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ON TWISTED SUBGROUPS AND BOL LOOPS OF ODD ORDER

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ABSTRACT. In the spirit of Glauberman's fundamental work in B-loops and Moufang loops [**18, 19**], we prove Cauchy and strong Lagrange theorems for Bol loops of odd order. We also establish necessary conditions for the existence of a simple Bol loop of odd order, conditions which should be useful in the development of a Feit-Thompson theorem for Bol loops. Bol loops are closely related to Aschbacher's twisted subgroups [**1**], and we survey the latter in some detail, especially with regard to the so-called Aschbacher radical.

1. Introduction. A *magma* $(\mathcal{L}, \cdot)$ consists of a set $\mathcal{L}$ together with a binary operation $\cdot$ on $\mathcal{L}$. For $x \in \mathcal{L}$, define the left, respectively right, translation by x by $L(x)y = x \cdot y$, respectively $R(x)y = y \cdot x$, for all $y \in \mathcal{L}$. A magma with a two-sided neutral element 1 such that all left translations bijective is called a *left loop*. A left loop in which all right translations are bijective is called a *loop*. For basic facts about loops, we refer the reader to [**5, 7, 9, 31**]. A loop satisfying the *left Bol identity*

$$(x \cdot (y \cdot x)) \cdot z = x \cdot (y \cdot (x \cdot z))$$

or equivalently

$$L(x \cdot (y \cdot x)) = L(x)L(y)L(x)$$

for all $x, y, z \in \mathcal{L}$, is called a *left Bol loop*. A loop satisfying the mirror identity $((x \cdot y) \cdot z) \cdot y = x \cdot ((y \cdot z) \cdot y)$ for all $x, y, z \in \mathcal{L}$ is called a *right Bol loop*, and a loop which is both left and right Bol is a *Moufang loop*. For the balance of this paper, the term "Bol loop" will refer to left Bol loop; all statements about left Bol loops dualize trivially to right Bol loops. For basic facts about Bol loops, we refer the reader to [**36**] and IV.6 in [**31**] (in both cases translating from right Bol to left Bol). A *Bruck loop* is a Bol loop with the *automorphic inverse property*, i.e., $x^{-1} \cdot y^{-1} = (x \cdot y)^{-1}$. (These are also known as K-loops [**22**] and

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gyrocommutative gyrogroups [**39**].) A loop is said to be *uniquely* 2-*divisible* if the squaring map $x \mapsto x \cdot x$ is a bijection; we will abuse terminology a bit and drop the "uniquely." A 2-divisible Bruck loop is called a B-*loop* [**18**]. (Glauberman's original definition was restricted to the finite case.)

In the fundamental papers [**18, 19**], Glauberman studied finite B-loops and finite Moufang loops of odd order. In [**18**], he proved Hall, Sylow, Cauchy and Lagrange theorems for finite B-loops. In [**19**], he used the B-loop results to establish similar results for Moufang loops. He also proved Feit-Thompson theorems for both finite B-loops and finite Moufang loops of odd order. This naturally raises the question as to how far these results extend to the general case of finite Bol loops of odd order. In this paper, we begin examining this question. We make use of the notion of a *twisted subgroup* of a group, adopting the terminology of Aschbacher [**1**]. This same idea can be found in Glauberman's papers [**18, 19**], and we use his results to establish Cauchy and strong Lagrange theorems for Bol loops of odd order. We also start an attack on a Feit-Thompson theorem for Bol loops of odd order. We were not able to prove a complete Feit-Thompson result, but we present some conditions that a simple Bol loop of odd order must satisfy which we think will be crucial in a proof, if there is indeed such a theorem. We also observe that certain varieties of Bol loops of odd order, such as those in which every left inner mapping is an automorphism, are necessarily solvable.

In the next section, we present a few preliminaries from loop theory. This can be safely skipped by those who are more interested in groups than in loops. Such readers will find Sections 3 and 4 to their taste. A minimal amount of loop theory is present in Section 4, so as not to abandon completely the spirit of [**18**], although it is possible in principle to avoid loops completely. In Sections 5 and 6, we apply the results of Sections 3 and 4, respectively, to Bol loops of odd order.

Throughout this paper, we state several open problems in the hope of stimulating research into Bol loops of odd order. In the Feit-Thompson direction, the existence of *any* finite simple (non-Moufang) Bol loop is widely considered to be the most important open problem in loop theory [**35**], and we think that focusing on Bol loops of odd order is a reasonable place to start.

2. Preliminaries. In this section we review a few necessary notions from loop theory and establish some notation conventions. Binary operations in left loops will be explicitly denoted, while group operations in groups will be denoted by juxtaposition. Permutations, such as left and right translations, will act on the left of their arguments.

For a set S, we let $S!$ denote the group of all permutations of S. The *multiplication group*, $\mathrm{Mlt}(\mathcal{L})$, of a loop $\mathcal{L}$ is the subgroup of $\mathcal{L}!$ generated by all right and left translations. The *left multiplication group*, $\mathrm{LMlt}(\mathcal{L})$, of a left loop $\mathcal{L}$ is the subgroup of $\mathrm{Mlt}(\mathcal{L})$ generated by left translations. The subgroup $\mathrm{LMlt}_1(\mathcal{L}) = \{\phi \in \mathrm{LMlt}(\mathcal{L}) : \phi 1 = 1\}$ is called the *left inner mapping group* of $\mathcal{L}$. This subgroup has trivial core (recall that the core $\ker_H(G) = \bigcap_{g\in G} gHg^{-1}$ of a subgroup H in a group G is the largest normal subgroup of G contained in H). The set $L(\mathcal{L}) = \{L(x) : x \in \mathcal{L}\}$ of left translations is a left transversal (complete set of coset representatives) to each conjugate of $\mathrm{LMlt}_1(\mathcal{L})$ in $\mathrm{LMlt}(\mathcal{L})$.

