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PERIODIC SOLUTIONS OF AN INFINITE DIMENSIONAL HAMILTONIAN SYSTEM

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ABSTRACT. We establish existence and multiplicity of periodic solutions to the infinite dimensional Hamiltonian system

$$\begin{cases} \partial_t u - \Delta_x u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v = H_u(t,x,u,v) \end{cases} \qquad \text{for } (t,x) \in \mathbf{R} \times \Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain or $\Omega = \mathbf{R}^N$. When Ω is bounded, we treat the situations where $H(t,x,z)$ is, with respect to $z=(u,v)$, sub- or superquadratic, or concave and convex, and discuss also the convergence to homoclinics of sequences of subharmonic orbits. If $\Omega = \mathbf{R}^N$, we handle the case of superquadratic nonlinearities.

1. Introduction. In this paper we are interested in existence and multiplicity of periodic orbits of the following system of partial differential equations

$$\text{(HS)} \qquad \begin{cases} \partial_t u - \Delta_x u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v = H_u(t,x,u,v) \end{cases} \qquad \text{for } (t,x) \in \mathbf{R} \times \Omega.$$

Here $\Omega \subset \mathbf{R}^N$ is a smoothly bounded domain or $\Omega = \mathbf{R}^N$, $z=(u,v) : \mathbf{R} \times \Omega \to \mathbf{R}^m \times \mathbf{R}^m$, and $H \in \mathcal{C}^1(\mathbf{R} \times \bar{\Omega} \times \mathbf{R}^{2m}, \mathbf{R})$, where $\overline{\Omega} = \Omega$ if $\Omega = \mathbf{R}^N$. Letting

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathcal{J}_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and $A = \mathcal{J}_0(-\Delta_x)$, (HS) can be rewritten as $\mathcal{J}(d/dt)z + Az = H_z(t,x,z)$. Certain linear and nonlinear problems connecting the operator $\mathcal{J}\partial_t - \mathcal{J}_0\Delta_x$ arise in optimal control of systems governed

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by partial differential equations. See, e.g., Lions [**16**], where the combination of the model $\partial_t - \Delta_x$ and its adjoint $-\partial_t - \Delta_x$ acts as a system for studying the control. Systems like (HS) are called unbounded Hamiltonian systems, cf. Barbu [**2**], or infinite-dimensional Hamiltonian systems, cf. [**3, 9, 10**]. Indeed, (HS) can be represented as

$$\mathcal{J}\frac{d}{dt}z = \nabla_z \mathcal{H}(t,z),$$

where $\mathcal{J}$ is obviously a symplectic structure on $L^2(\Omega, \mathbf{R}^{2m})$, $\mathcal{H}$ is a Hamiltonian

$$\mathcal{H}(t,z) := \int_\Omega (\nabla u \nabla v + H(t,x,z))\, dx$$

and ∇_z is the gradient operator, defined in (the infinite-dimensional Hilbert space) $L^2(\Omega, \mathbf{R}^{2m})$.

Assume that Ω is a smoothly bounded domain. Brézis and Nirenberg [**7**] considered the system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g, \end{cases}$$

where $f, g \in L^\infty(\Omega)$, subject to the boundary condition $z(t,\cdot)|_{\partial\Omega} = 0$ and the periodicity condition $z(0,\cdot) = z(T,\cdot) = 0$ for a given $T > 0$. They obtained a solution z with $u \in L^4$ and $v \in L^6$ by using Schauder's fixed point theorem. Clément, Felmer and Mitidieri considered in [**9**] and [**10**] the problem

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2}v \\ -\partial_t v - \Delta_x v = |u|^{p-2}u \end{cases} \tag{1.1}$$

with

$$\frac{N}{N+2} < \frac{1}{p} + \frac{1}{q} < 1.$$

Using their variational setting of Mountain Pass type, they proved that there is a $T_0 > 0$ such that, for each $T > T_0$, (1.1) has at least one positive solution $z_T = (u_T, v_T)$ satisfying the boundary condition $z_T(t,\cdot)|_{\partial\Omega} = 0$ for all $t \in (-T,\ T)$ and the periodicity condition $z_T(T,\cdot) = z_T(-T,\cdot)$ for $x \in \overline{\Omega}$. Moreover, by passing to the limit as $T \to \infty$ they obtained a positive homoclinic solution of (1.1).

For $\Omega = \mathbf{R}^N$, recently, Bartsch and Ding [**3**] dealt with the system

$$\begin{cases} \partial_t u + (-\Delta_x + V(x))u = H_v(t, x, u, v) \\ -\partial_t v + (-\Delta_x + V(x))v = H_u(t, x, u, v). \end{cases}$$

They established existence and multiplicity of solutions of the type $z(t, x) \to 0$ as $|t| + |x| \to \infty$. (They also considered solutions of the type $z(t, x) \to 0$ as $|t| \to \infty$ and $z(t, \cdot)|_{\partial\Omega} = 0$ if Ω is bounded.)

In the present paper we are interested in situations different from these works above. Unlike [**7**] and [**10**], the system we consider is not autonomous in the sense that the Hamiltonian depends also on time variable t. The growth rate of $H(t, x, z)$ as $|z| \to \infty$ can be subquadratic. We also deal with the case of $H(t, x, z)$ being superquadratic and with the case of concave and convex nonlinearities. For these cases we obtain infinitely many periodic solutions provided that $H(t, x, z)$ is even with respect to z. As in [**10**], if $H(t, x, z)$ is superquadratic, we obtain a family of subharmonic orbits which converges to a homoclinic solution as the period tends to the infinity. When $\Omega = \mathbf{R}^N$, we prove existence and multiplicity of periodic solutions for the superquadratic case. The method adopted for handling these cases is variational. An analytic setting is developed based on which critical point theorems of linking type and Calerkin approximation for strongly indefinite functionals are applied conveniently to our problems.

We organize the paper as follows. In next section we state our main results corresponding to the different situations. Then in Section 3 some preliminary results concerning the variational setting for our problems are given. Sections 4 and 5 are devoted to the case of Ω being bounded, proving the existence and multiplicity results corresponding to subquadratic, superquadratic and concave and convex nonlinearities, respectively. In Section 6 we obtain subharmonic orbits and discuss their convergence. Finally, in Section 7 we handle the problem on $\Omega = \mathbf{R}^N$.

2. Main results. We are interested in solutions of (HS) satisfying condition

$$\begin{cases} z(t, x) = 0 & \text{for all } (t, x) \in \mathbf{R} \times \partial\Omega \\ z(t, x) = z(t + T, x) & \text{for all } (t, x) \in \mathbf{R} \times \Omega \end{cases} \tag{2.1}$$

if $\Omega \subset \mathbf{R}^N$ is smoothly bounded, and

$$(2.2) \qquad \begin{cases} z(t,x) \to 0 & \text{uniformly in } t \in [0,T] \text{ as } |x| \to \infty \\ z(0,x) = z(T,x) = 0 & \text{for all } x \in \mathbf{R}^N \end{cases}$$

if $\Omega = \mathbf{R}^N$.

For stating our results, setting $Q_T = [0,T] \times \Omega$, for $r \geq 1$, let $B_{T,r}$ be the Banach space consisting of maps $z = z(t,x) : Q_T \to \mathbf{R}^{2m}$:

$$B_{T,r} = W^{1,r}((0,T), L^r(\Omega, \mathbf{R}^{2m})) \cap L^r((0,T), W^{2,r} \cap W_0^{1,r}(\Omega, \mathbf{R}^{2m}))$$

equipped with the norm

$$\|z\|_{B_{T,r}} := \left(\int_{Q_T} \left(|z|^r + \left| \frac{\partial z}{\partial t} \right|^r + \sum_{j=1}^N \left| \frac{\partial^2 z}{\partial x_j^2} \right|^r \right) \right)^{1/r}$$

(which is sometimes called anisotropic space). We are looking for solutions $z \in B_{T,r}$ with energies

$$I_T(z) := \int_{Q_T} \left(\frac{1}{2} \big(\mathcal{J} \partial_t z - \mathcal{J}_0 \Delta_x z \big) \cdot z - H(t,x,z) \right) < \infty.$$

First we assume Ω is bounded. Consider the situation where $H(t,x,z)$ is subquadratic in z. Let H satisfy

(h_0) $H(t,x,z)$ is T-periodic in t and $H(t,x,0) \equiv 0$;

(h_1) there exist $\alpha > 1$ such that $H(t,x,z) \geq a_1 |z|^\alpha$;

(h_2) there is $p \in (1,2)$ such that $|H_z(t,x,z)| \leq a_2(1 + |z|^{p-1})$;

(h_3) $H(t,x,z)$ is even in z and there is $\gamma > 0$ such that $H(t,x,z) \leq a_3 |z|^\gamma$ whenever $|z| \leq 1$.

