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ON THE CLASSIFICATION THEOREMS OF ALMOST-HERMITIAN OR HOMOGENEOUS KÄHLER STRUCTURES

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ABSTRACT. A proof by Young tableaux and symmetrizers is given of the classification theorems by Gray and Hervella of almost-Hermitian structures and by Abbena and Garbiero of homogeneous Kähler structures.

1. Introduction. As it is well known, representation theory has been applied to the classification of several geometric structures on differentiable manifolds, beginning with the almost-Hermitian structures [**10**].

An interesting case is that of homogeneous Kähler structures [**1, 4, 6**], both because of the importance of the manifolds under study and also as it gives some specific examples of representations of the unitary group $U(n)$. Moreover, Abbena-Garbiero's classification [**1**] has found an application [**8**] to spaces of negative constant holomorphic sectional curvature: The characterization of the complex hyperbolic space as the only connected simply-connected irreducible homogeneous Kähler manifold admitting a nonvanishing homogeneous Kähler structure in Abbena-Garbiero's class $\mathcal{K}_2 \oplus \mathcal{K}_4$, see [**1**] and Section 2 below. On the other hand, the almost-Hermitian case also has much interest, see [**5**] amongst many others.

The aim of the present paper is to give a proof of Gray-Hervella's [**10**] and Abbena-Garbiero's [**1**] theorems, by using Young tableaux and symmetrizers. Although other demonstrations have been given [**4**–**6**], we think that one more proof is in order due to the importance of both theorems and because the present proof can perhaps aid to a better understanding of the involved decompositions, and to solve some related questions: For instance, the expression of the tensors in the classes in the homogeneous quaternionic Kähler case, with relevant

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group Sp (n)Sp (1), see Fino [**6**], and thus the corresponding geometric properties.

2. The classification theorems.

2.1 *Gray-Hervella's and Abbena-Garbiero's theorems.* Let V be a $2n$-dimensional real vector space endowed with a complex structure J and a Hermitian inner product $\langle\ ,\ \rangle$; that is, $J^2 = -I$, $\langle JX, JY\rangle = \langle X, Y\rangle$, for any $X, Y \in V$, where I denotes the identity isomorphism of V. Let F denote the Kähler 2-form $F(X,Y) = \langle X, JY\rangle$.

From the geometric viewpoint, V is the model of the tangent space at any point of a differentiable manifold equipped with either an almost-Hermitian or a homogeneous Kähler structure.

In order to classify almost-Hermitian structures, the authors of [**10**] consider the space

$$\mathcal{S}(V)_- = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY} = -S_{XJYJZ}\} \tag{2.1}$$

of tensors satisfying the same symmetries as the covariant derivative ∇F of the Kähler form F with respect to the Levi-Civita connection of an almost-Hermitian manifold (M, g, J). By using, among other results and techniques, quadratic invariants, the authors obtain the following classification theorem

Theorem 2.1 (Gray and Hervella). *If* $\dim V \geq 6$, $\mathcal{S}(V)_-$ *decomposes into the direct sum of the following subspaces invariant and irreducible under the action of the group* $U(n)$:

$$\begin{aligned}
\mathcal{W}_1 &= \{S \in \mathcal{S}(V)_- : S_{XXZ} = 0\},\\
\mathcal{W}_2 &= \Big\{S \in \mathcal{S}(V)_- : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0\Big\},\\
\mathcal{W}_3 &= \{S \in \mathcal{S}(V)_- : S_{JXJYZ} = S_{XYZ},\ c_{12}(S) = 0\},\\
\mathcal{W}_4 &= \Big\{S \in \mathcal{S}(V)_- : S_{XYZ} = -\frac{1}{2(n-1)}\,(\langle X, Y\rangle c_{12}(S)(Z)\\
&\qquad - \langle X, Z\rangle c_{12}(S)(Y) - \langle X, JY\rangle c_{12}(S)(JZ)\\
&\qquad\qquad + \langle X, JZ\rangle c_{12}(S)(JY))\Big\},
\end{aligned}$$

$X, Y, Z \in V$, *where* c_{12} *is defined by* $c_{12}(S)(X) = \sum_{r=1}^{2n} S_{e_r e_r X}$, $X \in V$, *and* $\{e_1, \dots, e_{2n}\}$ *denotes an arbitrary orthonormal basis of* V. *If* $\dim V = 4$, *then* $\mathcal{S}(V)_- = \mathcal{W}_2 \oplus \mathcal{W}_4$. *If* $\dim V = 2$, *then* $\mathcal{S}(V)_- = \{0\}$.