These observations lead us to the following construction ([**3**]; see also [**24**]). Let G be a group, $H \le G$, and $T \subseteq G$ a left transversal of H. There is a natural G-action on T, which we denote by $\cdot$, defined by the equation $(g \cdot x)H = gxH$, that is, $g \cdot x$ is the unique representative in T of the coset gxH. This action restricted to T itself endows T with a binary operation. If $1 \in T$, then $(T, \cdot)$ turns out to be a left loop, which we call the *induced* left loop. If T is also a left transversal of each conjugate gHg^{-1}, $g \in G$, then $(T, \cdot)$ is a loop. All of the induced left loops we discuss in this paper turn out to be loops.

Proposition 2.1 [**32**]. *Let G be a group with subgroup $H \le G$ and a left transversal $T \subset G$ of H such that $\langle T\rangle = G$. The permutation representation $G \to T!$ defined by $(g \cdot x)H = gxH$, $g \in G$, $x \in T$, gives an epimorphism from G onto $\mathrm{LMlt}(T, \cdot)$. The sequence $1 \to \ker_H(G) \to G \to \mathrm{LMlt}(T, \cdot) \to 1$ is exact.*

If $\mathcal{L}$ is a Bol loop, then $\mathcal{L}$ is *power-associative*, that is, if $x^0 := 1$, $x^{n+1} := x \cdot x^n$, $x^{-n-1} := x^{-1} \cdot x^{-n}$, $n \ge 0$, then $x^m \cdot x^n = x^{m+n}$ for all $x \in \mathcal{L}$ and all integers m, n. Moreover, $\mathcal{L}$ is *left power-alternative*, which means that $L(x^n) = L(x)^n$ for all $x \in \mathcal{L}$ and all integers n. Taking $n = -1$ and $n = 2$, we obtain, respectively, the *left inverse* *property* (LIP) $L(x)^{-1} = L(x^{-1})$ and the *left alternative property* (LAP) $L(x)^2 = L(x^2)$.

The *left nucleus, middle nucleus, right nucleus*, and *nucleus* of a loop $\mathcal{L}$ are defined, respectively, by

$$\begin{aligned} \mathrm{Nuc}_l(\mathcal{L}) &:= \{x \in \mathcal{L} : x(yz) = (xy)z \ \forall\, y, z \in \mathcal{L}\} \\ \mathrm{Nuc}_m(\mathcal{L}) &:= \{y \in \mathcal{L} : x(yz) = (xy)z \ \forall\, x, z \in \mathcal{L}\} \\ \mathrm{Nuc}_r(\mathcal{L}) &:= \{z \in \mathcal{L} : x(yz) = (xy)z \ \forall\, x, y \in \mathcal{L}\} \\ \mathrm{Nuc}(\mathcal{L}) &:= \mathrm{Nuc}_l(\mathcal{L}) \cap \mathrm{Nuc}_m(\mathcal{L}) \cap \mathrm{Nuc}_r(\mathcal{L}) \end{aligned}$$

Each of these is an associative subloop of $\mathcal{L}$.

Lemma 2.2. *If $\mathcal{L}$ is a left loop, then*

$$\begin{aligned} L(\mathrm{Nuc}_l(\mathcal{L})) &= \bigcap_{x \in \mathcal{L}} L(\mathcal{L})L(x)^{-1} \\ L(\mathrm{Nuc}_m(\mathcal{L})) &= \bigcap_{x \in \mathcal{L}} L(x)^{-1}L(\mathcal{L}). \end{aligned}$$

Proof. If $g \in \bigcap_{x\in\mathcal{L}} L(\mathcal{L})L(x)^{-1}$, then $g = L(a)$ for some $a \in \mathcal{L}$, and for each $x \in \mathcal{L}$, there exists $y \in \mathcal{L}$ such that $L(y) = L(a)L(x)$. Applying both sides to 1 gives $y = a \cdot x$, and thus $L(a)L(x) = L(a \cdot x)$ for all $x \in \mathcal{L}$, i.e., $a \in \mathrm{Nuc}_l(\mathcal{L})$. Reversing the argument yields the other inclusion, and the argument for $\mathrm{Nuc}_m(\mathcal{L})$ is similar. □

Given a loop $\mathcal{L}$, a subloop $\mathcal{K}$ is said to be *normal* if, for all $x, y \in \mathcal{L}$, $x \cdot (y \cdot \mathcal{K}) = (x \cdot y) \cdot \mathcal{K}$, $x \cdot \mathcal{K} = \mathcal{K} \cdot x$, and $(\mathcal{K} \cdot x) \cdot y = \mathcal{K} \cdot (x \cdot y)$ ([**7**], p. 60, IV.1). These three conditions are clearly equivalent to the pair

$$x \cdot (\mathcal{K} \cdot y) = \mathcal{K} \cdot (x \cdot y) \qquad \text{and} \qquad x \cdot (\mathcal{K} \cdot y) = (x \cdot \mathcal{K}) \cdot y \tag{2.1}$$

for all $x, y \in \mathcal{K}$.

3. Twisted subgroups. Although the notion of a twisted subgroup of a group has been around for some time, see Remark 5.19, we follow here the terminology of Aschbacher [**1**], who proved one of the main structural results about twisted subgroups, our Proposition 3.9 below. Our definition is a trivial modification of his.

For a subset T of a group G, we use the notation $T^{-1} := \{x^{-1} : x \in T\}$ and $xTx := \{xyx : y \in T\}$ for $x \in T$.

Definition 3.1 [**1**]. A subset T of a group G is a *twisted subgroup* of G if

(i) $1 \in T$,

(ii) $T^{-1} = T$, and

(iii) $xTx \subseteq T$ for all $x \in T$.

Remark 3.2. One can replace (ii) and (iii) with the equivalent assertion

(ii′) $xy^{-1}x \in T$ for all $x, y \in T$.

A twisted subgroup T of a group G is said to be *uniquely* 2-*divisible* if each $x \in T$ has a unique square root in T, that is, a unique element $x^{1/2} \in T$ such that $(x^{1/2})^2 = x$. As we do with loops, we will abuse terminology slightly and drop the adverb "uniquely."

An easy induction argument shows the following

Proposition 3.3 [**1**, Lemma 1.2(1)]. *Let G be a group and let $T \subseteq G$ be a twisted subgroup. Then for each $x \in T$, $\langle x \rangle \subseteq T$.*

Remark 3.4. In some cases, portions of Definition 3.1 are redundant.