Here, and below, the symbols a_i stand for positive constants.

Theorem 2.1. *Assume Ω is smoothly bounded and H satisfies (h_0)–(h_2). Then* (HS)–(2.1) *has at least one nontrivial solution. If in addition (h_3) also holds,* (HS)–(2.1) *has a sequence (z_n) of solutions satisfying $0 > I_T(z_n) \to 0$ as $n \to \infty$. Moreover, these solutions are in $B_{T,r}$ for all $r \geq 1$.*

We remark from the conclusion above that $I_T(z_n) \to 0$, it seems reasonable to expect that the existence of such a sequence of solutions should be involved only with the local behavior of H around $z = 0$. Indeed, we have the following

Theorem 2.1$'$. *Assume Ω is smoothly bounded and that there are $r_0 > 0$ and $1 < \gamma \le \alpha < 2$ such that $H \in \mathcal{C}^1(\mathbf{R} \times \Omega \times B_{r_0}, \mathbf{R})$, even in z, and satisfies*

(2.3)

$$\begin{cases} |H_z(t,x,z)| \le a_1|z|^{\gamma-1} & \text{for } (t,x,z) \in \mathbf{R}\times\Omega\times B_{r_0} \\ 0 < H_z(t,x,z)z \le \alpha H(t,x,z) & \text{for } (t,x,z) \in \mathbf{R}\times\Omega\times B_{r_0} \text{ with } z \neq 0, \end{cases}$$

where $B_{r_0} := \{z \in \mathbf{R}^{2m} : \ |z| \le r_0\}$. Then (HS)–(2.1) *has a sequence (z_n) of solutions satisfying $0 > I_T(z_n) \to 0$ as $n \to \infty$. Moreover, these solutions are in $B_{T,r}$ for all $r \ge 1$.*

Next consider the superquadratic case. Assume that

(h_4) there are $\mu > 2$ and $R_0 > 0$ such that $0 < \mu H(t,x,z) \le H_z(t,x,z) \cdot z$ whenever $|z| \ge R_0$;

(h_5) there is $p \in (2, 2(2+N)/N)$ such that $|H_z(t,x,z)|^{p'} \le a_3(1 + H_z(t,x,z) \cdot z)$, where $p' = p/(p-1)$;

(h_6) $H_z(t,x,z) = o(|z|)$ uniformly in (t,x) as $z \to 0$.

Theorem 2.2. *Assume Ω is smoothly bounded and H satisfies (h_0), (h_4)–(h_5). If (h_6) holds, then* (HS)–(2.1) *has at least one nontrivial solution; if $H(t,x,z)$ is even in z, then* (HS)–(2.1) *has a sequence (z_n) of solutions satisfying $I_T(z_n) \to \infty$ as $n \to \infty$. In addition, these solutions are in $B_{T,r}$ for all $r \ge 1$.*

For the concave and convex case, we note that, since (2.3) describes only the behaviors of H near $z = 0$ and (h_4)–(h_5) state the behaviors around $z = \infty$, it is possible that H satisfies simultaneously all these hypotheses. Thus, as a consequence of Theorems 2.1$'$ and 2.2, we have

Theorem 2.3. *Let Ω be smoothly bounded. Suppose that H satisfies (h_0), (h_4)–(h_5) and* (2.3) *for some $r_0 > 0$, $1 < \gamma \le \alpha < 2$. If H is* *even in z, then* (HS)–(2.1) *has two sequences* (z_n) *and* $(\tilde{z}_n)$ *of solutions in* $B_{T,r}$ *for all* $r \geq 1$ *such that* $0 > I_T(z_n) \to 0$ *and* $I_T(\tilde{z}_n) \to \infty$ *as* $n \to \infty$.

In particular, we have

Theorem 2.3$'$. *Let* Ω *be smoothly bounded and suppose that*

(2.4)

$$\begin{cases} H(t,x,z) = a(t)\left(\xi|z|^\alpha + \eta|z|^p\right), & \text{where } a(t) > 0 \text{ is } T\text{-periodic,} \\ \xi \text{ and } \eta \text{ are real numbers, and } 1 < \alpha < 2 < p < 2(2+N)/N. \end{cases}$$

Then (HS)–(2.1) *has a sequence* (z_n) *of solutions satisfying* $0 > \xi I_T(z_n) \to 0$ *if* $\xi \neq 0$; *and a sequence* $(\tilde{z}_n)$ *of solutions satisfying* $\eta I_T(\tilde{z}_n) \to \infty$ *if* $\eta \neq 0$. *These solutions are in* $B_{T,r}$ *for all* $r \geq 1$. *In addition, if* T *is the minimal period of* $a(t)$, *then all solutions also have the minimal period* T.

Such a result extends previous study on elliptic problems, wave equations and classical Hamiltonian systems, see, for instance, [**1, 4, 5, 14**] and the references therein.

Notice that, since H is T-periodic in t, it is kT-periodic for any $k \in \mathbf{N}$. So we may seek solutions of (HS) satisfying

$$(2.1)_k \qquad \begin{cases} z(t,x) = 0 & \text{for all } (t,x) \in \mathbf{R} \times \partial\Omega \\ z(t,x) = z(t+kT,x) & \text{for all } (t,x) \in \mathbf{R} \times \Omega, \end{cases}$$

and discuss the convergence of such solutions as $k \to \infty$. For this aim we strengthen (h_4)–(h_5) by

(h_4') there are $\mu > 2$ such that $0 < \mu H(t,x,z) \leq H_z(t,x,z) \cdot z$ whenever $z \neq 0$;

(h_5') there is $p \in (2, 2(2+N)/N)$ such that $|H_z(t,x,z)|^{p'} \leq a_3 H_z(t,x,z) \cdot z$ where $p' = p/(p-1)$.

Let $B_{kT,r}$, respectively I_{kT}, be obtained as $B_{T,r}$, respectively I_T, with T replaced by kT.

Theorem 2.4. *Assume Ω is smoothly bounded and H satisfies* (h_0), (h_1) *with* $\alpha > 2$, (h_4') *with* $\mu \le \alpha$, *and* (h_5'). *Then, for each* $k \in \mathbf{N}$, (HS)–$(2.1)_k$ *has a solution* z_k *in* $B_{kT,r}$ *for all* $r \ge 1$ *such that*

(i) *there are* $0 < \delta < \Lambda$, *independent of* k, *satisfying* $\delta \le I_k(z_k) \le \Lambda$;

(ii) *for any sequence* $k_n \to \infty$, *there is a subsequence* $z_{k_{n_i}} \to z_\infty$ *in* $L^\infty_{loc}(\mathbf{R} \times \Omega, \mathbf{R}^{2m})$ *with* z_∞ *solving* (HS), $z_\infty \ne 0$ *and* $z_\infty(t,x) \to 0$ *uniformly in* x *as* $|t| \to \infty$.

Now we treat the case of $\Omega = \mathbf{R}^N$. Assume in addition

(h_7) $H_z(t,x,z)$ is 1-period with respect to x_i for $i = 1, \dots, N$.

(h_8) there are $q \in \big[2, (2(N+2))/N\big]$ and $\delta > 0$ such that $|H_z(t,x, z+w) - H_z(t,x,z)| \le a_2\big(1 + |z|^{p-1}\big)|w|$ whenever $|w| \le \delta$.

In what follows, two solutions z_1 and z_2 of (HS) are said to be geometrically distinct if $z_1(t,x) \ne k * z_2(t,x)$ for all $(t,x) \in Q_T$ and $0 \ne k = (k_1, \dots, k_N) \in \mathbf{Z}^N$, where

$$k * z(t,x) := z(t, x_1 + k_1, \dots, x_N + k_N).$$

Theorem 2.5. *Let* $\Omega = \mathbf{R}^N$ *and suppose* (h_0), (h_4'), (h_5') *and* (h_7) *hold. Then* (HS)–(2.2) *has at least one nontrivial solution. If, moreover, H is even in z and* (h_8) *also holds,* (HS)–(2.2) *has infinitely many geometrically distinct solutions. In addition, these solutions are in* $B_{T,r}$ *for all* $r \ge 2$.

