline{\nabla}_{\partial/(\partial x)} \frac{\partial}{\partial x} = \overline{\nabla}_{\partial/(\partial x)} \frac{\partial}{\partial y} = \overline{\nabla}_{\partial/(\partial y)} \frac{\partial}{\partial x} = \overline{\nabla}_{\partial/(\partial y)} \frac{\partial}{\partial y} = 0.$$

The simply connected covering space, $\widehat{U}$, of U is diffeomorphic to $\mathbf{R}^2$. Using polar coordinates r, θ on $\widehat{U}$ (with $(r, \theta) \in (0, \infty) \times \mathbf{R}$) we have the usual formula for the covering map: $x = r\cos\theta$ and $y = r\sin\theta$. In polar coordinates the connection is given by

$$\overline{\nabla}_{\partial/(\partial r)} \frac{\partial}{\partial r} = 0, \quad \overline{\nabla}_{\partial/(\partial r)} \frac{\partial}{\partial \theta} = \overline{\nabla}_{\partial/(\partial \theta)} \frac{\partial}{\partial r} = \frac{1}{r} \frac{\partial}{\partial \theta},$$
$$\overline{\nabla}_{\partial/(\partial \theta)} \frac{\partial}{\partial \theta} = -r \frac{\partial}{\partial r}.$$

(More precisely this is the pull back of the connection ∇ to $\widehat{U}$ by the covering map. We will still denote this connection by ∇.) The function $v = \|(x, y)\|^{-1}$ used in the definition (3.1) of the connection ∇ is given in polar coordinates a $v = r^{-1}$. Then $dv = -r^{-2}dr$. Using this in (3.1) gives

$$\nabla_X Y = \overline{\nabla}_X Y - \frac{1}{r}\,(dr(X)Y + dr(Y)X),$$

and therefore ∇ is given explicitly in polar coordinates as

$$\nabla_{\partial/(\partial r)}\,\frac{\partial}{\partial r} = \frac{-1}{r}\,\frac{\partial}{\partial r}, \quad \nabla_{\partial/(\partial r)}\,\frac{\partial}{\partial \theta} = \nabla_{\partial/(\partial \theta)}\,\frac{\partial}{\partial r} = \frac{1}{2r}\,\frac{\partial}{\partial \theta},$$
$$\nabla_{\partial/(\partial \theta)}\,\frac{\partial}{\partial \theta} = -r\,\frac{\partial}{\partial r}.$$

The formulas for ∇ simplify even further if we replace the coordinate r on $\widehat{U}$ by ρ related to r by $r = e^{\rho}$. The vector field $\partial/\partial\rho$ is related to the vector field $\partial/\partial r$ by $\partial/\partial\rho = r(\partial/\partial r)$ and $\partial/\partial r = e^{-\rho}(\partial/\partial\rho)$. Therefore, in the coordinates ρ, θ the connection ∇ is given by

$$\nabla_{\partial/(\partial \rho)}\,\frac{\partial}{\partial \rho} = 0, \quad \nabla_{\partial/(\partial \rho)}\,\frac{\partial}{\partial \theta} = \nabla_{\partial/(\partial \theta)}\,\frac{\partial}{\partial \rho} = \frac{1}{2}\,\frac{\partial}{\partial \theta},$$
$$\nabla_{\partial/(\partial \theta)}\,\frac{\partial}{\partial \theta} = -\frac{\partial}{\partial \rho}.$$

This explicit form of the connection ∇ makes it clear that it is invariant under translations $\rho \mapsto \rho + a$ and $\theta \mapsto \theta + b$. From the construction ∇ is complete and projectively flat.

Using the coordinates ρ and θ and letting $\mathbf{Z}$ be the integers, then the original open set U is naturally identified with the quotient group $\mathbf{R}^2/(\{0\}\times 2\pi\mathbf{Z})$ (that is, identify (ρ,θ) with $(\rho,\theta+2k\pi)$ for $k \in \mathbf{Z}$). As in the original set U, the ∇-geodesics are reparameterized line segments and it is not hard to see that a point $z \in U$ can be connected to a point z_0 on the positive real axis by a ∇-geodesic if and only if z is not on the negative real axis. That is, z can be connected to z_0 by a ∇-geodesic if and only if $|\theta(z)| < \pi$. (See Figure 2.) But, because of the homogeneity of the connection with respect to translations $\theta \mapsto \theta + b$, this implies:

3.2. Lemma. *Two points $z_1, z_2 \in \widehat{U}$ can be connected by a ∇-geodesic if and only if $|\theta(z_1) - \theta(z_2)| < \pi$. Therefore, if z_1, z_2 satisfy $|\theta(z_1) - \theta(z_2)| \geq m\pi$ for some positive integer m, any piecewise broken geodesic from z_1 to z_2 must have at least m breaks.* □

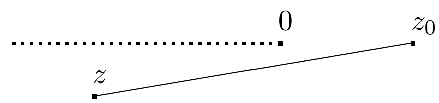

FIGURE 2. As the connection ∇ is projective with the usual flat connection, a point z in the set $U = \mathbf{R}^2 \setminus \{0\}$ can be connected to a point z_0 on the positive real axis by a ∇-geodesic if and only if $|\theta(z)| < \pi$.