1. For *finite* groups, the proof of Proposition 3.3 shows that a subset satisfying (i) and (iii) necessarily satisfies (ii) [**1**, Lemma 1.2(1)].

2. If $T \subseteq G$ is a left transversal of a subgroup $H \leq G$ such that (iii) holds, then (ii) holds. Indeed, for $x \in T$, let $x' \in T$ denote the representative of $x^{-1}H$. Then $x'xx' \in T$ and $x'xx'H = x'H$, which implies $x'xx' = x'$. Thus $x' = x^{-1}$. In this case, the induced left loop $(T, \cdot)$ is a Bol loop; see Proposition 5.2.

3. Glauberman showed that in the finite 2-divisible case, both (i) and (ii) are redundant, [**18**, Lemma 3]; [**19**, Remark 7]. More precisely, he showed that if T is a subset of a group G satisfying (iii) and such that every element of T has finite odd order, then (i) and (ii) hold, and every element of T has a unique square root.

Of course, any subgroup is a twisted subgroup, but the notion of twisted subgroup is modeled on the following example which is not a subgroup.

Example 3.5. Let G be a group, and fix $\tau \in \mathrm{Aut}\,(G)$. Define

$$K(\tau) := \{g \in G : g^{\tau} = g^{-1}\}.$$

Then $K(\tau)$ is a twisted subgroup of G. If $\tau^2 = 1$, define

$$B(\tau) := \{gg^{-\tau} : g \in G\}$$

Then $B(\tau)$ is a twisted subgroup of G and $B(\tau) \subseteq K(\tau)$.

Example 3.6. Let T be a twisted subgroup of a group G. For $x \in T$, define $\theta_x \in T!$ by $\theta_x y = xyx$ for all $y \in T$. Then $\theta_1 = 1_{T!}$, $\theta_{x^{-1}} = \theta_x^{-1}$, and $\theta_x \theta_y \theta_x = \theta_{xyx}$ for all $x, y \in T$. Thus $\hat{T} = \{\theta_x : x \in T\}$ is a twisted subgroup of $T!$. For later reference, we will denote by $\hat{G}$ the subgroup of $T!$ generated by $\hat{T}$.

The *associates* of a twisted subgroup T of a group G are the translates $aT = Ta^{-1}$, $a \in T$.

Proposition 3.7 [**1**, Lemma 1.5(1)]. *Every associate of a twisted subgroup is a twisted subgroup.*

Most interesting results about twisted subgroups are predicated upon the assumption that a twisted subgroup T generates its group G. In this case we will just say that T is a generating twisted subgroup of G. Contained in such T are important normal subgroups of G. First we consider the intersection of all associates.

Theorem 3.8. *Let G be a group with generating twisted subgroup T, and let*

$$T^{\#} = \bigcap_{x\in T} xT.$$

Then $T^{\#} \subseteq T$, $T^{\#} = \bigcap_{x\in T} Tx$, and $T^{\#} \triangleleft G$.

Proof. $T^{\#} \subseteq T$ is clear since $1 \in T$, while $T^{\#} = \bigcap_{x\in T} Tx$ follows from $xT = Tx^{-1}$ for $x \in T$. Now fix $a, b \in T^{\#}$ and $x \in T$. There exists $u, v \in T$ such that $a = xu \in xT$ and $b = u^{-1}v \in u^{-1}T$ (using $T^{-1} = T$). Hence $ab = xv \in xT$, and since $x \in T$ is arbitrary, $ab \in T^{\#}$. Thus $T^{\#}$ is a subgroup of G. For each $y \in T$,

$$yT^{\#}y^{-1} = \bigcap_{x\in T} yxTy^{-1} = \bigcap_{x\in T} (yxy)T = \bigcap_{x\in T} \theta_y xT = T^{\#}.$$

Since T generates G, $T^{\#}$ is normal in G. □

A more important normal subgroup sitting inside a twisted subgroup was introduced by Aschbacher [**1**, p. 117]. Our motivating discussion is a simplified version of his. Let G be a group and let $T \subseteq G$ be a generating twisted subgroup G. Consider the group $G_0 = \langle (x, x^{-1}) : x \in T\rangle < G \times G$ generated by the graph $\{(x, x^{-1}) : x \in T\}$ of the inversion mapping $x \mapsto x^{-1}$ on T. Let $\pi_i : G \times G \to G$ denote the projection onto the ith factor. As a subgroup of $G \times G$, G_0 is invariant under the action of the swapping automorphism $(x, y) \mapsto (y, x)$. This automorphism restricts to an isomorphism of the kernels $\mathrm{Ker}\,(\pi_i|_{G_0})$. Each kernel is obviously isomorphic to the following subgroup of G:

$$T' = \{x_1 \cdots x_n : x_1^{-1} \cdots x_n^{-1} = 1,\ x_i \in T\}. \tag{1}$$

We have $T' = \pi_1(\mathrm{Ker}\,(\pi_2|_{G_0})) \triangleleft \pi_1(G_0) = G$, since T generates G. From the preceding discussion, we see that G_0 is the graph of an automorphism τ of G if and only if $T' = \langle 1\rangle$. In other words, T is a subset of some $K(\tau)$ if and only if $T' = \langle 1\rangle$. This proves almost all of the following result.

Proposition 3.9 [**1**, Theorem 2.2]. *Let G be a group with generating twisted subgroup T. There exists $\tau \in \mathrm{Aut}\,(G)$ with $\tau^2 = 1$ such that*

$T \subseteq K(\tau)$ if and only if $T' = \langle 1 \rangle$. In this case the automorphism τ is uniquely determined.