3. Preliminary results. We start by recalling the following embedding [**6**]

$$B_{T,r} \hookrightarrow L^q_T := L^q(Q_T, \mathbf{R}^{2m}) \quad \text{for } r > 1, \qquad 0 \le \frac{1}{r} - \frac{1}{q} \le \frac{2}{2+N}. \tag{3.1}$$

In addition, setting

$$\tau(r) := \begin{cases} r(2+N)/(2+N-2r) & \text{if } 0 < r < (2+N)/2; \\ \infty & \text{if } r \ge (2+N)/2, \end{cases}$$

we have

$$B_{T,r} \hookrightarrow L^q_T \quad \text{compactly for } 1 < r \le q < \tau(r) \tag{3.2}$$

if Ω is smoothly bounded, and

$$B_{T,r} \hookrightarrow L^q_{T,loc} \quad \text{compactly for } 1 < r \le q < \tau(r) \tag{3.3}$$

if $\Omega = \mathbf{R}^N$, cf. [**10**].

Let $W_{T,r}$ denote the (closed) subspace of $B_{T,r}$ consisting of elements satisfying (2.1) if Ω is bounded, (2.2) if $\Omega = \mathbf{R}^N$.

Let $A_T := \mathcal{J}\partial_t - \mathcal{J}_0\Delta_x$. A_T is a self-adjoint operator acting on L^2_T with domain $\mathcal{D}(A_T) = W_{T,2}$. Note that

$$L^2_T = \left[L^2(Q_T, \mathbf{R})\right]^{2m} = L^2((0,T), \mathbf{R}^{2m}) \otimes L^2(\Omega, \mathbf{R})$$

with equivalent norms, where $\otimes$ is the tensor product.

Lemma 3.1. *Suppose Ω is smoothly bounded. Then A_T is an isomorphism from $W_{T,r}$ into L^r_T for all $r > 1$, and, in addition, there are $d_1, d_2 > 0$ independent of T such that*

$$d_2\|z\|^2_{W_{T,2}} \le \int_{Q_T} |A_T z|^2 \le d_1 \|z\|^2_{W_{T,2}} \quad \textit{for all } z \in W_{T,2}.$$

Proof. Let $\widetilde{W}_{T,r}$ denote the Banach space defined as $W_{T,r}$ with $\mathbf{R}^{2m}$ replaced by $\mathbf{R}^m$. Then $W_{T,r} = (\widetilde{W}_{T,r})^2$. With these notations, we write

$$A_T = \begin{pmatrix} 0 & L \\ L^* & 0 \end{pmatrix} \quad \text{where } L = -\partial_t - \Delta_x \quad \text{and} \quad L^* = \partial_t - \Delta_x. \tag{3.4}$$

By [**10**, Lemma 3.1], L and L^* are isomorphisms from $\widetilde{W}_{T,r}$ into $L^r(Q_T, \mathbf{R}^m)$. Thus A_T is an isomorphism from $W_{T,r}$ into L^r_T.

It is clear that there exists $d_1 > 0$ independent of T such that $\|A_T z\|^2_{L^2_T} \le d_1 \|z\|^2_{W_{T,2}}$ for all $z \in W_{T,2}$.

We remark that, for any $r > 1$ and any $z = f(t)g(x) \in W^{1,r}(S_T, \mathbf{R}^{2m}) \otimes \mathcal{C}_0^\infty(\Omega, \mathbf{R})$, where $S_T = \mathbf{R}/[0,T]$, we have

$$\begin{aligned}\|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r &\geq \|\partial_t f\|_{L^r(0,T)}^r \|g\|_{L^r(\Omega)}^r + C_r \|f\|_{L^r(0,T)}^r \|g\|_{L^r(\Omega)}^r \\ &= \|\partial_t z\|_{L_T^r}^r + C_r \|z\|_{L_T^r}^r\end{aligned}$$

where $C_r > 0$ depends only on r and the measure of Ω, by the Sobolev theorem. So

$$\|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r \geq \|\partial_t z\|_{L_T^r}^r + \left(1 - \frac{1}{r}\right) \|\Delta_x z\|_{L_T^r}^r + \frac{C_r}{r} \|z\|_{L_T^r}^r. \tag{3.5}$$

Since $W^{1,r}(S_T, \mathbf{R}^{2m}) \otimes \mathcal{C}_0^\infty(\Omega, \mathbf{R})$ is dense in $W_{T,r}$, (3.5) holds for all $z \in W_{T,r}$. On the other hand, since $\mathcal{J}' = -\mathcal{J}$ and $\mathcal{J}\mathcal{J}_0 = -\mathcal{J}_0\mathcal{J}$, we have, if $r = 2$,

$$\begin{aligned}\int_{Q_T} |A_T z|^2 &= \int_{Q_T} |\mathcal{J}\partial_t z - \mathcal{J}_0 \Delta_x z|^2 \\ &= \|\partial_t z\|_{L_T^2}^2 + \|\Delta_x z\|_{L_T^2}^2\end{aligned} \tag{3.6}$$

for all $z = f(t)g(x)$, then for all z as above. The last conclusion follows from (3.5)–(3.6). □

Lemma 3.2. *If $\Omega = \mathbf{R}^N$, A_T is an isomorphism from $W_{T,2}$ into L_T^2, i.e., $d_2 \|z\|_{W_{T,2}}^2 \leq \|A_T z\|_{L_T^2}^2 \leq d_1 \|z\|_{W_{T,2}}^2$ for all $z \in W_{T,2}$, where $d_1 > 0$ is independent of T and $d_2 > 0$ depends on T.*

Proof. We need only to show the first inequality. In fact, for $r > 1$ and $z = f(t)g(x) \in W_0^{1,r}((0,T), \mathbf{R}^{2m}) \otimes \mathcal{C}_0^\infty(\Omega, \mathbf{R})$, we have

$$\begin{aligned}\|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r &\geq \frac{r}{T^r} \|f\|_{L^r(0,T)}^r \|g\|_{L^r(\Omega)}^r + \|f\|_{L^r(0,T)}^r \|\Delta g\|_{L^r(\Omega)}^r \\ &= \frac{r}{T^r} \|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r,\end{aligned}$$

so

$$\|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r \geq \frac{1}{T^r} \|z\|_{L_T^r}^r + \frac{r-1}{r} \|\partial_t z\|_{L_T^r}^r + \|\Delta_x z\|_{L_T^r}^r.$$

This holds by density for all $z \in W_{T,r}$. On the other hand, as before, for $r = 2$,

$$\int_{Q_T} |A_T z|^2 = \|\partial_t z\|^2_{L^2_T} + \|\Delta_x z\|^2_{L^2_T}.$$

The proof is complete. $\square$

In what follows let $E_T := \mathcal{D}(|A_T|^{1/2})$, which is a Hilbert space equipped with the inner product

$$(z, w)_T := (|A_T|^{1/2} z, |A_T|^{1/2} w)_{L^2_T}$$

and norm $\|z\|^2_T = (z, z)_T = \||A_T|^{1/2} z\|^2_{L^2_T}$.

Lemma 3.3. *E_T imbeds continuously in L^r_T for any $r \geq 2$ if $N = 1$, and $r \in [2,\ 2(N+2)/N]$ if $N \geq 2$. Moreover,*

a) *If Ω is bounded, E_T imbeds compactly in L^r_T for any $r \geq 1$ if $N = 1$, and $r \in [1,\ 2(N+2)/N)$ if $N \geq 2$. In addition, if $r \geq 2$, there is $d_r > 0$ independent of T such that $\|z\|_{L^r_T} \leq d_r \|z\|_T$;*

b) *If $\Omega = \mathbf{R}^N$, E_T imbeds compactly in $L^r_{loc}(Q_T, \mathbf{R}^{2m})$ for any $r \geq 1$ if $N = 1$, and $r \in [1,\ 2(N+2)/N)$ if $N \geq 2$.*

Proof. We remark that the operator $B_T := \mathcal{J}\partial_t + \mathcal{J}_0(1 - \Delta_x)$ acting on L^2_T is self-adjoint with domain $W_{T,2}$, and there are $b_1, b_2 > 0$ independent of T such that

$$b_2 \|B_T z\|^2_{L^2_T} \leq \|z\|^2_{W_{T,2}} \leq b_1 \|B_T z\|^2_{L^2_T}. \tag{3.7}$$

Using Lemmas 3.1 and 3.2 and (3.7) we see that there are $b_3 > 0$ independent of T and $b_4 > 0$ such that

$$b_4 \|B_T z\|^2_{L^2_T} \leq \|A_T z\|^2_{L^2_T} \leq b_3 \|B_T z\|^2_{L^2_T}.$$

Here b_4 can be chosen to be also independent of T if Ω is bounded. Thus $\mathcal{D}(|B_T|^{1/2}) = \mathcal{D}(|A_T|^{1/2}) = E_T$ and

$$b_4 \||B_T|^{1/2} z\|^2_{L^2_T} \leq \|z\|^2_T \leq b_3 \||B_T|^{1/2} z\|^2_{L^2_T}.$$

Going to the complexification $L^2_T \times L^2_T \cong L^2_T + iL^2_T$ and using the (complex) interpolation $[\cdot,\cdot]_\theta$, see [**19**] or [**20**], one sees that

$$E_T = \mathcal{D}(|B_T|^{1/2}) \cong [\mathcal{D}(B_T), L^2_T]_{1/2}.$$

The embeddings

$$E_T \cong [\mathcal{D}(B_T), L^2_T]_{1/2} \hookrightarrow [L^r_T, L^2_T]_{1/2} \hookrightarrow L^q_T$$

are continuous for $r=\infty$ if $N=1,2$, and $r=2(N{+}2)/(N{-}2)$ if $N\geq 3$, and if q satisfies $1/q = \big(1/2+1/r\big)/2$, that is, if $q=2(N+2)/N$. For $r\in(2,q)$, the Hölder inequality implies

$$|z|_r \leq |z|_2^{1-\theta}|z|_q^{\theta} \quad \text{with } \theta = \frac{q(r-2)}{r(q-2)}.$$

Therefore the first conclusion of the lemma follows.

