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THE ARC LENGTH OF THE LEMNISCATE $|w^n+c|=1$

CHUNJIE WANG AND LIZHONG PENG

ABSTRACT. Let $s_n(c)$ be the arc length of the lemniscate $|w^n + c| = 1$, $c \in [0, \infty)$. We obtain some properties of the function $s_n(c)$. In particular, we prove that $s_n(c) \leq s_n(1)$, $c \in [0, \infty)$. We also give the sharp bound for $s_n(1)-2n$, that is,

$$4 \log 2 < s_n(1) - 2n \leq 2(\pi - 1).$$

1. Introduction. For a polynomial p of degree n, $\{z \in \mathbf{C} \mid |p(z)| = C\}$ is a curve in the plane known as a lemniscate, where C is a nonnegative constant. Lemniscates have a lot of interesting properties and applications, see, e.g., [**7**]. In 1958 Erdös, Herzog and Piranian proposed the following.

Conjecture A [**3**]. Suppose $p(z)$ is a monic polynomial of degree n, that is,

$$p(z) = \prod_{k=1}^{n} (z - \alpha_k),$$

where $\alpha_k \in \mathbf{C}$, $k = 1, 2, \dots, n$. Write

$$E_n(p) = \big\{z \in \mathbf{C} \,\big|\, |p(z)| = 1\big\}.$$

Then the length $|E_n(p)|$ is maximal when $p(z) = z^n + 1$, which is of length $2n + O(1)$.

This problem has been reposed by Erdös several times, see also [**2**]. Pommerenke obtained many important results on this problem, [**9**–**12**], and gave the first upper estimate [**12**] for the length of $E_n(p)$, namely $|E_n(p)| \leq 74n^2$. In 1995 Borwein [**1**] proved that $|E_n(p)| \leq 8\pi\, en$

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($\leq 69n$). In 1999 Eremenko and Hayman [**4**] improved Borwein's result: $|E_n(p)| \leq \alpha_0 n$, where $\alpha_0 < 9.173$. By the way, Conjecture A was proved in [**13**] in the case $n = 2$.

For any $c \in \mathbf{C}$, the lemniscate $|w^n + c| = 1$ has a parametric representation $w^n + c = e^{i\theta}$. Since the lemniscate $|w^n + c| = 1$ is n-fold symmetric, that is, invariant under the rotation $w \to e^{(2\pi i)/n} w$. Let $s_n(c)$ be the arc length of the lemniscate $|w^n + c| = 1$. Then we have $s_n(c) = s_n(|c|)$.

In the sequel we consider the lemniscate $|w^n + c| = 1$, where $c \geq 0$ and $n \geq 2$. Note that

$$dw = \frac{1}{n}\,(e^{i\theta} - c)^{(1/n)-1}\, ie^{i\theta}\, d\theta;$$

we obtain

$$\begin{aligned} s_n(c) &= n \int_{|w^n + c| = 1} |dw| \\ &= \int_0^{2\pi} \left| e^{i\theta} - c \right|^{(1/n)-1} d\theta \\ &= \int_0^{2\pi} \left(1 + c^2 - 2c\cos\theta\right)^{[1/(2n)]-1/2} d\theta. \end{aligned}$$

2. Some properties of the function $s_n(c)$. In this section we discuss some properties of the function $s_n(c)$. In particular, we prove that Conjecture A holds for the family of lemniscates $|w^n + c| = 1$.

In the following we consider the function

$$s_n(c) = \int_0^{2\pi} \Delta^{\alpha}\, d\theta, \quad c \geq 0, \tag{1}$$

where $\alpha \in (-1/2, -1/4]$ is a fixed real number, and

$$\Delta = 1 + c^2 - 2c\cos\theta = (c - \cos\theta)^2 + \sin^2\theta.$$

It is easy to see that

$$\cos\theta - c = \frac{1 - c^2 - \Delta}{2c}. \tag{2}$$

Lemma 1. *When* $0 < c < 1$ *we have*

$$\int_0^{2\pi} \Delta^{-1}\, d\theta = \frac{2\pi}{1-c^2},$$
$$\int_0^{2\pi} \Delta^{-2}\, d\theta = \frac{2\pi(1+c^2)}{(1-c^2)^3}.$$

Using the residue theorem, we can easily obtain Lemma 1, see also [**8**, p. 195] for a variant version.