Proof. All that remains is the uniqueness and the order. If $T \subseteq K(\sigma)$ for some $\sigma \in \mathrm{Aut}\,(G)$, then $\tau\sigma^{-1}$ centralizes T, but then $\sigma = \tau$ since T generates G. Since $T \subseteq K(\sigma)$ implies $T \subseteq K(\sigma^{-1})$, it follows that $\tau^2 = 1$. □

For a twisted subgroup T, whether it generates G or not, we define its (Aschbacher) *radical* to be the normal subgroup T' given by (1) [**1**, p. 117]. If $T' = \langle 1 \rangle$, then we say that T is *radical-free.*

If T is radical-free and generates G, we will refer to the uniquely determined $\tau \in \mathrm{Aut}\,(G)$ of order 2 such that $T \subseteq K(\tau)$ as being the corresponding *Aschbacher automorphism.*

Proposition 3.10. *Let G be a group with generating twisted subgroup T. Then T' is contained in every associate of T, and hence $T' \subseteq T^{\#}$.*

Proof. The principal assertion is [**1**, Theorem 2.1(3)], and the rest follows from the definition of $T^{\#}$. □

Remarks 3.11.

(1) If T is a *proper* generating twisted subgroup of G, then T' and $T^{\#}$ are proper normal subgroups of G. Thus twisted subgroups of simple groups are radical-free and the intersection of all their associates is trivial.

(2) If T is actually a subgroup of G, then the radical T' is the derived subgroup of T. This motivates our choice of notation, which is different from that of [**1**].

Besides the canonical projection of the previous proposition, there is another radical-free twisted subgroup associated with any twisted subgroup. Here we use the definitions and notation of Example 3.6.

Theorem 3.12. *Let G be a group, let $T \subseteq G$ be a twisted subgroup, and let $\hat{T} = \{\theta_x : x \in T\}$ and $\hat{G} = \langle \hat{T} \rangle$. Then $\hat{T}$ is a radical-free twisted subgroup of $\hat{G}$.*

Proof. If $\theta_{x_1} \cdots \theta_{x_n} \in \hat{T}'$ for some $x_i \in T$, $1 \le i \le n$, then $1 = \theta_{x_n} \cdots \theta_{x_1} 1 = x_n \cdots x_1^2 \cdots x_n$, and rearranging gives $x_1 \cdots x_n^2 \cdots x_1 = 1$. But then for all $y \in T$,

$$\theta_{x_1} \cdots \theta_{x_n} y = \theta_{x_1} \cdots \theta_{x_n} \theta_{x_n} \cdots \theta_{x_1} y = \theta_{x_1 \cdots x_n^2 \cdots x_1} y = y.$$

Thus $\theta_{x_1} \cdots \theta_{x_n} = 1_{\hat{G}}$, and therefore $\hat{T}' = \langle 1 \rangle$. □

In view of the preceding theorem, it is not surprising that the radical is exactly the obstruction to a natural permutation representation of a group G on a generating twisted subgroup T.

Theorem 3.13. *Let G be a group with generating twisted subgroup T. The mapping $\theta : T \to \hat{T}$ defined by $\theta_x y = xyx$, $x, y \in T$, extends to a homomorphism $\theta : G \to \hat{G}$ if and only if $T' = \langle 1 \rangle$. In this case,* $\ker(\theta) = Z(G) \cap C_G(\tau)$ *where $\tau \in \mathrm{Aut}(G)$ is the Aschbacher automorphism.*

Proof. The subgroup $\langle (x, \theta_x) : x \in T \rangle$ of $G \times T!$ is the graph of a homomorphism from G into $T!$ if and only if the group $K = \{\theta_{x_1} \cdots \theta_{x_n} : x_1 \cdots x_n = 1,\ x_i \in T\} = \langle 1 \rangle$. The mapping $K \to T$; $\phi \mapsto \phi 1$ is a homomorphism with image in T'. Indeed, if $\phi = \theta_{x_1} \cdots \theta_{x_n}$ for $x_i \in T$ with $x_1 \cdots x_n = 1$, then $\phi 1 = x_n \cdots x_1 \in T'$. This homomorphism is clearly onto, and if $\phi 1 = 1$, then $x_n \cdots x_1 = 1$, whence $\phi = \theta_{x_1} \cdots \theta_{x_n} = 1_{\hat{G}}$. Therefore K is isomorphic to T'. This establishes the first assertion. Assume now that $T' = \langle 1 \rangle$ and let $\tau \in \mathrm{Aut}(G)$ denote the Aschbacher automorphism. Fix $g = x_1 \cdots x_n \in \ker(\theta)$ for $x_i \in T$. Then $g y x_n \cdots x_1 = y$ for all $y \in T$. Taking $y = 1$, we have $x_n \cdots x_1 = g^{-1}$. Thus g centralizes T. Since T generates G, $g \in Z(G)$. Also, $g^\tau = x_1^\tau \cdots x_n^\tau = (x_n \cdots x_1)^{-1} = g$. Conversely, if $g = x_1 \cdots x_n \in Z(G) \cap C_G(\tau)$ for $x_i \in T$, then $g^{-1} = g^{-\tau} = x_n^{-\tau} \cdots x_1^{-\tau} = x_n \cdots x_1$, and so $g \in \ker(\theta)$. □

Corollary 3.14. *Let G be a simple group with generating twisted subgroup T. The mapping $\theta : T \to \hat{T}$ defined by $\theta_x y = xyx$, $x, y \in T$, extends to an isomorphism $\theta : G \to \hat{G}$.*

4. 2-divisible twisted subgroups and B-loops. We now focus our attention on 2-divisible twisted subgroups and their associated B-loops. Much, though not all, of this section is an adumbration of Glauberman's fundamental results [**18, 19**]. We give, often simpler, proofs of some of his results to make the exposition self-contained.

Lemma 4.1. *Let G be a group, and let $T \subseteq G$ be a twisted subgroup in which every element has finite order. The following are equivalent*:

(i) *T is 2-divisible*;

(ii) *every element of T has odd order*;

(iii) *no element of T has order 2. If, in addition, T has finite order, then these conditions imply*:

(iv) *$|T|$ is odd.*

Proof. If T is 2-divisible, then obviously no elements of T have even order, and so (i) implies (ii) and (iii). The equivalence of (ii) and (iii) is trivial. Now assume (ii) and let $z \in T$ be given with order $2k+1$. Then z^{k+1} is a square root of z in T. If $y \in T$ were another square root of z, then $y^{2(2k+1)} = 1$, and so $y^{2k+1} = 1$. But then $z^{2k+1} = y^{2k+1} = z^k y$ so that $y = z^{k+1}$. Thus (i) holds. For the remaining assertion, note that the inversion mapping $x \mapsto x^{-1}$ is a permutation of the set $T \backslash \{1\}$. If T had even order, then this mapping would necessarily fix some $a \neq 1$. But then $a^2 = 1$, whence T is not 2-divisible. Thus (i) implies (iv). □

Example 4.2. In general, condition (iv) of Lemma 4.1 does not imply the other conditions. Indeed, let $G = S_3$, the symmetric group on 3 letters and let T be the set of transpositions. Then $|T| = 3$, but every element of T has order 2.