Finally, the conclusions a) and b) follow from the above argument and (3.1)–(3.3). □

By virtue of Lemmas 3.1 and 3.2, we have the decomposition

$$L^2_T = (L^2_T)^- \oplus (L^2_T)^+, \quad z = z^- + z^+$$

with $(L^2_T)^\pm$ invariant subspaces of A_T such that A_T is negative (positive, respectively) on $(L^2_T)^-$ (on $(L^2_T)^+$, respectively). Accordingly,

$$E_T = E^-_T \oplus E^+_T \quad \text{with } E^\pm_T = E_T \cap (L^2_T)^\pm.$$

On E_T the energy functional I_T has the form

$$I_T(z) := \frac{1}{2}(\|z^+\|^2_T - \|z^-\|^2_T) - \int_{Q_T} H(t,x,z). \tag{3.8}$$

Lemma 3.4. *Assuming that $H\in\mathcal{C}^1(Q_T,\mathbf{R}^{2m})$ satisfying for some $\gamma\geq 2$*

$$|H_z(t,x,z)| \leq \begin{cases} c_1 + c_2|z|^{\gamma-1} & \text{if } \Omega \textit{ is bounded,} \\ |z| + c|z|^{\gamma-1} & \text{if } \Omega = \mathbf{R}^N. \end{cases} \tag{3.9}$$

Then the functional $I_T \in \mathcal{C}^1(E_T, \mathbf{R})$, and critical points of I are solutions of (HS) *in $W_{T,r}$ for any $r \geq 2$.*

Proof. Using Lemma 3.3 and the assumption (3.9), a standard argument shows that $I_T \in \mathcal{C}^1(E_T)$, and critical points of I_T are weak solutions of (HS). It remains to show that if z is a weak solution then $z \in \cap_{2\leq r<\infty} W_{T,r}$.

If $\Omega = \mathbf{R}^N$, we may regard $W_{T,r}$ as a subspace of $Z_r := W^{1,2}(\mathbf{R}, L^r \times (\mathbf{R}^N, \mathbf{R}^{2m})) \cap L^r(\mathbf{R}, W^{2,r}(\mathbf{R}^N, \mathbf{R}^{2m}))$ equipped with the norm

$$\|z\|_{Z_r} = \left(\int_{\mathbf{R}\times\mathbf{R}^N} \left(|z|^r + \left|\frac{\partial z}{\partial t}\right|^r + \sum_{j=1}^{N} \left|\frac{\partial^2 z}{\partial x_j^2}\right|^r \right) \right)^{1/r}.$$

Then the desired conclusion follows from [**3**, Lemma 4.7].

Assume Ω is smoothly bounded. By (3.1),

$$\|z\|_{L_T^q} \leq C_{q,r}\|z\|_{W_{T,r}} \quad \text{for all } z \in W_{T,r},$$

where $C_{q,r} > 0$ is a constant depending only on q and r, for $1 < r \leq q < \tau(r)$ and also for $q = \tau(r)$ if $\tau(r) < \infty$. Now using (HS) and (3.9), a standard bootstrap argument, cf. [**10**] or [**3**, Lemma 4.7], gives the desired result . $\square$

4. Subquadrature case. From this section to Section 6, we always assume that Ω is a smoothly bounded domain. By virtue of Lemma 3.3 a), the spectrum $\sigma(A_T)$ consists of eigenvalues of finite multiplicity denoted by $\cdots \leq \lambda_2^- \leq \lambda_1^- < 0 < \lambda_1^+ \leq \lambda_2^+ \leq \cdots$ with $\lambda_j^\pm \to \pm\infty$ as $j \to \infty$. Let $(e_j^\pm)_{j\in\mathbf{N}}$ denote the corresponding set of eigenfunctions with $|e_j^\pm|_2 = 1$. $(e_j^\pm)_{j\in\mathbf{N}}$ is an orthogonal basis for both $(L_T^2)^\pm$ and $E_T^\pm$. Define

$$X_\ell^0 := \operatorname{span}\{e_\ell^-, \dots, e_1^-\},\ X_0^n := \operatorname{span}\{e_1^+, \dots, e_n^+\},\ X_\ell^n := X_\ell^0 \oplus X_0^n$$

and

$$X_\ell := X_\ell^0 \oplus X^+, \quad X^n := X^- \oplus X_0^n.$$

We remark that $\|z\|_{L_T^2}^2 \leq \lambda_n^{-1}\|z\|_T^2$ for all $z \in (X^{n-1})^\perp$. Therefore, by Hölder inequality, for all $q \in (1,\ 2)$, there is a $c_q > 0$ such that

$$\|z\|_{L_T^q}^q \leq c_q\lambda_n^{-q/2}\|z\|_T^q \quad \text{for all } z \in (X^{n-1})^\perp, \tag{4.1}$$

and, for $q \in (2,\ 2(2+N)/N)$ there is $c_q > 0$ such that, letting $\theta_q := 1 - N(q-2)/4$,

$$\|z\|_{L^q_T}^q \le c_q \lambda_n^{-\theta_q} \|z\|_T^q \quad \text{for all } z \in (X^{n-1})^{\perp}. \tag{4.2}$$

For a functional $f \in \mathcal{C}^1(E_T)$ let f_ℓ denote the restriction of f on X_ℓ. A sequence $z_\ell \in X_\ell$, $\ell \in \mathbf{N}$, $z_\ell \in E_T$, respectively, is said to be a $(\mathrm{PS})_c^*$, $(\mathrm{PS})_c$, respectively, sequence if $f(z_\ell) \to c$ and $f_\ell'(z_\ell) \to 0$ (and $f'(z_\ell) \to 0$, respectively) as $\ell \to \infty$. f is said to satisfy the $(\mathrm{PS})_c^*$, $(\mathrm{PS})_c$, respectively, condition if any $(\mathrm{PS})_c^*$, $(\mathrm{PS})_c$, respectively, sequence has a convergent subsequence.

Proof of Theorem 2.1–I *Existence.* Observe that, by (h_2), $H(t,x,z) \le c(1+|z|^p)$ for all (t,x,z), which implies that in (h_1) the constant $\alpha < 2$.

First we check that the functional $J(z) = -I_T(z)$ satisfies $(\mathrm{PS})_c^*$ condition. Let $z_\ell \in X_\ell$ be such that $J(z_\ell) \to c$ and $J_\ell'(z_\ell) \to 0$. Then it follows from (h_2) that

$$\begin{aligned}
\|z_\ell\|_T^2 &= -J_\ell'(z_\ell)(z_\ell^+ - z_\ell^-) + \int_{Q_T} H_z(t,x,z_\ell)(z_\ell^+ - z_\ell^-) \\
&\le c_0 \|z_\ell\|_T + c_1 \|z_\ell\|_{L^p_T}^{p-1} \|z_\ell^+ - z_\ell^-\|_{L^p_T} \\
&\le c_2 (1 + \|z_\ell\|_T^{p-1}) \|z_\ell\|_T,
\end{aligned}$$

which implies that (z_ℓ) is bounded, since $1 < p < 2$. Now, using Lemma 3.3, a standard argument shows that (z_ℓ) has a convergent subsequence.

Similarly, it is easy to verify that J and J_ℓ satisfy $(\mathrm{PS})_c$ condition.