In turn, in order to classify homogeneous Kähler structures, the authors of [**1**] consider the space

$$\mathcal{S}(V)_+ = \{S \in \otimes^3 V^* : S_{XYZ} = -S_{XZY} = S_{XJYJZ}\} \tag{2.2}$$

of tensors fulfilling the same symmetries as a homogeneous almost-Hermitian structure S on a connected homogeneous Kähler manifold $(M = G/H, g, J)$; that is, a $(1,2)$ tensor on M satisfying the Ambrose-Singer-Sekigawa equations [**3, 12**]

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0, \quad \widetilde{\nabla} J = 0,$$

where $\widetilde{\nabla} = \nabla - S$, ∇ denotes the Levi-Civita connection, and R its curvature tensor. By using, among other results and techniques, quadratic invariants, the authors obtain the following classification theorem

Theorem 2.2 (Abbena and Garbiero). *If* $\dim V \geq 6$, $\mathcal{S}(V)_+$ *decomposes into the direct sum of the following subspaces invariant and irreducible under the action of the group* $U(n)$:

$$\mathcal{K}_1 = \big\{S \in \mathcal{S}(V)_+ : S_{XYZ} = \tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \\ c_{12}(S) = 0\big\},$$

$$\mathcal{K}_2 = \big\{S \in \mathcal{S}(V)_+ : S_{XYZ} = \langle X, Y\rangle \alpha(Z) - \langle X, Z\rangle \alpha(Y) \\ + \langle X, JY\rangle \alpha(JZ) - \langle X, JZ\rangle \alpha(JY) \\ - 2\langle JY, Z\rangle \alpha(JX), \ \ \alpha \in V^*\big\},$$

$$\mathcal{K}_3 = \big\{S \in \mathcal{S}(V)_+ : S_{XYZ} = -\tfrac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \\ c_{12}(S) = 0\big\},$$

$$\mathcal{K}_4 = \big\{S \in \mathcal{S}(V)_+ : S_{XYZ} = \langle X, Y\rangle \beta(Z) - \langle X, Z\rangle \beta(Y) \\ + \langle X, JY\rangle \beta(JZ) - \langle X, JZ\rangle \beta(JY) \\ + 2\langle JY, Z\rangle \beta(JX), \ \ \beta \in V^*\big\},$$

$X, Y, Z \in V$, where c_{12} is defined as in the previous theorem, and

$$\alpha(X) = \frac{1}{2(n-1)}\, c_{12}(S)(X), \quad \beta(X) = \frac{1}{2(n+1)}\, c_{12}(S)(X), \quad X \in V.$$

If $\dim V = 4$, then $\mathcal{S}(V)_+ = \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \mathcal{K}_4$. If $\dim V = 2$, then $\mathcal{S}(V)_+ = \mathcal{K}_4$.

Denoting complexifications by a superscript c, we now consider the decompositions in $(\pm i)$-eigenspaces $V^c = V^{1,0} \oplus V^{0,1}$ and $V^{*c} = \lambda^{1,0} \oplus \lambda^{0,1}$, with respect to the complex structure J^c. In Salamon's notation [**11**], let $\lambda^{p,q}$ denote the space of forms of type (p, q). One has an isomorphism $\lambda^{p,q} \approx \Lambda^p \lambda^{1,0} \otimes \Lambda^q \lambda^{0,1}$. We can decompose the space

$$\mathcal{S}(V)^c = \{S \in \otimes^3 V^{*c} : S_{XYZ} = -S_{XZY}\},$$

$X, Y, Z \in V^c$, into subspaces invariant under the action of $U(n)$, as follows:

$$\begin{aligned} V^{*c} \otimes \Lambda^2 V^{*c} &= (\lambda^{1,0} \otimes \Lambda^2 \lambda^{1,0}) \oplus (\lambda^{1,0} \otimes \Lambda^1 \lambda^{1,0} \otimes \Lambda^1 \lambda^{0,1}) \\ &\quad \oplus (\lambda^{1,0} \otimes \Lambda^2 \lambda^{0,1}) \oplus (\lambda^{0,1} \otimes \Lambda^2 \lambda^{1,0}) \qquad (2.3) \\ &\quad \oplus (\lambda^{0,1} \otimes \Lambda^1 \lambda^{1,0} \otimes \Lambda^1 \lambda^{0,1}) \oplus (\lambda^{0,1} \otimes \Lambda^2 \lambda^{0,1}) \\ &\approx [V^{*c} \otimes (\lambda^{2,0} \oplus \lambda^{0,2})] \oplus (V^{*c} \otimes \lambda^{1,1}). \qquad (2.4) \end{aligned}$$

Now, since $J^c X = iX$ if $X \in V^{(1,0)}$ and $J^c X = -iX$ if $X \in V^{(0,1)}$, the space

$$\mathcal{S}(V)^c_- = \{S \in \otimes^3 V^{*c} : S_{XYZ} = -S_{XZY} = -S_{XJ^cYJ^cZ}\}, \qquad (2.5)$$

$X, Y, Z \in V^c$, complexified of Gray-Hervella's space $\mathcal{S}(V)_-$ in (2.1), is the first summand in (2.4):

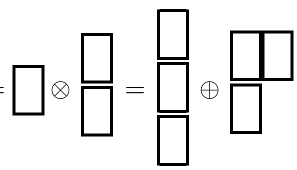
$$\mathcal{S}(V)^c_- = V^{*c} \otimes (\lambda^{2,0} \oplus \lambda^{0,2}). \qquad (2.6)$$

Similarly, the space

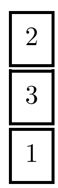
$$\mathcal{S}(V)^c_+ = \{S \in \otimes^3 V^{*c} : S_{XYZ} = -S_{XZY} = S_{XJ^cYJ^cZ}\},$$

$X, Y, Z \in V^c$, complexified of Abbena-Garbiero's space $\mathcal{S}(V)_+$ in (2.2), is the second summand in (2.4), $\mathcal{S}(V)^c_+ = V^{*c} \otimes \lambda^{1,1}$. The further decompositions of either $\mathcal{S}(V)^c_-$ or $\mathcal{S}(V)^c_+$ into subspaces invariant and irreducible under the action of $U(n)$, have a somewhat different treatment, as we shall see.

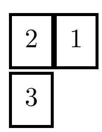
2.2 *The primitive classes* $\mathcal{W}_1, \dots, \mathcal{W}_4$ *of almost-Hermitian structures.* As usual in the theory of Young diagrams [**9**], let us denote our basic vector space by a box, that is, $\square = V^{*c}$. Then

(2.1) $\mathcal{S}(V)^c_\mp \subset \mathcal{S}(V)^c =$ 

In the almost-Hermitian case, only ordinary Young tableaux do appear. Those "standard with respect to the order 231" and having 23-skew-symmetry, that is,



and



have respective invariant and irreducible subspaces of tensors [**9**, Theorem 9.3.9] given by

$$\left\{S \in \otimes^3 V^{*c} : S_{XYZ} = \frac{1}{3} \underset{XYZ}{\mathfrak{S}} S_{XYZ}, \quad X, Y, Z \in V^c\right\}, \tag{2.2}$$

$$\left\{S \in \otimes^3 V^{*c} : \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0, \quad X, Y, Z \in V^c\right\}. \tag{2.3}$$