Radical-free, generating, 2-divisible twisted subgroups are "rigid" in the sense that they are uniquely determined by the Aschbacher automorphism. For a subset S of a group G, we denote $S^2 = \{x^2 : x \in S\}$.

Theorem 4.3. *Let G be a group, let $T \subseteq G$ be a radical-free, generating twisted subgroup, and let $\tau \in \mathrm{Aut}(G)$ be the corresponding Aschbacher automorphism. Then*

$$K(\tau)^2 \subseteq B(\tau) \subseteq T \subseteq K(\tau).$$

In particular, if T is 2-divisible, then $B(\tau) = T = K(\tau)$, and T is a left transversal of $C_G(\tau)$ in G.

Proof. For $g \in K(\tau)$, $g^2 = gg^{-\tau} \in B(\tau)$, and so $K(\tau)^2 \subseteq B(\tau)$. For $g \in G$, $g = x_1 \cdots x_n$ for some $x_i \in T$. Thus $gg^{-\tau} = x_1 \cdots x_n x_n \cdots x_1 \in T$, since T is a twisted subgroup. Thus $B(\tau) \subseteq T$. The equality in the 2-divisible case follows immediately. Now, for $g \in G$, we have $a := (gg^{-\tau})^{1/2} \in T$, and it is easy to check that $h := a^{-1}g \in C_G(\tau)$. The uniqueness of the decomposition $g = ah$ is obvious. □

Definition 4.4. Let G be a group with a 2-divisible twisted subgroup T. Define a binary operation $\odot : T \times T \to T$ by

$$x \odot y := (xy^2x)^{1/2}$$

for $x, y \in T$. We follow Glauberman's notation [**19**] and denote the magma $(T, \odot)$ by $T(1/2)$. We denote the left multiplication maps for $T(1/2)$ by $b_x y := (xy^2x)^{1/2}$ for $x, y \in T$.

Lemma 4.5. *Let G be a group with a 2-divisible twisted subgroup T.*

1. *$T(1/2)$ is a B-loop.*

2. *Integer powers of elements in T formed in G agree with those in $T(1/2)$. Thus an element has finite order in T if and only if it has the same order in $T(1/2)$.*

3. *If T is radical-free and generates G, then $T(1/2)$ agrees with the left loop structure induced on T as a left transversal.*

Proof. (1) and (2) follow from Lemma 3 in [**18**] and the following remark. For (3), let $\tau \in \mathrm{Aut}(G)$ be the Aschbacher automorphism, and note that for $x, y \in T$, $x \odot y = ((xy)(xy)^{-\tau})^{1/2}$. □

Remark 4.6. It is slightly more common in the loop theory literature to use the operation

$$x \odot' y := x^{1/2} y x^{1/2}$$

for $x, y \in T$. Clearly the squaring map $x \mapsto x^2$ is an isomorphism of $(T, \odot)$ onto $(T, \odot')$. That some authors prefer $\odot'$ is partly because the B-loop $(T, \odot')$ is isotopic to a quasi-group structure on T given by $(x, y) \mapsto xy^{-1}x$. (For the notion of isotopy, see any of the standard references [**5, 7, 31**].) With different terminology than that used here, the preceding construction on 2-divisible twisted subgroups (using either $\odot$ or $\odot'$) can be found in Foguel and Ungar [**16**], Glauberman [**18**, Lemma 3], Kikkawa [**23**, Theorem 5], Kreuzer [**25**], and Kiechle [**22**, Chapter 6D]. There is a related construction in uniquely 2-divisible loops $\mathcal{L}$ which goes as follows: for $x, y \in \mathcal{L}$, define $x \odot'' y = x^{1/2} \cdot (y \cdot x^{1/2})$. For certain loops $(\mathcal{L}, \cdot)$, the new magma $(\mathcal{L}, \odot'')$ turns out to be a B-loop. For 2-divisible Moufang loops, this construction is due to Bruck [**7**, VII.5.2, p. 121]. For uniquely 2-divisible Bol loops, it is implicit in the work of Belousov [**4, 5**], and is spelled out in work of Nagy and Strambach [**29**, p. 301, Theorem 7] as well as the recent dissertation of Nagy [**28**, Tétel 2.3.6]. As it turns out, these loop-based constructions are no more general than the construction for twisted subgroups, because they all depend on the fact that the loop in question can be identified in a natural way with a twisted subgroup. The B-loop structure is then transferred from the twisted subgroup to the loop. We will see how this works for Bol loops in Section 6.

The following is clear from the definitions.

Lemma 4.7. *Let G be a group with 2-divisible twisted subgroup T, and let $s \in T!$ denote the squaring map on T: $s(x) = x^2$. Then $b_x = s^{-1}\theta_x s$ for all $x \in T$. Thus $\mathrm{LMlt}(T(1/2))$ is conjugate in $T!$ to $\hat{G} = \langle \theta_x : x \in T \rangle$.*

Theorem 4.8. *Let G be a group with a generating, 2-divisible twisted subgroup T. The mapping $b : T \to T!$ defined by $b_x y = (xy^2x)^{1/2}$, $x, y \in T$, extends to a homomorphism $b : G \to T!$ if and only if $T' = \langle 1 \rangle$. In this case, there is an exact sequence*

$$1 \longrightarrow Z(G) \cap C_G(\tau) \longrightarrow G \longrightarrow \mathrm{LMlt}(T(1/2)) \longrightarrow 1$$

where $\tau \in \mathrm{Aut}\,(G)$ *is the Aschbacher automorphism, and* $Z(G) \cap C_G(\tau)$ *is the core of* $C_G(\tau)$ *in* G.