Next, there are $r > 0$, $\delta > 0$ and $0 \ne z_0 \in E_T^+$ such that

$$J(z) \le 0 \quad \text{for all } z \in E_T^+ \text{ with } \|z\|_T \ge r, \tag{4.3}$$

$$J(z + z_0) \ge \delta \quad \text{for all } \ z \in E_T^-. \tag{4.4}$$

In fact, by (h_2), for $z \in E_T^+$,

$$\begin{aligned}
J(z) &= \int_{Q_T} H(t,x,z) - \frac{1}{2}\|z\|_T^2 \\
&\le c_3 (1 + \|z\|_{L^p_T}^p) - \frac{1}{2}\|z\|_T^2,
\end{aligned}$$

hence, (4.3) holds. By virtue of (h_1), for $z \in E_T^-$ and $z_0 \in E_T^+$,

$$\begin{aligned} J(z+z_0) &\geq a_1 \|z+z_0\|_{L_T^\alpha}^\alpha + \frac{1}{2}\|z\|_T^2 - \frac{1}{2}\|z_0\|_T^2 \\ &\geq c_3(1-\|z_0\|_T^{2-\alpha})\|z_0\|_T^\alpha + \frac{1}{2}\|z\|_T^2 \end{aligned}$$

and we get (4.4).

Now a standard deformation argument [**8, 18**] yields a sequence $z_\ell \in X_\ell$ satisfying

$$J_\ell'(z_\ell) \to 0 \quad \text{and} \quad J(z_\ell) \to c \quad \text{with } \delta \leq c \leq M := \sup J(B_r E_T^+).$$

Finally, the $(\mathrm{PS})_c^*$ condition implies that J has a critical point z with $\delta \leq J(z) \leq M$. This proves that (HS)–(2.1) has at least one nontrivial solution which is in $W_{T,r}$ for all $r \geq 1$ by Lemma 3.4. □

Proof of Theorem 2.1–II *Multiplicity.* By assumptions, J is even and $J(0) = 0$. We claim that for each $n \in \mathbf{N}$:

(i) There are $r_n > 0$ and $\alpha_n > 0$ such that $J(z) \geq \alpha_n$ for all $z \in X^n$ with $\|z\|_T = r_n$;

(ii) There is $\beta_n > 0$ such that $J(z) \leq \beta_n$ for all $z \in (X^{n-1})^\perp$. Moreover, $\beta_n \to 0$ as $n \to \infty$.

As above, noting that $\dim(X_0^n) < \infty$, for $z \in X^n$,

$$\begin{aligned} J(z) &\geq a_1\|z\|_{L_T^\alpha}^\alpha + \frac{1}{2}\|z^-\|_T^2 - \frac{1}{2}\|z^+\|_T^2 \\ &\geq \left(c_n - \frac{1}{2}\|z^+\|_T^{2-\alpha}\right)\|z^+\|_T^\alpha + \frac{1}{2}\|z^-\|_T^2, \end{aligned}$$

so claim (i) follows. The combination of (h_1) and (h_3) implies $\gamma < 2$, so $H(t,x,z) \leq a(|z|^\gamma + |z|^p)$ for all (t,x,z). Thus it follows from (4.1) that, for $z \in (X^{n-1})^\perp$,

$$\begin{aligned} J(z) &\leq c_1 \int_{Q_T} (|z|^\gamma + |z|^p) - \frac{1}{2}\|z\|_T^2 \\ &\leq \left(c_2\lambda_n^{-\gamma/2}\|z\|_T^\gamma - \frac{1}{4}\|z\|_T^2\right) + \left(c_2\lambda_n^{-p/2}\|z\|_T^p - \frac{1}{4}\|z\|_T^2\right). \end{aligned}$$

Set

$$\beta_n := \frac{c_2(2-\gamma)}{2}\left(2\gamma c_2\lambda_n^{-1}\right)^{\gamma/(2-\gamma)} + \frac{c_2(2-p)}{2}\left(2pc_2\lambda_n^{-1}\right)^{p/(2-p)}$$

which verifies claim (ii).

Now, by virtue of [**12**, Proposition 2.2], see also [**13**], J has a sequence (z_n) of critical points satisfying $\alpha_n \le J(z_n) \le \beta_n$. The proof is finished. $\square$

Proof of Theorem 2.1$'$. Let (2.3) be satisfied. Let $\chi = \chi(s) \in \mathcal{C}^\infty(\mathbf{R}, [0, 1])$ be such that $\chi(s) = 0$ for $s \le r_0/2, \chi(s) = 1$ for $s \ge r_0$, and $\chi'(s) > 0$ for all $s \in (r_0/2, r_0)$. Set $M = \inf\left\{H(t,x,z)/r_0^\alpha : t \in \mathbf{R}\right.$ and $\left.|z| = r_0\right\}$. Consider $\widetilde{H} : \mathbf{R} \times \bar{\Omega} \times \mathbf{R}^{2m} \to \mathbf{R}$ defined by

$$\widetilde{H}(t,x,z) = (1-\chi(|z|))H(t,x,z) + \chi(|z|)M|z|^\alpha.$$

Then, by definition and (2.3), $\widetilde{H}$ is even in z, and

$$\widetilde{H}(t,x,z) \ge M|z|^\alpha; \tag{4.5}$$

$$0 < \widetilde{H}_z(t,x,z)z \le \alpha\widetilde{H}(t,x,z) \quad \text{whenever } z \ne 0; \tag{4.6}$$

$$|\widetilde{H}_z(t,x,z)| \le a_2(|z|^{\gamma-1} + |z|^{\alpha-1}). \tag{4.7}$$

Therefore, $\widetilde{H}$ satisfies the assumptions (h_0)–(h_3). It follows from Theorem 2.1 that the functional

$$J(z) := \int_{Q_T} \widetilde{H}(t,x,z) + \frac{1}{2}\,\|z^-\|_T^2 - \frac{1}{2}\,\|z^+\|_T^2$$

on E_T has a sequence (z_n) of critical points satisfying

$$0 < \alpha_n \le c_n := J(z_n) \le \beta_n \quad \text{with } \beta_n \to 0. \tag{4.8}$$

z_n solves

$$A_T z_n = \widetilde{H}_z(t,x,z). \tag{4.9}$$

By (4.6) and (4.8),

$$\beta_n \ge J(z_n) \ge \frac{2-\alpha}{2}\int_{Q_T} \widetilde{H}(t,x,z_n) \ge \frac{2-\alpha}{2\alpha}\int_{Q_T} \widetilde{H}_x(t,x,z_n)z_n;$$

hence, it follows from (4.5) that

$$\|z_n\|_{L_T^\alpha}^\alpha \le b_n := \frac{2M\beta_n}{2-\alpha}.$$

Since $b_n \to 0$, without loss of generality, assume $b_n \le 1$. Using (4.7) and (4.9),

$$\|z_n\|_T^2 = \int_{Q_T} \widetilde{H}_z(t,x,z_n)(z_n^+ - z_n^-) \le d_1 \left(\|z_n\|_{L_T^\alpha}^\gamma + \|z_n\|_{L_T^\alpha}^\alpha \right) \le 2d_1 b_n^{\gamma/\alpha}.$$

Using Lemma 3.2,

$$\|z_n\|_{L_T^{q_1}} \le d_2 b_n^{\gamma/2\alpha} \quad \text{where } q_1 = 2(2+N)/N.$$

By (4.9) and (4.7) again,

$$\|A_T z_n\|_{L_T^{r_1}} = \left(\int_{Q_T} |\widetilde{H}_z(t,x,z_n)|^{r_1} \right)^{1/r_1} \le d_3 b_n^{\gamma(\gamma-1)/2\alpha}$$
$$\text{where } r_1 = q_1/(\alpha-1),$$

consequently, by Lemma 3.1,

$$\|z_n\|_{W_{T,r_1}} \longrightarrow 0.$$

Repeating the above process, the bootstrap argument implies that $\|z_n\|_{W_{T,r}} \to 0$ for each $r > 1$. Now applying Sobolev embedding we arrive at

$$\|z_n\|_{L_T^\infty} \longrightarrow 0.$$

Thus, for all n large, z_n is a solution of (HS)–(2.1). The proof is complete. $\square$

5. Superquadrature case. In this section we consider superquadratic nonlinearities and prove Theorems 2.2 and 2.3′.

Proof of Theorem 2.2–I *Existence.* In the following we write simply $I(z) = I_T(z)$, etc. Firstly, we check that I satisfies $(\mathrm{PS})_c^*$ condition. Let $z_\ell \in X_\ell$ be such that $I(z_\ell) \to c$ and $I_\ell'(z_\ell) \to 0$. By (h_4)

$$I(z_\ell) - \frac{1}{2} I'(z_\ell) z_\ell \ge \int_{Q_T} \left(\frac{1}{2} - \frac{1}{\mu} \right) H_z(t,x,z_\ell) z_\ell - c_1$$

or

$$\int_{Q_T} H_z(t,x,z_\ell)z_\ell \le c_2(1+\|z_\ell\|_T). \tag{5.1}$$

It follows from the form of $I'(z_\ell)(z_\ell^+ - z_\ell^-)$ and (h_5), (5.1) that

$$\begin{aligned}\|z_\ell\|_T^2 &\le o(\|z\|_T) + c_3\bigg(\int_{Q_T} |H_z(t,x,z_\ell)|^{p'}\bigg)^{1/p'}\|z_\ell\|_T \\ &\le c_4\left(1+\|z_\ell\|_T^{1/p'}\right)\|z_\ell\|_T.\end{aligned}$$

Hence, (z_ℓ) is bounded in E_T. Using Lemma 3.3, a standard argument shows that (z_ℓ) has convergent subsequence.