Lemma 2. *For* $\alpha - 1 < \beta < \alpha < 0$*, we have*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \Delta^{\beta}\, d\theta\right)^{1/-\beta} \int_0^{2\pi} \Delta^{\alpha}\, d\theta \le \int_0^{2\pi} \Delta^{\alpha-1}\, d\theta.$$

Proof. Applying Hölder's inequality, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \Delta^{\alpha}\, d\theta \le \left(\frac{1}{2\pi}\int_0^{2\pi} \Delta^{\beta}\, d\theta\right)^{-\alpha/-\beta},$$

and

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \Delta^{\beta}\, d\theta\right)^{(1-\alpha)/-\beta} \le \frac{1}{2\pi}\int_0^{2\pi} \Delta^{\alpha-1}\, d\theta.$$

Taking products on both sides yields the desired result. □

Theorem 1. $s_n'(c) \ge 0$, $0 < c < 1$; $s_n'(c) \le 0$, $c > 1$.

Proof. Differentiating (1) under the integral sign, we have

$$s_n'(c) = -2\alpha \int_0^{2\pi} \Delta^{\alpha-1}(\cos\theta - c)\, d\theta. \tag{3}$$

If $0 < c < 1$, combining (3) with (2), we conclude that

$$\frac{1}{-\alpha}\, s_n'(c) = \frac{1-c^2}{c}\left[\int_0^{2\pi} \Delta^{\alpha-1}\, d\theta - \frac{1}{1-c^2}\int_0^{2\pi} \Delta^{\alpha}\, d\theta\right].$$

An application of Lemma 2 for $\beta = -1$ and Lemma 1 yields $s_n'(c) \geq 0$.

If $c > 1$ it follows easily from (3) that $s_n'(c) \leq 0$. This completes the proof of Theorem 1. $\square$

Note that $s_n(c)$ is continuous on $[0, \infty)$. Theorem 1 implies that

$$s_n(c) \leq s_n(1), \quad c \in [0, \infty).$$

Thus Conjecture A holds for the special family of lemniscates $|w^n + c| = 1$. In particular, when $0 \leq c \leq 1$, we have

$$2\pi = s_n(0) \leq s_n(c) \leq s_n(1).$$

Theorem 2. *For any $0 < c < 1$ or $c > 1$, we have $s_n''(c) \geq 0$.*

Proof. Differentiating (3) under the integral sign and integrating by parts, we have

$$\begin{aligned}
\frac{s_n''(c)}{-2\alpha} &= -\int_0^{2\pi} (\alpha-1)\Delta^{\alpha-2} \cdot 2\,(c-\cos\theta)^2\,d\theta - \int_0^{2\pi} \Delta^{\alpha-1}\,d\theta \\
&= -2\int_0^{2\pi} (\alpha-1)\Delta^{\alpha-1}\,d\theta + 2\int_0^{2\pi} (\alpha-1)\Delta^{\alpha-2}\sin^2\theta\,d\theta \\
&\quad - \int_0^{2\pi} \Delta^{\alpha-1}\,d\theta \\
&= (1-2\alpha)\int_0^{2\pi} \Delta^{\alpha-1}\,d\theta + c^{-1}\int_0^{2\pi} \sin\theta\, d\left(\Delta^{\alpha-1}\right) \\
&= (1-2\alpha)\int_0^{2\pi} \Delta^{\alpha-1}\,d\theta - c^{-1}\int_0^{2\pi} \Delta^{\alpha-1}\cos\theta\,d\theta.
\end{aligned}$$

That is,

$$\frac{s_n''(c)}{-2\alpha} = c^{-1}\int_0^{2\pi} \Delta^{\alpha-1}(c-\cos\theta)\,d\theta - 2\alpha\int_0^{2\pi} \Delta^{\alpha-1} d\theta. \tag{4}$$

If $0 < c < 1$, combining (4) with (2), we conclude that

$$\begin{aligned}
\frac{s_n''(c)}{-2\alpha} = \frac{1}{2c^2}\Bigg[(-4\alpha c^2)\int_0^{2\pi} \Delta^{\alpha-1}\,d\theta + \int_0^{2\pi} \Delta^{\alpha}\,d\theta& \\
- (1-c^2)\int_0^{2\pi} \Delta^{\alpha-1}\,d\theta&\Bigg].
\end{aligned}$$