Proof. This follows from Lemma 4,.7, Theorem 3.13 and Proposition 2.1. □

One reason it is particularly convenient to work with the B-loop associated with a 2-divisible twisted subgroup is the following.

Lemma 4.9 (cf. [**18**, p. 379, Lemma 4]. *Let* G *be a group, and let* $T \subseteq G$ *be a* 2*-divisible twisted subgroup. Then* $K \subseteq T$ *is a twisted subgroup of* G *if and only if* $K(1/2) := (K, \odot)$ *is a subloop of* $T(1/2)$.

Proof. This is immediate from the definition of $\odot$. □

The second corollary to the following result is [**18**, p. 384, Corollary 3].

Theorem 4.10. *Let* G *be a finite group, let* $T \subseteq G$ *be a* 2*-divisible twisted subgroup, and let* $A \subseteq T$ *be a subgroup of* G.

1. *If* A *is normal in* G, *then* $|A|$ *divides* $|T|$.
2. *If* A *is abelian, then* $|A|$ *divides* $|T|$.

Proof. (1) For each $x \in T$, note that $xA = x^{1/2}Ax^{1/2} \subseteq T$, and thus $\{xA : x \in T\}$ partitions T into subsets of equal cardinality.

(2) $A(1/2)$ is an abelian group isomorphic to A. The restriction of

$$b : T \longrightarrow \mathrm{LMlt}(T(1/2));\, x \longmapsto (y \longmapsto x \odot y)$$

to A is a homomorphism of $A(1/2)$ onto its image. The orbits $\{\{b_x y : x \in A\} : y \in T\}$ clearly partition T, and the orbit through $1 \in T$ is A itself since A is 2-divisible. The action of A on any orbit is regular since $T(1/2)$ is a loop. □

Corollary 4.11. *Let* G *be a group and let* $T \subseteq G$ *be a finite* 2*-divisible twisted subgroup. Then* $|T^{\#}|$ *and* $|T'|$ *divide* $|T|$.

Corollary 4.12 (Lagrange's theorem). *Let G be a group and let $T \subseteq G$ be a finite 2-divisible twisted subgroup. Then for every $x \in T$, the order of x divides $|T|$.*

Proof. Apply Theorem 4.10 (1) to the subgroup $\langle x \rangle \subseteq T$. □

Remark 4.13. As Example 4.2 indicates, Lagrange's theorem does not hold for all twisted subgroups.

The following is a distilled version of [**19**, Theorem 14].

Theorem 4.14. *Let G be a finite group, and let $T \subseteq G$ be a 2-divisible, generating twisted subgroup. Then G has odd order.*

Proof. Assume first that T is radical-free, and let $\tau \in \mathrm{Aut}\,(G)$ denote the Aschbacher automorphism. By Theorem 4.3, $T = B(\tau)$. By Glauberman's Z^* theorem [**20**, Theorem 1], there exists a normal subgroup N of $G\langle \tau \rangle$ such that $|N|$ is odd and $\tau N \in Z(G\langle \tau \rangle / N)$. But then for all $g \in G$, $g\tau g^{-1}\tau = gg^{-\tau} \in N$. Thus $T \subseteq N$. Since T generates G, $G = N$. For the general case, G/T' must have odd order, and thus by Corollary 4.11, $|G| = |G/T'|\,|T'|$ is odd. □

Definition 4.15. Let π be a set of primes. A positive integer n is a π-*number* if $n = 1$ or if n is a product of primes in π. For every positive integer n, let n_π denote the largest π-number that divides n. As usual, a finite group G is a π-*group* if $|G| = |G|_\pi$. If $T \subseteq G$ is a twisted subgroup, then we say that T is a *twisted* π-*subgroup* of G if $|T| = |T|_\pi$. We say that T satisfies the *Hall* π-*condition* if there exists a twisted π-subgroup S of G such that $S \subset T$ and $|S| = |T|_\pi$. If $\pi = \{p\}$, we say that T satisfies the *Sylow* p-*condition* if T satisfies the Hall $\{p\}$-condition.

Lemma 4.16. *Let G be a finite group of odd order, let π be a set of primes, and let $\beta \in \mathrm{Aut}\,(G)$ have order 2. Then every π-subgroup of G fixed by β is contained in a Hall π-subgroup of G fixed by β.*

Proof. Since G is solvable [**13**], this is just [**18**, p. 391, Lemma 11]. □

Glauberman remarked that the following result can be established by purely group-theoretical means [**19**, p. 413, Remark 7].

Theorem 4.17 [**19**, Theorem 15]. *Let G be a finite group, let $T \subseteq G$ be a 2-divisible, generating twisted subgroup, and let π be a set of odd primes. Then T is a twisted π-subgroup if and only if G is a π-group.*

Proof. By Theorem 4.3, $|T|$ divides $|G|$, and so if G is a π-group, then T is certainly a twisted π-subgroup. For the converse, assume first that T is radical-free, let $\tau \in \mathrm{Aut}\,(G)$ denote the Aschbacher automorphism, and set $H := C_G(\tau)$. Let P_0 be a Hall π-subgroup of H. By Theorem 4.3, $|G|$ is odd, and so by Lemma 4.16, P_0 is contained in some Hall π-subgroup P of G which is fixed by τ. By Theorem 4.3, $S := P \cap T$ is a left transversal of $P_0 = P \cap H$ in P, and hence $|S| = |P|/|P_0| = |G|_\pi/|H|_\pi = [G : H]_\pi = |T|_\pi = |T|$. Thus $T \subseteq P$, and since T generates G, we have $G = P$. In the general case, G/T' is a π-group, and so by Corollary 4.11, $|G| = |G/T'|\,|T'|$ is a π-number. □

Theorem 4.18 (Hall's theorem, cf. [**18**, p. 392, Theorem 8]). *Let G be a finite group, and let $T \subseteq G$ be a 2-divisible twisted subgroup. For every set π of primes, T satisfies the Hall π-condition, and thus $T(1/2)$ has a Hall π-subloop.*

Proof. Without loss of generality, assume T generates G. Since T can be identified with its radical-free image $b(T) \subseteq \mathrm{LMlt}(T(1/2))$, there is no loss of generality in assuming that T is radical-free. Repeating the proof of Theorem 4.17, we obtain a Hall π-subgroup P of G fixed by τ such that $S := P \cap T$ satisfies $|S| = |T|_\pi$. $S(1/2)$ is a Hall π-subloop of $T(1/2)$ by Lemma 4.9. □

Remark 4.19. Using Theorem 4.17, i.e., [**19**, Theorem 15], Glauberman showed that the Hall π-subloops of $T(1/2)$ are all conjugate under $C_G(\tau)$, that every prime dividing the number of such subloops also divides $|T|$ and is not in π, and that every π-subloop of $T(1/2)$, that is, every twisted π-subgroup of G contained in T, is contained in a Hall π-subloop; see [**18**, Theorem 8].

