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NEW CONGRUENCES FOR ODD PERFECT NUMBERS

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ABSTRACT. We present some new congruences for odd perfect numbers improving on a congruence modulo 2 of Ewell.

1. Introduction. Our notation is classical. First of all, for a positive integer n we denote by $\sigma(n)$ the sum of all positive divisors of n; secondly we say that such an integer n is perfect if one has

$$2\,n = \sigma(n).$$

The main result of Ewell's paper [**2**] is the following. If n is an odd perfect number, then

$$n^2 + \sum_{k=1}^{(n-1)/2} \sigma(2k-1)\ \sigma(2n-(2k-1)) \equiv 0 \pmod 2. \tag{1}$$

The proof is intricate. It turns out that there is a simple proof of this result, see Theorem 2.6. It is a consequence of an easy counting argument and some formulae from Touchard [**5**] involving the "convolution" sums

$$S_r(n) = \sum_{k=1}^{n-1} k^r\,\sigma(k)\ \sigma(n-k)$$

This will be the first part of our paper.

In the second part, we will show that there is a simple relation between Ewell's sum as well as the "odd part" of the convolution sums for $r=0$, when computed over $2\,n$ instead of over n, i.e.,

$$S_0^*(2\,n) = \sum_{\substack{k=1\\ k \text{ odd}}}^{2\,n-1} \sigma(k)\ \sigma(2\,n-k)$$

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and the sums of the kth powers of divisors function

$$\sigma_k(n) = \sum_{d \mid n} d^{\,k}$$

for $k \in \{1, 3\}$.

These relations together with some classical formulae from Liouville, Glaisher, Lehmer and Touchard lead to new congruences modulo 32, see Corollary 3.7, that generalize Ewell's result.

2. Some results on the S_rs. We denote, as usual, by $\mathbf{N}$ the set of nonnegative integers. For n, $r \in \mathbf{N}$ with $n \geq 2$, put

$$S_r(n) = \sum_{k=1}^{n-1} k^r \ \sigma(k) \ \sigma(n-k).$$

Now, using the well known lemma:

Lemma 2.1. *$\sigma(n)$ is odd if and only if either n is a square or it is twice a square.*

We get the following result:

Proposition 2.2. *If n is odd and if it is not a square, then for $r \in \mathbf{N}$, $S_r(2n) \equiv 0 \pmod 2$.*

Proof.

Case $r = 0$. We clearly have:

$$S_0(2n) = 2 \sum_{k=1}^{n-1} \sigma(k) \ \sigma(2n-k) + \sigma(n) \ \sigma(n).$$

To conclude, note that $\sigma(n)$ is even, see Lemma 2.1.

Case $r \geq 1$. It is enough to establish the case $r = 1$.

Let Λ_0 be the set $\{k \in [1, 2n-1] \cap \mathbf{N} \ / \ k, \sigma(k)$ and $\sigma(2n-k)$ are odd$\}$.

We have $S_1(2n) \equiv \sum_{k\in\Lambda_0} 1 \equiv \ \mathrm{card}\,(\Lambda_0) \pmod 2$.

By Lemma 2.1, there exist a, b, α, $\beta \in \mathbf{N}$ such that ($k = a^2$ or $k = 2\alpha^2$) and ($2n-k = b^2$ or $2n-k = 2\beta^2$).

The condition on n implies that : $k \in \Lambda_0$ if and only if k and $2n-k$ are both odd squares. So that $2\,n$ is a sum of two distinct squares.

Set $\Lambda_1 = \{(a^2, b^2) \in \mathbf{N}^2 \ / \ 2n = a^2 + b^2\}$, $f : \Lambda_0 \to \Lambda_1$, $g : \Lambda_1 \to \Lambda_0$ such that $f(k) = (k, 2n-k)$ and $g(a^2, b^2) = a^2$. These two maps are bijections with $g = f^{-1}$. Thus, we have $\mathrm{card}\,(\Lambda_0) = \mathrm{card}\,(\Lambda_1)$.

Moreover, $(a^2, b^2) \in \Lambda_1 \Rightarrow [(b^2, a^2) \in \Lambda_1$ and $a^2 \neq b^2]$.