Similarly, I and I_ℓ satisfy $(\mathrm{PS})_c$ condition.

By Lemma 3.3 and (h_6) it is easy to verify that there are $\delta > 0$ and $\rho > 0$ satisfying

$$I(z) \ge \delta \quad \text{for all } z \in \partial B_\rho E_T^+. \tag{5.2}$$

It follows from (h_4) and (h_6) that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$H(t,x,z) \ge C_\varepsilon |z|^\mu - \varepsilon|z|^2 \quad \text{for all } (t,x,z). \tag{5.3}$$

Let $e = e_1^+$. Using (5.3) and Lemma 3.3, we see that there are s_0, $R > 0$ and $M > 0$ such that

$$\sup I(\mathcal{N}) \le M \quad \text{and} \quad I_{\partial\mathcal{N}} \le 0 \tag{5.4}$$

where $\mathcal{N} := \{z + se : \ z \in B_R E_T^-,\ 0 < s < s_0\}$.

Define the following minimax value

$$c_\ell := \inf_{\lambda\in\Gamma_\ell} \max_{z\in\mathcal{N}_\ell} I(\lambda(z)),$$

where $\mathcal{N}_\ell = \mathcal{N} \cap X_\ell$ and $\Gamma_\ell = \{\lambda \in \mathcal{C}(\mathcal{N}_\ell, X_\ell) : \ \lambda|_{\partial\mathcal{N}_\ell} = id\}$. The linking theorem, see [**18, 21**], implies that

$$\delta \le c_\ell \le M,$$

and c_ℓ is a critical value of I_ℓ. Let $z_\ell \in X_\ell$ be such that $I'_\ell(z_\ell) = 0$ and $I(z_\ell) = c_\ell$. Then, up to a subsequence, (z_ℓ) is a $(\mathrm{PS})_c^*$ sequence with $\delta \le c \le M$. Therefore, by $(\mathrm{PS})_c^*$ condition, I has a nontrivial critical point, and the first conclusion of Theorem 2.2 follows. $\square$

Proof of Theorem 2.2–II *Multiplicity.* In this case, I is even and $I(0) = 0$.

Firstly, *there is $R_n > 0$ such that $I(z) \le 0$ for all $z \in X^n$ with $\|z\|_T \ge R_n$.* Indeed, by (h_4), $H(t,x,z) \ge c_1|z|^\mu - c_2$ for all (t,x,z), hence, for $z \in X^n$,

$$\begin{aligned} I(z) &\le \frac{1}{2}\|z^+\|_T^2 - \frac{1}{2}\|z^-\|_T^2 - c_1\int_{Q_T}|z|^\mu + c_3 \\ &\le \left(\frac{1}{2} - c_4\|z^+\|_T^{\mu-2}\right)\|z^+\|_T^2 - \frac{1}{2}\|z^-\|_T^2 + c_3 \end{aligned}$$

which implies clearly the desired conclusion.

Next, *for all n large, there are $r_n > 0$ and $a_n \to \infty$, $n \to \infty$, such that $I(z) \ge a_n$ for all $z \in (X^{n-1})^\perp$ with $\|z\|_T = r_n$.* By (h_5), $H(t,x,z) \le c_5(1 + |z|^p)$ for all (t,x,z). Using (4.2), for $z \in (X^{n-1})^\perp$, one has

$$\begin{aligned} I(z) &\ge \frac{1}{2}\|z\|_T^2 - c_5\int_{Q_T}|z|^p - c_6 \\ &\ge \frac{1}{2}\|z\|_T^2 - c_7\lambda_n^{-\theta_p}\|z\|_T^p - c_6. \end{aligned}$$

It is easy to check that the numbers

$$r_n = \left(pc_7\lambda_n^{-\theta_p}\right)^{-1/(p-2)} \quad \text{and} \quad a_n = \left(\frac{1}{2} - \frac{1}{p}\right)r_n^2 - c_6$$

satisfy the requirement.

Now, an application of symmetric mountain pass theorem, cf. [**12, 13**] or [**21**], completes the proof. $\square$

Proof of Theorem 2.3′. If $\xi > 0$, then (2.3) holds for some $r_0 > 0$ and the existence of (z_n) follows from Theorem 2.1′. Assume $\xi < 0$. Then $-H$ satisfies (2.3) and we have the modification $\widetilde{H}$ of $-H$ as that in proof of Theorem 2.1′. Applying the argument for Theorem 2.1′ to the functional

$$\tilde{I}(z) = \int_{Q_T} \widetilde{H}(t,x,z) + \frac{1}{2}\left(\|z^+\|_T^2 - \|z^-\|_T^2\right)$$

with exchanging the positions of E_T^+ and E_T^-, e.g., setting $X_\ell^n = \operatorname{span}\{e_n^-, \dots, e_1^-\} \oplus E_T^+$ etc., we see that it has a critical point sequence (z_n) satisfying, for n large, $I_T(z_n) = \tilde{I}(z_n) > 0$ and $0 > \xi I_T(z_n) \to 0$ as $n \to \infty$.

Similarly, applying the argument for Theorem 2.2 we obtain the sequence $\tilde{z}_n$ of critical points of I_T such that $\eta I_T(\tilde{z}_n) \to \infty$ as $n \to \infty$.

Finally, letting z stand for one of the solutions, according to standard existence and uniqueness theory for linear differential systems, $z(t,x) \neq$ any equilibrium everywhere. Let T_0 denote the minimal period of z. Using (HS) and (2.4) one obtains $a(t+T_0) = a(t)$ for all t, which implies clearly that $T_0 = T$. □

6. Convergence of subharmonic orbits. Throughout this section, let the assumptions of Theorem 2.4 be satisfied. For each $k \in \mathbf{N}$, let $Q_k = (0, kT) \times \Omega$, $L_k^r = L^r(Q_k, \mathbf{R}^{2m})$ and A_k, the operator $\mathcal{J}\partial_t - \mathcal{J}_0\Delta_x$ acting in L_k^2. Replacing Q_T by Q_k, define as before E_k and $I_k \in \mathcal{C}^1(E_k, \mathbf{R})$, etc. Critical points of I_k are solutions of

$$(\mathrm{HS})_k \qquad \begin{cases} \mathcal{J}\partial_t z - \mathcal{J}_0\Delta_x z = H_z(t,x,z) & (t,x) \in \mathbf{R} \times \Omega \\ z(t,x) = 0 & (t,x) \in \mathbf{R} \times \partial\Omega \\ z(t+kT, x) = z(t,x) & (t,x) \in \mathbf{R} \times \Omega. \end{cases}$$

The existence of a nontrivial solution z_k of $(\mathrm{HS})_k$ follows easily from the proof of Theorem 2.2. However, in order to treat the convergence of (z_k) as $k \to \infty$ we need some estimates with boundedness independent of k. This requires more details on the existence proof.

Lemma 6.1. *There are $\delta > 0$ and $\rho > 0$, both independent of k, such that $I_k(z) \geq \delta$ for all $z \in E_k^+$ with $\|z\|_k = \rho$.*

Proof. By (h_5'), $H(t,x,z) \le c_1|z|^p$ for all (t,x,z). This, together with Lemma 3.3 a), induces that there is $c_p > 0$ independent of k such that, for $z \in E_k^+$,

$$I_k(z) \ge \frac{1}{2}\,\|z\|_k^2 - c_1 \int_{Q_k} |z|^p \ge \frac{1}{2}\,\|z\|_k^2 - c_p\|z\|_k^p.$$

The conclusion follows. $\square$

Below, for each $z \in \mathcal{C}_0^\infty := \mathcal{C}_0^\infty(Q_1, \mathbf{R}^{2m})$, we extend z to $Q_k \to \mathbf{R}^{2m}$ by setting $z = 0$ on $[T, kT)$ and regard it as a kT-periodic map on $\mathbf{R}\times\Omega$. Clearly, $\mathcal{C}_0^\infty$ is invariant under the A_k-action, i.e., $A_k\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$. Let $\widetilde{E}_k$ denote the completion of $\mathcal{C}_0^\infty$ under the norm $\|\cdot\|_k$. Then $\widetilde{E}_k = \widetilde{E}_1$ for all $k \ge 1$. Take $e_k \in \widetilde{E}_k \cap (L_k^2)^+$ with $\|e_k\|_{L_k^2} = 1$. Clearly $e_k = e_1$ for all $k \in \mathbf{N}$. Set $\mathcal{M}_k(R,r) = \{z + se_1 : z \in E_k^- \text{ with } \|z\|_k < R,\ 0 < s < r\}$.