By virtue of (2.5), see also (2.3), we can write

$$
\begin{aligned}
\mathcal{S}(V)^{c}_{-} &= (\lambda^{1,0}\otimes\Lambda^2\lambda^{1,0}) \oplus (\lambda^{1,0}\otimes\Lambda^2\lambda^{0,1}) \\
&\quad \oplus (\lambda^{0,1}\otimes\Lambda^2\lambda^{1,0}) \oplus (\lambda^{0,1}\otimes\Lambda^2\lambda^{0,1}) \\
&= \left(\begin{array}{|c|}\hline \lambda^{1,0}\\ \hline \lambda^{1,0}\\ \hline \lambda^{1,0}\\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \lambda^{1,0} & \lambda^{1,0}\\ \hline \lambda^{1,0} \\ \cline{1-1} \end{array} \right) \\
&\quad \oplus (\lambda^{1,0}\otimes\lambda^{0,1}\wedge\lambda^{0,1}) \oplus (\lambda^{0,1}\otimes\lambda^{1,0}\wedge\lambda^{1,0}) \\
&\quad \oplus \left(\begin{array}{|c|}\hline \lambda^{0,1}\\ \hline \lambda^{0,1}\\ \hline \lambda^{0,1}\\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \lambda^{0,1} & \lambda^{0,1}\\ \hline \lambda^{0,1} \\ \cline{1-1} \end{array} \right) \qquad (2.4)\\
&= \left\{ \mathrm{Re} \left(\begin{array}{|c|}\hline \lambda^{1,0}\\ \hline \lambda^{1,0}\\ \hline \lambda^{1,0}\\ \hline \end{array} \oplus \begin{array}{|c|}\hline \lambda^{0,1}\\ \hline \lambda^{0,1}\\ \hline \lambda^{0,1}\\ \hline \end{array} \right) \right. \\
&\quad \oplus \mathrm{Re} \left(\begin{array}{|c|c|}\hline \lambda^{1,0} & \lambda^{1,0}\\ \hline \lambda^{1,0} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline \lambda^{0,1} & \lambda^{0,1}\\ \hline \lambda^{0,1} \\ \cline{1-1} \end{array} \right) \\
&\quad \oplus \mathrm{Re}\left[\left((\lambda^{1,0}\otimes\lambda^{0,1})^0\wedge\lambda^{0,1}\right) \oplus \left((\lambda^{0,1}\otimes\lambda^{1,0})^0\wedge\lambda^{1,0}\right)\right] \\
&\quad \left. \oplus \mathrm{Re}\left[(\lambda^{1,0}\otimes\lambda^{0,1})^{0\perp}\wedge(\lambda^{1,0}\oplus\lambda^{0,1})\right] \right\}^{c},
\end{aligned}
$$

where we have ordered the four primitive classes as in Gray-Hervella's Theorem 2.1, and where $(\lambda^{1,0}\otimes\lambda^{0,1})^{0\perp}$ denotes the orthogonal complement, with respect to the induced Hermitian metric, of the space of tr_{12}-traceless tensors, denoted in turn by a zero superscript. It is immediate that the tensors in the two first classes satisfy

$$S_{JXJYZ} = -S_{XYZ} \tag{2.5}$$

and the tensors in the two last classes fulfill $S_{JXJYZ} = S_{XYZ}$.

The class $\mathcal{W}_1$ corresponds to the first summand in (2.4); that is, to the representation of $U(n)$ with highest weight $(1,1,1,0,\ldots,0)$. According to [**9**, Theorem 5.2.1] and [**7**, Proposition 26.24], this representation (note the different notation for highest weights in [**7**]) is complex. Then, by (2.2), we have that

$$\begin{aligned} S_{XYZ} &= \frac{1}{24} \operatorname{Re} \underset{XYZ}{\mathfrak{S}} (S^c_{X-iJX,Y-iJY,Z-iJZ} + S^c_{X+iJX,Y+iJY,Z+iJZ}) \\ &= \frac{1}{6} \underset{XYZ}{\mathfrak{S}} (S_{XYZ} - S_{JXJYZ}); \end{aligned}$$

that is, Gray-Hervella's formula in [**10**, p. 42]. Thus, from (2.5) we obtain that $S_{XYZ} = (1/3)\mathfrak{S}_{XYZ} S_{XYZ}$, which is equivalent to the property characterizing the tensors in the class $\mathcal{W}_1$.

The second subspace in (2.4) corresponds to the irreducible representation of $U(n)$ with highest weight $(2,1,0,\ldots,0)$. As the one above, this representation is complex. By (2.2), it consists, see (2.3), of tensors S satisfying

$$\operatorname{Re} \underset{XYZ}{\mathfrak{S}} (S^c_{X-iJX,Y-iJY,Z-iJZ} + S^c_{X+iJX,Y+iJY,Z+iJZ}) = 0.$$

Thus, on account of (2.5), we deduce that $\mathfrak{S}_{XYZ} S_{XYZ} = 0$; that is, the condition for $\mathcal{W}_2$. The third summand in (2.4) clearly corresponds to the class $\mathcal{W}_3$ and the fourth summand in (2.4) to the class $\mathcal{W}_4$.