It follows that $S_1(2n) \equiv \mathrm{card}\,(\Lambda_0) \equiv \mathrm{card}\,(\Lambda_1) \equiv 0 \pmod 2$. □

Let $r(n)$ be the cardinality of the set $\{(a, b) \in \mathbf{Z}^2 \ / \ 2n = a^2 + b^2\}$. For an integer $s = \prod_{\substack{p|s \\ p \text{ prime}}} p^{j_p}$, consider the two multiplicative functions:

$$\tau(s) = \sum_{d|s} 1, \quad \mu(s) = \prod_{\substack{p|s \\ p\equiv 1 \pmod 4}} p^{j_p}.$$

We have the following classical lemma, see [**3**]:

Lemma 2.3. *For all positive integers n*

$$r(n) = 4\ \tau(\mu(n)).$$

Proposition 2.4. *If for some nonnegative integer k, $n = p^{4k+1}S^2$ where $p \equiv 1 \pmod 4$ is a prime number not dividing S, and if S is odd then*

$$S_r(2n) \equiv 0 \pmod 2, \quad \textit{for all} \quad r \geq 1.$$

Proof. As in 2.2, it is enough to prove the case $r = 1$.

We denote $\Lambda_2 = \{(a,b) \in \mathbf{Z}^2 \ / \ 2n = a^2+b^2\}$ and $r(2n) = \mathrm{card}\,(\Lambda_2)$. Noting that $(a^2,b^2) \in \Lambda_1$ if and only if $(\pm\, a, \pm\, b) \in \Lambda_2$, we see that $r(2n) = 4\ \mathrm{card}\,(\Lambda_1)$.

Set $S = \prod_{q|S} q^{j_q}$; thus, since $n = p^{4k+1}S^2$, we have:

$$\mu(2n) = \mu(2)\ \mu(p^{4k+1})\ \prod_{q|S} \mu(q^{2j_q})$$

which implies

$$\tau(\mu(2n)) = \tau(p^{4k+1}) \prod_{\substack{q|s \\ q\equiv 1 \pmod 4}} \tau(q^{2j_q}) = (4k+2) \prod_{\substack{q|s \\ q\equiv 1 \pmod 4}} (2j_q+1).$$

Hence, $\tau(\mu(2n)) \equiv 2 \pmod 4$. We conclude, by Lemma 2.3 that

$$S_1(2n) \equiv \mathrm{card}\,(\Lambda_0) \equiv \mathrm{card}\,(\Lambda_1) = \frac{r(2n)}{4} = \tau(\mu(2n)) \equiv 0 \pmod 2. \qquad \square$$

Proposition 2.5. *If n is odd and if it is not a square, then $S_2(2n) \equiv \sigma(n) \pmod 4$.*

Proof. We use the following Touchard's relation, see [**5**]:

$$\frac{n^2(n-1)}{6}\ \sigma(n) = 3n^2 S_0(n) - 10 S_2(n), \quad \forall\, n \in \mathbf{N} \setminus \{0,1\}.$$

By 2.1 and 2.2, $\sigma(n)$ and $S_2(2n)$ are both even. Set $\sigma(n) = 2m$ and $S_2(2n) = 2N$. Applying the relation above to $2n$, we have:

$$n^2(2n-1)m = 3n^2 S_0(2n) - 5N.$$

But, $S_0(2n)$ is even, see (2.2), so that one has: $m \equiv N \pmod 2$. The proposition follows. $\square$

We are now ready to prove Ewell's result given in formula (1) above and to show a new congruence satisfied by odd perfect numbers.

Theorem 2.6. *If n is an odd perfect number, then*

$$E_w \equiv 1 \equiv N_w \pmod 2$$

where

$$E_w = \sum_{k=1}^{(n-1)/2} \sigma(2k-1)\ \sigma(2n-(2k-1))$$

and

$$N_w = \sum_{k=(n+1)/2}^{n} \sigma(2k-1)\ \sigma(2n-(2k-1))$$

Proof. Observe that n can be written as $n = p^{4k+1}S^2$ for some nonnegative integer k where $p \equiv 1 \pmod 4$ is a prime and $\gcd(p, S) = 1$. We have: $n \equiv 1 \pmod 4$ and n is not a square. Hence by Proposition 2.4, $S_1(2n) \equiv 0 \pmod 2$.

But, modulo 2 we have:

$$\begin{aligned} S_1(2\,n) &= \sum_{k=1}^{2\,n-1} k\ \sigma(k)\ \sigma(2\,n-k) \\ &\equiv \sum_{\substack{k=1 \\ k \text{ odd}}}^{2\,n-1} \sigma(k)\ \sigma(2\,n-k) \\ &\equiv \sum_{k=1}^{n} \sigma(2k-1)\ \sigma(2\,n-(2k-1)) \\ &\equiv \sum_{k=1}^{(n-1)/2} \sigma(2k-1)\ \sigma(2\,n-(2k-1)) \\ &\quad + \sum_{k=(n+1)/2}^{n} \sigma(2k-1)\ \sigma(2\,n-(2k-1)) \end{aligned}$$

so that we obtain

$$S_1(2\,n) \equiv E_w + N_w \pmod 2.$$

Set

$$\begin{aligned}\Lambda_3 &= \left\{(a^2,\, b^2) \in ([1, n-2] \times [n+2,\, 2n-1]) \cap \mathbf{N}^2 \ / \ 2n = a^2 + b^2\right\},\\ \Lambda_4 &= \left\{(a^2,\, b^2) \in ([n, 2n-1] \times [1, n]) \cap \mathbf{N}^2 \ / \ 2n = a^2 + b^2\right\},\\ \Lambda \ &= \Lambda_3 \cup \Lambda_4 \quad \text{(disjoint union)}.\end{aligned}$$