Remark 3.3. There is a less geometric, but possibly more informative, proof of this lemma. Using the coordinates ρ, θ on $\widehat{U}$ and the coordinates x, y on U, the covering map from $\widehat{U}$ to U is given by $x = e^\rho \cos\theta$ and $y = e^\rho \sin\theta$. In U the ∇-geodesics are reparameterizations of straight lines, and thus along a ∇-geodesic the coordinates x and y are related by $ax + by = 0$ (if geodesic goes through the origin) or $ax+by = 1$ (if it does not pass through the origin). The first case leads to a relation between ρ and θ of the form $e^\rho(a\cos\theta + b\sin\theta) = 0$ along the geodesic which implies $\theta = \theta_0$ on the geodesic, for some constant θ_0. In the second case we get $e^\rho(a\cos\theta+b\sin\theta) = 1$ along the geodesic. Let $A = \sqrt{a^2+b^2}$ and let α be so that $A\cos\alpha = a$ and $A\sin\alpha = b$. Then the equation between ρ and θ becomes $e^\rho A\cos(\theta-\alpha) = 1$. From this it follows that, given a point in $\widehat{U}$ with coordinates (ρ_0, θ_0), the ∇-geodesics of $\widehat{U}$ through this point are the line $\theta = \theta_0$ and the curves defined for $|\theta - \alpha| < \pi/2$ by the equation

$$e^\rho \cos(\theta - \alpha) = e^{\rho_0} \cos(\theta_0 - \alpha) \tag{3.2}$$

where α varies over real numbers with $|\alpha - \theta_0| < \pi/2$. This makes it clear that a point (ρ_1, θ_1) with $|\theta_1 - \theta_0| \geq \pi$ cannot be on a geodesic through (ρ_0, θ_0). And, conversely, if $|\theta_1 - \theta_0| < \pi$, then either $\theta_1 = \theta_0$, and the points are both on the geodesic $\theta = \theta_0$, or $\theta_1 \neq \theta_0$ and straightforward calculus argument shows that there is a unique $\alpha \in (\theta_0 - \pi/2, \theta_0 + \pi/2) \cap (\theta_1 - \pi/2, \theta_1 + \pi/2)$ so that $e^{\rho_1}\cos(\theta_1 - \alpha) = e^{\rho_1}\cos(\theta_0 - \alpha)$. For this choice of α, both of the points (ρ_0, θ_0) and (ρ_1, θ_1) will be on the ∇-geodesic defined by (3.2). $\square$

We now complete the proof of Theorem 1. Given the positive integer m, let k be an integer with $k \geq m$. Let T^2 be the torus

$$T^2 = \widehat{U}/(\mathbf{Z} \times 2\pi k\mathbf{Z})$$

(that is, identify (ρ,θ) with $(\rho+j,\theta+2\pi kl)$ for $j,l\in\mathbf{Z}$). As the connection ∇ is translation invariant it well defined as a connection on T^2 and will be invariant under translations of T^2 when T^2 is viewed as a Lie group. We have already seen that ∇ is complete and projectively flat. Let $\varpi{:}\widehat{U}\to T^2$ be the covering map. We now claim that any broken ∇-geodesic in T^2 from $\varpi(\rho_0,\theta_0)$ to $\varpi(\rho_0,\theta_0+m\pi)$ must have at least m breaks. For, let $c{:}[a,b]\to T^2$ be such a broken geodesic. By the Path Lifting theorem, [**2**, p. 22] or [**5**, p. 67], there is a unique curve $\hat{c}{:}[a,b]\to M$ with $\hat{c}(a)=(\rho_0,\theta_0)$ and $\varpi\circ\hat{c}=c$. This curve will also be a broken geodesic. Also $\varpi(\hat{c}(b))=c(b)=\varpi(\rho_0,\theta_0+m\pi)$, and therefore $\hat{c}(b)=(\rho_0+j,\theta_0+m\pi+2\pi kl)$ for some $j,l\in\mathbf{Z}$. The difference in the θ coordinates of the ends of $\hat{c}$ is

$$|\theta_0+m\pi+2\pi kl-\theta_0|=|m+2kl|\pi\geq m\pi$$

as $k\geq m$. By Lemma 3.2 this implies that $\hat{c}$ has at least m breaks. But then $c=\varpi\circ\hat{c}$ also has at least m breaks. As $\varpi(\rho_0,\theta_0)$ was an arbitrary point of T^2 this completes the proof of Theorem 1. □

Remark 3.4. The connection ∇ has another property worth noting. If $c(t)=(\rho(t),\theta(t))$ is a smooth curve in $\widehat{U}$, then the equations for c to be a ∇-geodesic are

$$\ddot{\rho}=\dot{\theta}^2,\quad \ddot{\theta}=-\dot{\rho}\dot{\theta}.$$

These imply

$$\frac{1}{2}\frac{d}{dt}(\dot{\rho}^2+\dot{\theta}^2)=\dot{\rho}\ddot{\rho}+\dot{\theta}\ddot{\theta}=\dot{\rho}\dot{\theta}^2-\dot{\theta}\dot{\rho}\dot{\theta}=0.$$

Therefore, $\dot{\rho}^2+\dot{\theta}^2$ is constant along ∇-geodesics. Thus all ∇-geodesics have constant speed with respect to the flat Riemannian metric $ds^2=d\rho^2+d\theta^2$ on $\widehat{U}$. As this metric is translation invariant, it is also well defined on the torus $T^2=\widehat{U}/(\mathbf{Z}\times 2\pi k\mathbf{Z})$, and the ∇-geodesics on T^2 will also have constant speed with respect to this metric. This can be used to give another proof that ∇ is complete. □

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