Corollary 4.20 (Sylow's theorem, [**18**, p. 394, Corollary 3]). *Let G be a finite group with a 2-divisible twisted subgroup T. For every prime p, T satisfies the Sylow p-condition, and thus $T(1/2)$ has a Sylow p-subloop.*

Remark 4.21. In [**18**], Glauberman originally gave separate proofs under different hypotheses of the Sylow and Hall theorems for B-loops, because at the time it was not known that the group generated by a 2-divisible twisted subgroup must have odd order. In light of his later result [**19**, Theorem 14], our Theorem 4.14, the Sylow result easily follows from the Hall result. The additional properties mentioned in Remark 4.19 obviously hold in the Sylow case as well.

Corollary 4.22 (Cauchy's Theorem, cf. [**18**, p. 394, Corollary 1]). *Let G be a finite group, and let $T \subseteq G$ be a 2-divisible twisted subgroup. If a prime $p \mid |T|$, then T contains an element of order p.*

Proof. By Corollary 4.20, there is a twisted p-subgroup S of G such that $S \subseteq T$ and $|S| = |T|_p$. The rest follows from Lagrange's theorem (Corollary 4.12). □

Remark 4.23. There is no hope of extending Cauchy's theorem to all twisted subgroups. There exists a twisted subgroup of order 180 which does not have an element of order 5. We will discuss this further in Remark 6.3.

Proposition 4.24 (Strong Lagrange theorem). *Let G be a finite group, and let $T \subseteq G$ be a 2-divisible twisted subgroup. If $A \subseteq B \subseteq T$ are twisted subgroups of G, then $|A|$ divides $|B|$.*

Proof. ([**18**, p. 395, Corollary 4]). □

Remark 4.25. Feder [**11**] recently extended Proposition 4.24 to *strong near subgroups*, which include twisted subgroups of odd order as a special case. Roughly speaking, strong near subgroups are twisted subgroups in which the 2-elements are well-behaved.

5. Bol loops. We now apply the results of Section 3 to Bol loops. In fact, Bol loops are related to twisted subgroups in more than one way.

Example 5.1. Let $\mathcal{L}$ be a loop, and let $L(\mathcal{L}) = \{L(x) : x \in \mathcal{L}\}$ denote its set of left translations. Then $\mathcal{L}$ is a Bol loop if and only $L(\mathcal{L})$ is a twisted subgroup of $\mathrm{LMlt}(\mathcal{L})$. Also, if $\mathcal{K}$ is a subloop of $\mathcal{L}$, then $L(\mathcal{K})$ is a twisted subgroup of $\mathrm{LMlt}(\mathcal{L})$.

More generally, we have the following.

Proposition 5.2 [**24**, Remark 4.4 (2)]. *Let G be a group, $H \leq G$, and $T \subseteq G$ a transversal of H. If T is a twisted subgroup, then $(T, \cdot)$ is a Bol loop. Conversely, if H is core-free and $(T, \cdot)$ is a Bol loop, then T is a twisted subgroup.*

Example 5.3. Let $\mathcal{L}$ be a Bol loop. For each $x \in \mathcal{L}$, set $P(x) = L(x)R(x)$, and let $P(\mathcal{L}) = \{P(x) : x \in \mathcal{L}\}$. Then $P(\mathcal{L})$ is a twisted subgroup of the group $\mathrm{PMlt}(\mathcal{L}) := \langle P(x) : x \in \mathcal{L}\rangle$. This is really just a special case of Example 3.6. Indeed, for $x, y \in \mathcal{L}$, we have

$$\theta_{L(x)} L(y) = L(P(x)y). \tag{5.1}$$

Thus for $x, y, z \in \mathcal{L}$, we compute

$$\begin{aligned} L(P(x \cdot (y \cdot x))z) &= \theta_{L(x \cdot (y \cdot x))} L(z) \\ &= \theta_{L(x)L(y)L(x)} L(z) \\ &= \theta_{L(x)} \theta_{L(y)} \theta_{L(x)} L(z) \\ &= L(P(x)P(y)P(x)z). \end{aligned}$$

Thus $P(x \cdot (y \cdot x)) = P(x)P(y)P(x)$ as claimed. The other properties of twisted subgroups follow similarly.

Example 5.4. Let $\mathcal{L}$ be a Bol loop. Then for each $x \in \mathcal{L}$, the triple

$$B(x) = (P(x), L(x^{-1}), L(x))$$

is an autotopism of $\mathcal{L}$. (For the notion of autotopism, see any of the standard references [**5, 7, 31**]). Conversely, if $\mathcal{L}$ is a loop in which each $B(x)$ is an autotopism, then $\mathcal{L}$ is a Bol loop. Let $\mathrm{Btp}(\mathcal{L}) = \langle B(x) : x \in \mathcal{L} \rangle$ denote the group of all *Bol autotopisms* of $\mathcal{L}$. Then from Examples 5.1 and 5.3, we see that the set $B(\mathcal{L}) = \{B(x) : x \in \mathcal{L}\}$ is a twisted subgroup of $\mathrm{Btp}(\mathcal{L})$ (or of the entire autotopism group of $\mathcal{L}$). Geometrically, the Bol autotopism group $\mathrm{Btp}(\mathcal{L})$ is isomorphic to a subgroup of the collineation group of the associated 3-net, namely the direction-preserving collineation group generated by Bol reflections [**17**].