We have, as in the proof of Proposition 2.2:

$$\begin{aligned}E_w &\equiv \sum_{(a^2,\, b^2) \in \Lambda_3} 1 = \ \text{card}\, \Lambda_3 \pmod 2,\\ N_w &\equiv \sum_{(a^2,\, b^2) \in \Lambda_4} 1 = \ \text{card}\, \Lambda_4 \pmod 2.\end{aligned}$$

Moreover, by the parity of a, b and by the fact that n is not a square, we have:

(i) $(a^2,\, b^2) \in \Lambda \ \Rightarrow \ (a^2,\, b^2 \notin \{n-1,\, n,\, n+1\})$,

(ii) $(a^2,\, b^2) \in \Lambda \ \Rightarrow \ (a^2 \neq b^2)$.

So:

$$\Lambda_3 \subset [1, n-2] \times [n+2, 2n-1], \ \ \Lambda_4 \subset [n+2, 2n-1] \times [1, n-2],$$

and

$$(a^2,\, b^2) \in \Lambda_3 \quad \text{if and only if} \quad (b^2,\, a^2) \in \Lambda_4.$$

Thus, it is sufficient to show that:

$$N_w = \sum_{k=(n+1)/2}^{n} \sigma(2k-1)\ \sigma(2n-(2k-1)) \equiv 1 \pmod 2,$$

i.e., to show that $\text{card}\,(\Lambda_4) \equiv 1 \pmod 2$.

But (see proof of Proposition 2.2),

$$2\ \text{card}\,(\Lambda_4) = \text{card}\,(\Lambda_3) + \text{card}\,(\Lambda_4) = \text{card}\,(\Lambda) = \text{card}\,(\Lambda_1) \equiv 2 \pmod 4.$$

So, we are done. □

The next proposition presents our first improvement on Ewell's result.

Proposition 2.7. *If n is an odd perfect number, then*:

$$E_w \equiv N_w \pmod 4.$$

Proof. We have seen that:

$$\begin{cases} S_2(2n) \equiv E_w + N_w \equiv 0 \pmod 2, \\ E_w \equiv N_w \equiv 1 \pmod 2, \\ \quad \text{and } S_2(2n) \equiv 2 \pmod 4, \quad \text{see 2.5.} \end{cases}$$

Furthermore,

$$S_2(2n) \equiv \sum_{\substack{k=1 \\ k \text{ odd}}}^{2\,n-1} \sigma(k)\ \sigma(2\,n-k) \equiv E_w + N_w \pmod 4.$$

The proposition follows. □

3. Some classical formulae as well as some new congruences for odd perfect numbers. First of all, we recall some classical results from Liouville, Glaisher, Lehmer and Touchard. One of Touchard's formulae was already used in the proof of Proposition 2.5.

Lemma 3.1. *Let $n > 0$ be an integer. Then we have*

(a) $n^2\,(n-1)\,\sigma(n) = 18\,n^2\,S_0(n) - 60\,S_2(n)$.

(b) $n^3\,(n-1)\,\sigma(n) = 48\,n\,S_2(n) - 72\,S_3(n)$.

(c) $S_0^*(2\,n) = (1/8)\,(\sigma_3(2\,n) - \sigma_3(n))$.

(d) $24\,(2\,S_3(n) - n\,S_2(n)) = n^3\,(\sigma_3(n) - (2\,n-1)\,\sigma(n))$.

Proof. The first two formulae (a) and (b) appear in [**5**].

Formula (c) is from Glaisher [**1**, p. 300], while the case n odd is from Liouville [**1**, p. 287].