Lemma 6.2. *There are positive constants R, r and Λ, all independent of k, such that $I_k \le 0$ on $\partial\mathcal{M}_k(R,r)$ and* $\sup I_k(\mathcal{M}_k(R,r)) \le \Lambda$.

Proof. The conclusion follows easily from the facts that, for $z \in E_k^-$ and $s > 0$,

$$s(e_1,e_1)_{L_1^2} = (e_1, z^- + se_1)_{L_k^2} \le \|e_1\|_{L_1^{\alpha'}} \|z^- + se_1\|_{L_k^\alpha},$$

hence,

$$s^\alpha \le c_1\|z^- + se_1\|_{L_k^\alpha}^\alpha.$$

Consequently, using (h_1),

$$I_k(z^- + se_1) \le c_2 s^2 - \frac{1}{2}\,\|z^-\|_k^2 - c_3 s^\alpha,$$

where $c_i > 0$ are independent of k. $\square$

Now, as proving Theorem 2.2, we obtain, for each k, a solution z_k of $(\mathrm{HS})_k$ satisfying

$$\delta \le C_k := I_k(z_k) \le \Lambda \quad \text{and} \quad I_k'(z_k) = 0. \tag{6.1}$$

It follows from (6.1) and (h_5') that there is $\sigma > 0$ independent of k such that

$$\|z_k\|_{L^\infty} \geq \sigma \quad \text{for all } k \in \mathbf{N}.$$

By periodicity, we find a sequence $\ell_k \in \mathcal{N}$ such that (recalling that $Q_1 = Q_T$)

$$\max_{Q_1} |z_k(t + \ell_k T, x)| = \max_{\mathbf{R}\times\Omega} |z_k(t,x)|.$$

So, up to rescalings of t-variable, we may assume

$$\|z_k\|_{L^\infty} = \max_{Q_1} |z_k(t,x)| \geq \sigma. \tag{6.2}$$

By assumptions on H and (6.1), standard arguments show that there is $\Lambda_1 > 0$ independent of k such that

(6.3)

$$\int_{Q_k} H(t,x,z_k) \leq \Lambda_1, \quad \int_{Q_k} H_z(t,x,z_k)\cdot z_k \leq \Lambda_1 \quad \text{and} \quad \|z_k\|_k \leq \Lambda_1.$$

A bootstrap argument implies that for each $r \in [2,\infty)$, there is Λ_r independent of k such that

$$\|z_k\|^r_{W_{k,r}} \leq \Lambda_r \quad \text{and} \quad \|z_k\|^r_{L^r_k} \leq \Lambda_r, \tag{6.4}$$

cf. [**3, 10**] or the proof of Theorem 2.1′. As a consequence, $z_k \in \mathcal{C}^{0,\kappa}(Q_k, \mathbf{R}^{2m})$ for $0 \leq \kappa \leq 1$, by the Sobolev embedding theorem.

It is clear from (6.4) that, for any $r \geq 2$ and $-\infty < a < b < \infty$, there is $C(r;a,b) > 0$ depending on r and $b - a$ such that

$$\|Az_k\|_{L^r(Q_{a,b},\mathbf{R}^{2m})} \leq C(r,a,b), \tag{6.5}$$

where $Q_{a,b} = (a,b) \times \Omega$ and $Az = \mathcal{J}\partial_t z - \mathcal{J}_0 \Delta_x z$. Recall that A is an isomorphism from $B_{a,b;r}$ into $L^r(Q_{a,b}, \mathbf{R}^{2m})$ where $B_{a,b;r}$ denotes the Banach space defined as $B_{T,r}$ with Q_T replaced by $Q_{a,b}$, see [**10**]. Consequently,

$$\|z_k\|_{B_{a,b;r}} \leq C(r,a,b),$$

independent of k. Therefore, by the Sobolev embedding theorem,

$$\|z_k\|_{\mathcal{C}^{0,\kappa}(Q_{a,b},\mathbf{R}^{2m})} \le C(a,b) \tag{6.6}$$

with $C(a,b)$ depending on $b-a$.

Proof of Theorem 2.4. Taking into account (6.6) we extract a subsequence $k_n \to \infty$ such that

$$z_{k_n} \to z_\infty \quad \text{in } \mathcal{C}_{loc}(\mathbf{R}\times\Omega, \mathbf{R}^{2m}).$$

It is easy to see that z_∞ solves (HS). By (6.2), $z_\infty \in L^\infty(\mathbf{R}\times\Omega,\mathbf{R}^{2m})$ and $z_\infty \neq 0$.

Let $r \in [2,\infty)$. By (6.4), for any $a > 0$,

$$\begin{aligned}\int_{Q_{-a,a}} |z_\infty|^r &= \lim_{n\to\infty}\int_{Q_{-a,a}} |z_{k_n}|^r \\ &\le \limsup_{k\to\infty} 2\|z_k\|_{L_k^r}^r \le 2\Lambda_r.\end{aligned}$$

Letting $a \to \infty$ we see that $z_\infty \in L^r(\mathbf{R}\times\Omega,\mathbf{R}^{2m})$ for $r \in [2,\infty)$. Similarly,

$$\int_{\mathbf{R}\times\Omega} |Az_\infty|^r \le 2\Lambda_r$$

for all $r \ge 2$. From this, we find

$$\lim_{a\to\infty}\int_{Q_{a,a+1}} |Az_\infty|^r = 0,$$

so by the Sobolev embedding theorem, $z_\infty(t,\cdot) \to 0$ as $t \to \infty$. Similarly, $z_\infty(t,\cdot) \to 0$ as $t \to -\infty$. This completes the proof. $\square$

7. The case on $\mathbf{R}^N$. We now turn to the case $\Omega = \mathbf{R}^N$. Throughout the section, suppose H satisfies $(h_0), (h_4'), (h_5')$ and (h_7). Setting

$$\Psi(z) = \int_{Q_T} H(t,x,z),$$

we write

$$I(z) := I_T(z) = \frac{1}{2}\left(\|z^+\|_T^2 - \|z^-\|_T^2\right) - \Psi(z).$$

We are looking for critical points of I on E_T.

Lemma 7.1. *Ψ is bounded below and weakly sequentially lower semi-continuous with $\Psi' : X \to X$ weakly sequentially continuous.*

Proof. Since $H(t,x,x) \geq 0$, we have $\Psi(z) \geq 0$ for $z \in E_T$. Let $z_n \rightharpoonup z$. Then, by Lemma 3.3, $z_n \to z$ in L^2_{loc}, thus almost everywhere in (t,x). By a Fatou lemma,

$$\liminf_{n\to\infty} \int_{\mathbf{R}^{1+N}} H(t,x,z_n) \geq \int_{\mathbf{R}^{1+N}} \lim_{n\to\infty} H(t,x,z_n) = \int_{\mathbf{R}^{1+N}} H(t,x,z)$$

proving the lower semi-continuity of Ψ. For any $w \in \mathcal{C}_0^\infty$,

$$\Psi'(z_n)w = \int_{\mathbf{R}^{1+N}} H_z(t,x,z_n)w \longrightarrow \Psi'(z)w,$$

which, together with the boundedness of $\Psi'(z_n)$, by (h_5'), implies that Ψ' is weakly sequentially continuous. $\square$

As before, we have

Lemma 7.2. *There are constants $\delta, \rho > 0$ such that $\Phi(z) \geq \delta$ for $z \in E_T^+$, $\|z\|_T = \rho$.*

Lemma 7.3. *Let $e \in E_T^+$, $\|e\|_T = 1$. There are $R > \rho$ such that $\Phi(z) \leq 0$ for $z \in \partial M$, and $\Lambda := \sup I(M) < \infty$, where $M = \{z = z^- + se : z^- \in X^-, \|z\| < R, s > 0\}$.*

The existence part of one solution of Theorem 2.5 will be proved by virtue of the following critical point theorem due to Kryszewski and Szulkin [**15**].