Invoking Jensen's inequality, see, e.g., [**5**], we get

$$\left(\frac{\int_0^{2\pi}\Delta^{1+\alpha}\Delta^{-2}d\theta}{\int_0^{2\pi}\Delta^{-2}d\theta}\right)^{1/(1+\alpha)} \leq \frac{\int_0^{2\pi}\Delta^{-1}d\theta}{\int_0^{2\pi}\Delta^{-2}d\theta}.$$

It follows from a straightforward calculation and Lemma 1 that

$$\begin{aligned}(1-c^2)\int_0^{2\pi}\Delta^{\alpha-1}\,d\theta &\leq 2\pi(1+c^2)^{-1-3\alpha}(1-c^4)^{1+2\alpha}(1-c^2)^{-1}\\ &< 2\pi(1+c^2)^{-1-3\alpha}(1-c^2)^{-1}.\end{aligned}$$

On the other hand, Lemma 2 and $2\pi = s_n(0) \leq s_n(c)$ give

$$\begin{aligned}-4\alpha c^2\int_0^{2\pi}\Delta^{\alpha-1}\,d\theta + \int_0^{2\pi}\Delta^{\alpha}\,d\theta &\geq \frac{-4\alpha c^2}{1-c^2}\int_0^{2\pi}\Delta^{\alpha}\,d\theta + \int_0^{2\pi}\Delta^{\alpha}\,d\theta\\ &\geq \frac{2\pi}{1-c^2}\,(1-c^2-4\alpha c^2).\end{aligned}$$

To prove $s_n''(c) \geq 0$ for $0 < c < 1$, it suffices to show that

$$1-c^2-4\alpha c^2 \geq (1+c^2)^{-1-3\alpha}. \tag{5}$$

Note that, for a fixed $r \in [0,1)$, we have the following inequality

$$(1+x)^r \leq 1+rx, \quad x > 0.$$

Using the above inequality for $0 < -1-3\alpha < 1/2$, we obtain that

$$(1+c^2)^{-1-3\alpha} \leq 1+(-1-3\alpha)\,c^2 \leq 1-c^2-4\alpha c^2.$$

If $-1-3\alpha \leq 0$, note that

$$1-c^2-4\alpha c^2 \geq 1, \qquad (1+c^2)^{-1-3\alpha} \leq 1,$$

and (5) follows.

If $c > 1$ it follows easily from (4) that $s_n''(c) \geq 0$. This completes the proof of Theorem 2. $\square$

Remark. Many problems in the analysis of Bergman spaces involve estimating integral operators whose kernel is a power of the Bergman kernel. For any real β, let

$$J_\beta(z) = \int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}}, \quad |z| < 1.$$

Then we have the following estimate for $J_\beta(z)$.

Proposition [**6**]. *When* $|z| \to 1^-$ *we have*

$$J_\beta(z) \sim \begin{cases} 1 & \beta < 0, \\ \log\left[1/\left(1 - |z|^2\right)\right] & \beta = 0, \\ [1/\left(1 - |z|^2\right)^\beta] & \beta > 0. \end{cases}$$

Note that $s_n(c)$ is precisely $J_\beta(z)$ when $z = c \geq 0$ and $-1/2 < \alpha = -[(1+\beta)/2] \leq -1/4$.

3. Estimate of $s_n(1) - 2n$. In this section we will give the sharp bound for $s_n(1) - 2n$, that is,

$$4 \log 2 < s_n(1) - 2n \leq 2\,(\pi - 1).$$

Write $x = 1/n$, where $0 < x \leq 1$. Then it follows easily from (1) that

$$\begin{aligned} s_n(1) &= 2 \int_0^\pi (2 - 2\cos\theta)^{(x/2)-(1/2)}\, d\theta \\ &= 2 \int_0^\pi \left(2 \sin \frac{\theta}{2}\right)^{x-1} d\theta \\ &= 4 \int_0^{\pi/2} (2 \sin\theta)^{x-1}\, d\theta. \end{aligned}$$

Write

$$\frac{s_n(1) - 2n}{2} = 2 \int_0^{\pi/2} (2\sin\theta)^{x-1}\, d\theta - \frac{1}{x} := f(x). \tag{6}$$