Formula (d) is due to Lehmer [**4**, p. 680]. We correct here a misprint in the exponent of n in the original formula. □

First of all we show some simple relations that hold between $S_0^*(2\,n)$, $\sigma_3(n)$, $\sigma(n)$, E_w and N_w:

Lemma 3.2. *Let $n > 1$ be an odd integer. Then we have*

$$S_0^*(2n) = 2\,E_w + (\sigma(n))^2 = E_w + N_w = \sigma_3(n)$$

Proof.

$$\begin{aligned}
S_0^*(2\,n) &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{2\,n-1} \sigma(k)\ \sigma(2\,n-k) \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-2} \sigma(k)\ \sigma(2n-k) + \sum_{\substack{k=n+2 \\ k \text{ odd}}}^{2n-1} \sigma(k)\ \sigma(2n-k) + (\sigma(n))^2 \\
&= 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-2} \sigma(k)\ \sigma(2n-k) + (\sigma(n))^2 \\
&= 2 \sum_{j=1}^{(n-1)/2} \sigma(2j-1)\ \sigma(2n-(2j-1)) + (\sigma(n))^2 \\
&= 2\,E_w + (\sigma(n))^2.
\end{aligned}$$

$S_0^*(2n) = E_w + N_w$: see proof of Theorem 2.6.
$S_0^*(2n) = \sigma_3(n)$: by Lemma 3.1 part (c) since n is odd. □

An immediate consequence is the following:

Corollary 3.3. *If n is an odd perfect number, then*

$$E_w = \frac{\sigma_3(n) - 4\,n^2}{2}, \quad N_w = \frac{\sigma_3(n) + 4\,n^2}{2}.$$

We now state a key proposition to obtain our congruences:

Proposition 3.4. *If* $n \equiv 1 \pmod 4$, *then*:

$$S_2(n) \equiv S_3(n) \pmod 8.$$

Proof.

$$S_2(n) - S_3(n) = \sum_{k=1}^{n-1} (k^2 - k^3)\ \sigma(k)\ \sigma(n-k).$$

We note that:

if $l \equiv 3 \pmod 4$, then l is not a square and $\sigma(l) \equiv 0 \pmod 4$,

if $l \equiv 5 \pmod 8$, then l is not a square and $\sigma(l)$ is even.

Thus, in the above sum:

if k is even and if $n - k \equiv 1 \pmod 4$, then $k \equiv 0 \pmod 4$ and $(k^2 - k^3) \equiv 0 \pmod{16}$,

if k is even and if $n - k \equiv 3 \pmod 4$, then $(k^2 - k^3) \equiv 0 \pmod 4$ and $\sigma(n-k) \equiv 0 \pmod 4$,

if $k \equiv 3 \pmod 4$, then $(k^2 - k^3)$ is even and $\sigma(k) \equiv 0 \pmod 4$,

if $k \equiv 1 \pmod 8$, then $(k^2 - k^3) \equiv 0 \pmod 8$,

if $k \equiv 5 \pmod 8$, then $(k^2 - k^3) \equiv 0 \pmod 4$ and $\sigma(k)$ is even.

So, we are done. $\square$

An easy consequence is:

Proposition 3.5. *If* n *is an odd perfect number, then*:

$$S_2(n) \equiv 5\ \frac{n-1}{4} \pmod 8.$$

Proof. By Touchard's formula, in Lemma 3.1 part (b), we obtain:

$$n^4\ \frac{n-1}{4} - 6\,n\,S_2(n) + 9\,S_3(n) = 0.$$

So that, $6\,S_2(n) - S_3(n) \equiv (n-1)/4 \pmod 8$.

It follows, by Proposition 3.4, that:

$$5\,S_2(n) \equiv \frac{n-1}{4} \pmod 8,$$

or, equivalently:

$$S_2(n) \equiv 5\ \frac{n-1}{4} \pmod 8. \qquad \square$$

Using Lehmer's formula in Lemma 3.1 part (d) we obtain our second corollary:

Corollary 3.6. *If n is an odd perfect number, then*:

$$\sigma_3(n) \equiv 4n-2 \pmod{64}.$$

Proof. By Propositions 3.4 and 3.5 we have, modulo 8:

$$2S_3(n) - nS_2(n) \equiv (2-n)S_2(n) \equiv nS_2(n) \equiv 5n\ \frac{n-1}{4}.$$

So that, we obtain from Lehmer's formula in Lemma 3.1 part (d), the following congruence modulo 64:

$$\begin{aligned} 30n(n-1) &\equiv n^3\left(\sigma_3(n) - 4n^2 + 2n\right) \\ n^{13}\ .\ 30n(n-1) &\equiv \sigma_3(n) - 4n^2 + 2n \\ n^5\ .\ 30n(n-1) &\equiv \sigma_3(n) - 4n^2 + 2n. \end{aligned}$$

Thus:

$$\begin{aligned} \sigma_3(n) &\equiv 30n^7 - 30n^6 + 4n^2 - 2n \pmod{64} \\ &\equiv 56n^2 + 52n + 22 \pmod{64} \\ &\equiv 4n - 2 \pmod{64}. \qquad \square \end{aligned}$$

Our main new congruences now follow:

Corollary 3.7. *If n is an odd perfect number, then*:

$$\begin{aligned} E_w &\equiv -2n + 1 \pmod{32}, \\ N_w &\equiv 6n - 3 \pmod{32}. \end{aligned}$$

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