2.3 *The primitive classes $\mathcal{K}_1, \ldots, \mathcal{K}_4$ of homogeneous Kähler structures.* In order to study the further decomposition of the other subspace, $\mathcal{S}(V)^c_+$, we follow Salamon's notations [**11**], but denoting by Re the "real part," as follows: Wedging with the Kähler form $F = -i\sum_{k=1}^n \theta^k \wedge \bar{\theta}^k$ on V, where $\{\theta^k\}$ stands for a basis of $\lambda^{1,0}$, determines a $U(n)$-equivariant map $L\colon \lambda^{p-1,q-1} \to \lambda^{p,q}$. The orthogonal complement of the image of L with respect to the induced Hermitian metric is denoted by $\lambda^{p,q}_0$. The complex $U(n)$-modules $\lambda^{p,q}_0$ are irreducible. In particular, the Kähler form is a member of $\operatorname{Re}\lambda^{1,1}$ and its orthogonal complement in $\operatorname{Re}\lambda^{1,1}$ is denoted by $(\operatorname{Re}\lambda^{1,1})_0$. Let F^c denote the complexified Kähler form. One has the orthogonal decomposition $\lambda^{1,1} = \lambda_0^{1,1} \oplus \langle F^c\rangle$.

Consider the first summand

$$\Lambda^3(\lambda^{1,0} \oplus \lambda^{0,1}) = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$$

at the right-hand side in (2.1). Denoting by $(\Lambda^3(\lambda^{1,0} \oplus \lambda^{0,1}))'$ the subspace of $\Lambda^3(\lambda^{1,0} \oplus \lambda^{0,1})$ of tensors satisfying moreover $S_{XYZ} = S_{XJ^cYJ^cZ}$, we have

$$\begin{aligned}
&\big(\Lambda^3(\lambda^{1,0} \oplus \lambda^{0,1})\big)' \\
&\quad = \big\{(\Lambda^3\lambda^{1,0}) \oplus (\Lambda^2\lambda^{1,0} \otimes \Lambda^1\lambda^{0,1}) \oplus (\Lambda^1\lambda^{1,0} \otimes \Lambda^2\lambda^{0,1}) \oplus (\Lambda^3\lambda^{0,1})\big\}' \\
&\quad \approx \lambda^{2,1} \oplus \lambda^{1,2} \\
&\quad = \lambda_0^{2,1} \oplus (\lambda^{1,0} \otimes \langle F^c\rangle) \oplus \lambda_0^{1,2} \oplus (\lambda^{0,1} \otimes \langle F^c\rangle) \\
&\quad = \begin{array}{|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \multicolumn{1}{c|}{} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \cline{2-2} \end{array} \oplus \begin{array}{|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \ (V^{*c} \otimes \langle F^c\rangle).
\end{aligned}$$

In the last line:

(1) We have used composite Young tableaux, see for instance [**2**, pp. 157, 160], corresponding to mixed tensors which are traceless with respect to the second and third component, and we have put either a $\boldsymbol{\lambda}^{\mathbf{0,1}}$ or a $\boldsymbol{\lambda}^{\mathbf{1,0}}$ in boldface for the sake of visualization of those tableaux;

(2) We have used the commutativity of the tensor product, that is, that $\lambda^{1,2} \approx \Lambda^2\lambda^{0,1} \otimes \Lambda^1\lambda^{1,0}$, in order to write the second summand as the "conjugate" of the first one.

Since

$$\begin{aligned}
\big(\Lambda^2(\lambda^{1,0} \oplus \lambda^{0,1})\big)' &= \big\{(\Lambda^2\lambda^{1,0}) \oplus (\Lambda^1\lambda^{1,0} \otimes \Lambda^1\lambda^{0,1}) \oplus (\Lambda^2\lambda^{0,1})\big\}' \\
&= \lambda_0^{1,1} \oplus \langle F^c\rangle,
\end{aligned}$$

the second summand in (2.1) can be written as

$$\begin{array}{|c|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \end{array} \oplus \begin{array}{|c|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \end{array} \oplus (V^{*c} \otimes \langle F^c \rangle).$$