Differentiating under the integral sign, we have

$$f'(x) = 2\int_0^{\pi/2} (2\sin\theta)^{x-1}\log(2\sin\theta)\,d\theta + \frac{1}{x^2}, \tag{7}$$

and

$$f''(x) = 2\int_0^{\pi/2} (2\sin\theta)^{x-1}\log^2(2\sin\theta)\,d\theta - \frac{2}{x^3}. \tag{8}$$

Lemma 3. *For any fixed $x \in (0,1]$ and $a > 1$, let*

$$\begin{aligned} \lambda(t) &= t^{x-1}\log^2 t, \quad 0 < t < 1; \\ \eta(s) &= a^s\left(s^2\log^2 a - 2s\log a + 2\right), \quad s > 0. \end{aligned}$$

Then we have the following inequalities:

$$\begin{aligned} \lambda'(t) &< 0, \quad 0 < t < 1; \\ \eta'(s) &> 0, \quad s > 0. \end{aligned}$$

Proof. It follows from a straightforward calculation that

$$\lambda'(t) = t^{x-2}\log t\,[(x-1)\log t + 2] < 0,$$

and

$$\eta'(s) = s^2 a^s \log^3 a > 0.$$

This completes the proof. □

Theorem 3. *For any $0 < x < 1$, we have $f''(x) > 0$.*

Proof. Note that

$$\frac{2}{\pi}\theta < \sin\theta < \theta, \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

From the above inequality, Lemma 3 and (8), we have

$$\begin{aligned}
f''(x) &> 2\int_0^{\pi/6} (2\sin\theta)^{x-1}\log^2(2\sin\theta)\,d\theta - \frac{2}{x^3}\\
&> 2\int_0^{\pi/6} (2\theta)^{x-1}\log^2(2\theta)\,d\theta - \frac{2}{x^3}\\
&= \frac{1}{x^3}\left[\left(\frac{\pi}{3}\right)^x\left(x^2\log^2\frac{\pi}{3} - 2x\log\frac{\pi}{3} + 2\right) - 2\right]\\
&> 0.
\end{aligned}$$

This completes the proof of Theorem 3. $\square$

Theorem 4. *For any* $0 < x < 1$, *we have* $f'(x) > 0$.

Proof. From (7) and L'Hospital's rule, we get

$$\begin{aligned}
&f'(0)\\
&= \lim_{x\to 0^+}\int_0^{\pi/2}\left[2(2\sin\theta)^{x-1}\log(2\sin\theta) + x(2\sin\theta)^{x-1}\log^2(2\sin\theta)\right]\,d\theta.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\int\left[2\,(2\sin\theta)^{x-1}\log(2\sin\theta) + x(2\sin\theta)^{x-1}\log^2(2\sin\theta)\right]\,d\theta\\
&\quad= \int 2\,(2\sin\theta)^{x-1}\log(2\sin\theta)\,d\theta + \int\frac{\log^2(2\sin\theta)}{2\cos\theta}\,d\left[(2\sin\theta)^x\right]\\
&\quad= \frac{\log^2(2\sin\theta)}{2\cos\theta}(2\sin\theta)^x - \int\frac{\log^2(2\sin\theta)}{(2\cos\theta)^2}(2\sin\theta)^{x+1}\,d\theta\\
&\quad= \frac{\log^2(2\sin\theta)}{2\cos\theta}(2\sin\theta)^x - \frac{1}{4}\int(2\sin\theta)^{x+1}\log^2(2\sin\theta)\,d(\tan\theta)\\
&\quad= \frac{1}{2}\cos\theta(2\sin\theta)^x\log^2(2\sin\theta)\\
&\qquad+ \frac{1}{4}\int(2\sin\theta)^{x+1}\left[(x+1)\log^2(2\sin\theta) + 2\log(2\sin\theta)\right]\,d\theta.
\end{aligned}$$

From the above we obtain

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0^+}\left[\frac{1}{4}\int_0^{\pi/2}(2\sin\theta)^{x+1}[(x+1)\log^2(2\sin\theta)+2\log(2\sin\theta)]\,d\theta\right]\\
&= \frac{1}{4}\int_0^{\pi/2}(2\sin\theta)\left[\log^2(2\sin\theta)+2\log(2\sin\theta)\right]\,d\theta\\
&> \frac{1}{4}\int_0^{\pi/2}2\,(2\sin\theta)\log(2\sin\theta)\,d\theta\\
&= 2\log 2-1>0.
\end{aligned}
$$

Thus Theorem 3 implies the desired result. $\square$

Remark. From the proof of Theorem 4 we can obtain

$$f'(0) = 2\log^2 2 - \frac{\pi^2}{24}.$$

But we have not used this sharp value of $f'(0)$.