Theorem 7.1. *Let X be a separable Hilbert space with the orthogonal decomposition $X = X^- \oplus X^+$ and suppose $\Phi \in \mathcal{C}^1(X, \mathbf{R})$ satisfies the hypotheses*:

(i) $\Phi(z) = \left(\|z^+\|^2 - \|z^-\|^2\right)/2 - \Psi(z)$ *where Ψ is bounded below, weakly sequentially lower semi-continuous with $\Psi' : X \to X$ weakly sequentially continuous*;

(ii) *There are constants $\delta, \rho > 0$ such that $\Phi(z) \geq \delta$ for $z \in X^+$, $\|z\| = \rho$*;

(iii) *There exist $e \in X^+$, $\|e\| = 1$, and $R > \rho$ such that $\Phi(z) \leq 0$ for $z \in \partial M$ where $M = \{z = z^- + se : z^- \in X^-, \|z\| < R, s > 0\}$.*

Then there exists a sequence (z_k) such that $\Phi'(z_k) \to 0$ and $\Phi(z_k) \to c$ for some $c \in [\delta, \ \sup \Phi(\overline{M})]$.

Proof of Theorem 2.5–I *Existence.* By Lemmas 7.1–7.3, the functional $\Phi = I$ on E_T satisfies (i)–(iii). Now by Theorem 7.1, let (z_k) be such that $I'(z_k) \to 0$ and $I(z_k) \to c$ with $\delta \leq c \leq \Lambda$. As before, it is easy to verify that (z_k) is bounded. Using the Lions' concentration compactness lemma [**17**], it is not difficult to see that, along a subsequence, there exist $a > 0$ and $(y_k) \subset \mathbf{R}^N$ such that

$$\lim_{k \to \infty} \int_{[0,T] \times B(y_k, 1)} |z_k|^2 \geq a, \tag{7.1}$$

cf., e.g., [**3, 21**]. By (7.1), we may assume that there exist $(k_1, \dots, k_N) \in \mathbf{Z}^N$ satisfying

$$\int_{[0,T] \times B(y'_k, 1+\sqrt{N})} |z_k|^2 > a/2 \tag{7.2}$$

where $y'_k := (k_1, \dots, k_N)$. Set $\bar{z}_k(t, x) := z_k(t, x_1 + k_1, \dots, x_N + k_N)$. Then $\|\bar{z}_k\|_T = \|z_k\|_T$ and we may suppose that $\bar{z}_k \to z$ weakly in E_T and locally in L^2_T. By (7.2) and the periodic dependence of H on x, $z \neq 0$ and $I'(z) = 0$. The proof is hereby completed. $\square$

We now turn to the multiplicity. Let us recall an abstract theorem on multiplicity of critical point of even functional from [**3**].

Let X be a reflexive Banach space with the direct sum decomposition $X = X^- \oplus X^+, z = z^- + z^+$ for $z \in X$, and suppose that X^- has a countable Schauder basis. Write X_w for the space X with the weak topology and similarly X_w^-. For a functional Φ on X we set $\Phi_a^b = \{z \in X : \ a \leq \Phi(z) \leq b\}$. Given an interval $J \subset \mathbf{R}$, we call a set $\mathcal{A} \subset X$ a $(\mathrm{PS})_J$-attractor if, for any $(\mathrm{PS})_c$-sequence (z_n) with $c \in J$, and any $\varepsilon, \delta > 0$ one has $z_n \in \mathcal{N}_\varepsilon(\mathcal{A} \cap \Phi_{c-\delta}^{c+\delta})$ provided n is large enough, where $\mathcal{N}_\varepsilon(F)$ denotes the ε-neighborhood of F in X. We consider a functional Φ satisfying the hypotheses:

(Φ_1) $\Phi \in \mathcal{C}^1(X, \mathbf{R})$ is even and $\Phi(0) = 0$;

(Φ_2) there exist $\delta, \rho > 0$ such that $\Phi(z) \geq \delta$ for every $z \in \partial B_\rho X^+$;

(Φ_3) there exists a strictly increasing sequence of finite dimensional subspaces $Y_n \subset X^+$ such that $\sup \Phi(X_n) < \infty$ where $X_n := X^- \oplus Y_n$, and an increasing sequence of real numbers $R_n > 0$ with $\sup \Phi(X_n \setminus B_{R_n}) < \inf \Phi(B_\rho X)$;

(Φ_4) $\Phi(z) \to -\infty$ as $\|z^-\| \to \infty$ and $\|z^+\|$ bounded;

(Φ_5) $\Phi' \colon X_\omega^- \times X^+ \to X_w^*$ is sequentially continuous, and $\Phi \colon X_\omega^- \times X^+ \to \mathbf{R}$ is sequentially upper semi-continuous;

(Φ_6) for any compact interval $J \subset (0, \infty)$ there exists a $(\mathrm{PS})_J$-attractor $\mathcal{A}$ such that $\inf\{\|z^+ - w^+\| : z, w \in \mathcal{A},\ z^+ \neq w^+\} > 0$.

Theorem 7.2. *If Φ satisfies (Φ_1)–(Φ_6), then it has an unbounded sequence (c_n) of positive critical values.*

Proof of Theorem 2.5.–II *Multiplicity.* We will apply Theorem 7.2 to $\Phi = I$ on $X = E_T$.

Clearly (Φ_1) is satisfied since H is even in z and $H(t, x, 0) = 0$. (Φ_2) follows from Lemma 7.2 and (Φ_5) from Lemma 7.1. (Φ_4) follows from the form of I and the fact that $\Psi(z) \geq 0$.

Let (e_n) be an orthonormal basis for E_T^+ and set $Y_n := \operatorname{span}\{e_1, \dots, e_n\}$ and $X_n = E_T^- \oplus Y_n$. (h_4') and (h_5') imply that for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that $H(t, x, z) \geq c_\varepsilon |z|^\mu - \varepsilon |z|^2$ for all (t, x, z). For $z \in X^n$,

$$\begin{aligned} I(z) &\leq \frac{1}{2}\left(\|z^+\|_T^2 - \|z^-\|_T^2\right) + \varepsilon \|z\|_{L_T^2}^2 - c_\varepsilon \|z\|_{L_T^\mu}^\mu \\ &\leq \left(\frac{1}{2} + \varepsilon c_1\right)\|z^+\|_T^2 - \left(\frac{1}{2} - \varepsilon c_1\right)\|z^-\|_T^2 - c_2 c_\varepsilon \|z^+\|_T^\mu. \end{aligned}$$

Now (Φ_3) follows easily.

The proof will be complete by showing that if

(†) (HS) *has only finitely many geometrically distinct solutions*,

then the condition (Φ_6) is satisfied. So we assume (†). We remark that there is $\alpha > 0$ satisfying

$$\inf I\,(\mathcal{K}\backslash\{0\}) > \alpha.$$

where $\mathcal{K} := \{z \in E_T :\ I'(z) = 0\}$. Let $\mathcal{F} \subset \mathcal{K}$ consist of arbitrarily chosen representatives of the orbits of $\mathcal{K}$ under the action of $\mathbf{Z}^N$. Since I is even, we may assume that $\mathcal{F} = -\mathcal{F}$. Let $[r]$ denote the integer part of r for any $r \in \mathbf{R}$. Using the assumptions on H it is not difficult to check the following claim, cf., [**11, 15**]:

(‡) *Let* $(z_n) \subset E_T$ *be a* $(\mathrm{PS})_c$*-sequence. Then* $c \geq 0$, (z_n) *is bounded, and either* $z_n \to 0$, *corresponding to* $c = 0$; *or* $c \geq \alpha$ *and there are* $l \leq [c/\alpha]$, $w_i \in \mathcal{F}\backslash\{0\}$, $i = 1, \dots, l$, *a subsequence denoted again by* (z_n), *and* l *sequences* $(a_{in})_n$ *in* $\mathbf{Z}^N$, $i = 1, \dots, l$ *such that*

$$\left\| z_n - \sum_{i=1}^{l} a_{in} * w_i \right\| \longrightarrow 0 \quad \text{as } n \to \infty,$$
$$|a_{in} - a_{jn}| \longrightarrow \infty \quad \text{as } n \to \infty, \ \text{if } i \neq j,$$

and

$$\sum_{i=1}^{l} I(w_i) = c.$$

Given a compact interval $J \subset (0, \infty)$ with $d := \max J$ we set $l := [d/\alpha]$ and

$$[\mathcal{F}, l] := \left\{ \sum_{i=1}^{j} k_i * w_i;\ 1 \leq j \leq l, k_i \in \mathbf{Z}^N, w_i \in \mathcal{F} \right\}.$$

As a consequence of (‡) we see that $[\mathcal{F}, l]$ is a $(\mathrm{PS})_J$-attractor. It is easy to check that

$$\inf\left\{\|u^+ - v^+\| :\ u, v \in [\mathcal{F}, l],\ u^+ \neq v^+\right\} > 0,$$

see, e.g., [**11**]. Therefore (Φ_6) is satisfied and Theorem 2.5 is proved. $\square$

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