Consequently,

(2.1)

$$\begin{aligned} \mathcal{S}(V)^c_+ = \Bigg\{ \mathrm{Re} & \left(\begin{array}{|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \multicolumn{1}{c|}{} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \cline{2-2} \end{array} \oplus \begin{array}{|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right) \oplus (V^* \otimes \langle F \rangle) \\ & \oplus \mathrm{Re} \left(\begin{array}{|c|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \end{array} \oplus \begin{array}{|c|c|c|}\hline \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{0,1}} & \boldsymbol{\lambda}^{\mathbf{1,0}} \\ \hline \end{array} \right) \\ & \oplus (V^* \otimes \langle F \rangle)^c \Bigg\}. \end{aligned}$$

The first summand in (2.1) corresponds to the irreducible representation of $U(n)$ with highest weight $(1,1,0,\ldots,0,-1)$ and consists of tensors satisfying two conditions:

(1) The tensors are skew-symmetric in the two first indices and the block of the two first indices is symmetric with respect to the last index. Notice that this condition guarantees the final 23-skew-symmetry.

(2) The two first slots in each of the four summands corresponding to the tensors following rule (1) corresponding to the first, respectively second, composite Young tableau in (2.1) belong to $V^{(1,0)}$, respectively $V^{(0,1)}$, and the last slot belongs to $V^{(0,1)}$, respectively $V^{(1,0)}$.

That is, the tensors corresponding to the first summand are given by

$$\begin{aligned} S_{XYZ} = \frac{1}{16} \mathrm{Re}\, (& S^c_{X-iJX,Y-iJY,Z+iJZ} - S^c_{Y-iJY,X-iJX,Z+iJZ} \\ & + S^c_{Z-iJZ,X-iJX,Y+iJY} - S^c_{X-iJX,Z-iJZ,Y+iJY} \\ & + S^c_{X+iJX,Y+iJY,Z-iJZ} - S^c_{Y+iJY,X+iJX,Z-iJZ} \\ & + S^c_{Z+iJZ,X+iJX,Y-iJY} - S^c_{X+iJX,Z+iJZ,Y-iJY}) \\ = \frac{1}{2} & (S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \end{aligned} \tag{2.2}$$

which is the expression of the tensors in the class $\mathcal{K}_1 \oplus \mathcal{K}_2$. If we moreover take zero trace one obtains the tensors in the first class.

Similarly, the space of tensors corresponding to the irreducible representation of $U(n)$ with highest weight $(2,0,\dots,0,-1)$, is (the real part of) that of tensors which are symmetric in the two first indices and such that the block of the two first indices is skew-symmetric with respect to the last index, satisfying moreover the second condition above. A computation similar to the one in (2.2) gives us the space of tensors

$$S_{XYZ} = -\frac{1}{2}\left(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}\right), \tag{2.3}$$

that is, the expression of the tensors in the class $\mathcal{K}_3 \oplus \mathcal{K}_4$. If we moreover take zero trace one obtains the tensors in the third class. One has the

Proposition 2.3.

$$\mathcal{K}_1 \oplus \mathcal{K}_2 = \{S \in \mathcal{S}(V)_+ : S_{XYZ} = \frac{1}{4}\big(\mathop{\mathfrak{S}}_{XYZ} S_{XYZ} + \mathop{\mathfrak{S}}_{XJYJZ} S_{XJYJZ}\big)\},$$
$$\mathcal{K}_3 \oplus \mathcal{K}_4 = \{S \in \mathcal{S}(V)_+ : \mathop{\mathfrak{S}}_{XYZ} S_{XYZ} = 0\}.$$

Proof. The expression for $\mathcal{K}_1 \oplus \mathcal{K}_2$ is immediate from (2.2). As for $\mathcal{K}_3 \oplus \mathcal{K}_4$, if S satisfies Abbena-Garbiero's expression (2.3), then it satisfies $\mathfrak{S}_{XYZ} S_{XYZ} + \mathfrak{S}_{XJYJZ} S_{XJYJZ} = 0$, from which we obtain that

$$0 = \mathop{\mathfrak{S}}_{XYZ}\big(\mathop{\mathfrak{S}}_{XYZ} S_{XYZ} + \mathop{\mathfrak{S}}_{XJYJZ} S_{XJYJZ}\big) = 4 \mathop{\mathfrak{S}}_{XYZ} S_{XYZ}.$$

The converse is immediate. □

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