Theorem 5. *For any positive integer n, we have*

$$4\log 2 < s_n(1) - 2n \le 2\,(\pi - 1).$$

Proof. From (6) we can easily get $f(1) = \pi - 1$. Now we turn to the calculation on

$$f(0) = \lim_{x\to 0^+} f(x).$$

From (6) we obtain

$$
\begin{aligned}
f(x) &= 2^x\int_0^{\pi/2}(\sin\theta)^{x-1}\,d\theta - \int_0^{\pi/2}(\sin\theta)^{x-1}\cos\theta\,d\theta\\
&= (2^x-1)\int_0^{\pi/2}(\sin\theta)^{x-1}\,d\theta + \int_0^{\pi/2}(\sin\theta)^{x-1}(1-\cos\theta)\,d\theta\\
&:= I_1(x) + I_2(x).
\end{aligned}
$$

It is easy to see that

$$\lim_{x\to 0^+} I_2(x) = \int_0^{\pi/2} \frac{1-\cos\theta}{\sin\theta}\, d\theta = \int_0^{\pi/2} \tan\frac{\theta}{2}\, d\theta = \log 2.$$

Let $t = \tan(\theta/2)$. We have

$$\begin{aligned} I_1(x) &= (2^x - 1)\int_0^1 \left(\frac{2t}{1+t^2}\right)^{x-1} \frac{2}{1+t^2}\, dt \\ &= \frac{2^x\,(2^x-1)}{x}\int_0^1 \frac{1}{(1+t^2)^x}\, d(t^x) \\ &= \frac{2^x\,(2^x-1)}{x}\left[\frac{1}{2^x} + 2x\int_0^1 \left(\frac{t}{1+t^2}\right)^{x+1} dt\right]. \end{aligned}$$

Taking limits on both sides, we obtain

$$\lim_{x\to 0^+} I_1(x) = \log 2.$$

Thus we conclude that

$$f(0) = \lim_{x\to 0^+} f(x) = 2\log 2.$$

Observe that $2\log 2 = f(0) < f(1) = \pi - 1$. It follows from Theorem 4 and (6) that

$$4\log 2 < s_n(1) - 2n \le 2\,(\pi - 1).$$

This completes the proof of Theorem 5. □

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REFERENCES

1. P. Borwein, *The arc length of the lemniscate* $\{|p(z)| = 1\}$, Proc. Amer. Math. Soc. **123** (1995), 797–799.

2. P. Erdös, *Extremal problems on polynomials*, in *Approximation theory* II, Academic Press, New York, 1976, pp. 347–355.

3. P. Erdös, F. Herzog and G. Piranian, *Metric properties of polynomials*, J. Analyse Math. **6** (1958), 125–148.

4. A. Eremenko and W. Hayman, *On the length of lemniscates*, Michigan Math. J. **46** (1999), 409–415.

5. G.H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

6. H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces*, Springer-Verlag, New York, 2000.

7. E. Hille, *Analytic function theory*, Vol. II, Ginn and Co., Boston, 1962.

8. S. Lang, *Complex analysis*, 2nd ed., Springer-Verlag, New York, 1985.

9. Ch. Pommerenke, *On some problems of Erdös, Herzog and Piranian*, Michigan Math. J. **6** (1959), 221–225.

10. ———, *On some metric properties of polynomials with real zeros*, Michigan Math. J. **6** (1959), 377–384.

11. ———, *On some metric properties of polynomials* II, Michigan Math. J. **8** (1961), 49–54.

12. ———, *On metric properties of complex polynomials*, Michigan Math. J. **8** (1961), 97–115.

13. Chunjie Wang, *The arc length of the lemniscate* $\left|w^2+c\right|=1$, Acta Math. Sci. **18** (1998), 297–301 (in Chinese).

Department of Mathematics, Hebei University of Technology, Tianjin 300130, P.R. China
E-mail address: wcj498@eyou.com

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China
E-mail address: lzpeng@pku.edu.cn