Recall that for a group G with twisted subgroup T, the group $\hat{G} \subseteq T!$ is defined by $\hat{G} = \langle \theta_x : x \in T \rangle$, see Example 3.6.

Lemma 5.5. *Let $\mathcal{L}$ be a Bol loop. Then* $\mathrm{PMlt}(\mathcal{L}) \cong \widehat{\mathrm{LMlt}}(\mathcal{L})$. *The isomorphism is defined on generators by $P(x) \mapsto \theta(L(x))$. In case $\mathcal{L}$ is* 2*-divisible, we also have* $\mathrm{PMlt}(\mathcal{L}) \cong \mathrm{LMlt}(\mathcal{L}(1/2))$.

Proof. The first assertion follows from (5.1) in Example 5.3. The second follows from Lemma 4.7. □

The distinction, therefore, between $\mathrm{PMlt}(\mathcal{L})$ and $\widehat{\mathrm{LMlt}}(\mathcal{L})$ is that the former acts directly on the loop $\mathcal{L}$, while the latter acts on the transversal $L(\mathcal{L})$.

Corollary 5.6. *Let $\mathcal{L}$ be a* Bol *loop. Then $P(\mathcal{L})$ is a radical-free twisted subgroup of* $\mathrm{PMlt}(\mathcal{L})$.

Proof. This follows from Lemma 5.5 and Theorem 3.12. □

First we consider the interpretation in $\mathcal{L}$ of the normal subgroup $T^{\#}$ for $T = \mathrm{LMlt}(\mathcal{L})$.

Theorem 5.7. *If* $\mathcal{L}$ *is a* Bol *loop, then* $L(\mathcal{L})^{\#} = L(\mathrm{Nuc}_l(\mathcal{L})) = L(\mathrm{Nuc}_m(\mathcal{L}))$.

Proof. $\mathcal{L}$ has LIP, and so by Lemma 2.2, $L(\mathrm{Nuc}_m(\mathcal{L})) = \bigcap_{x\in\mathcal{L}} L(x^{-1})L(\mathcal{L}) = L(\mathcal{L})^{\#}$. The other equality follows Theorem 3.8. □

Remark 5.8. The equality $\mathrm{Nuc}_l(\mathcal{L}) = \mathrm{Nuc}_m(\mathcal{L})$ for left loops with LIP is well-known, e.g., [**22**, p. 62]. Expressed in terms of a subset T, such as $L(\mathcal{L})$, of a group (such as $\mathrm{LMlt}(\mathcal{L})$), this just says that the equality $\bigcap_{x\in T} xT = \bigcap_{x\in T} Tx$ holds provided that $T^{-1} = T$.

Corollary 5.9 [**27**, p. 405, Lemma 1]. *Let* $\mathcal{L}$ *be a Bol loop. Then* $\mathrm{Nuc}_l(\mathcal{L})$ *is a normal subloop.*

Proof. Using Theorems 5.7 and 3.8, the conditions of (2.1) are easily checked. □

Next we turn to the radical.

Definition 5.10. Let $\mathcal{L}$ be a Bol loop. The *radical* of $\mathcal{L}$ is the set $\mathcal{L}' := \{x \in \mathcal{L} : L(x) \in L(\mathcal{L})'\}$. In case $\mathcal{L}' = \{1\}$, i.e., $L(\mathcal{L})' = \langle 1\rangle$, we will say that $\mathcal{L}$ is *radical-free.*

Corollary 5.11. *Let* $\mathcal{L}$ *be a Bol loop with radical* $\mathcal{L}'$. *Then* $\mathcal{L}'$ *is an associative normal subloop of* $\mathcal{L}$ *contained in* $\mathrm{Nuc}_l(\mathcal{L})$.

Proof. That $\mathcal{L}' \subseteq \mathrm{Nuc}_l(\mathcal{L})$ follows from $L(\mathcal{L})' \subseteq L(\mathcal{L})^{\#}$ and Theorem 5.7. Thus $\mathcal{L}'$ is associative and the conditions of (2.1) follow easily. □

Remarks 5.12.

(1) As isomorphic abstract groups, there is, of course, no meaningful distinction between the radical $\mathcal{L}'$ of a Bol loop $\mathcal{L}$ and the radical $L(\mathcal{L})'$ of the twisted subgroup $L(\mathcal{L})$ of left translations, particularly if one

identifies the loop $\mathcal{L}$ with the induced loop structure on the transversal $L(\mathcal{L})$. However, the distinction does help clarify the normality of $\mathcal{L}'$ as a sub*loop* of $\mathcal{L}$ versus the normality of $L(\mathcal{L})'$ as a *subgroup* of $\mathrm{LMlt}(\mathcal{L})$.

(2) Let $\mathcal{L}$ be a Bruck loop, and let $\tau \in \mathrm{Aut}\,(\mathrm{LMlt}(\mathcal{L}))$ denote conjugation by the inversion mapping $x \mapsto x^{-1}$. Then the automorphic inverse property is equivalent to $L(x)^{\tau} = L(x^{-1})$ for all $x \in \mathcal{L}$. Thus τ is the Aschbacher automorphism of $\mathrm{LMlt}(\mathcal{L})$, and hence $\mathcal{L}$ is a radical-free Bol loop.

Theorem 5.13. *Let $\mathcal{L}$ be a Bol loop, and let $G = \mathrm{LMlt}(\mathcal{L})$. The mapping $L(\mathcal{L}) \to P(\mathcal{L})$; $L(x) \mapsto P(x)$ extends to a homomorphism from G onto $\mathrm{PMlt}(\mathcal{L})$ if and only if $\mathcal{L}' = \{1\}$. In this case, the kernel of the homomorphism is $Z(G) \cap C_G(\tau)$ where $\tau \in \mathrm{Aut}\,(G)$ is the Aschbacher automorphism.*

Proof. This follows from Lemma 5.5 and Theorem 3.13. □

Next we consider the Bol autotopism group $\mathrm{Btp}\,(\mathcal{L})$ of a Bol loop $\mathcal{L}$. Let $\Phi_i : \mathrm{Btp}\,(\mathcal{L}) \to \mathrm{Mlt}(\mathcal{L})$; $(f_1, f_2, f_3) \mapsto f_i$ denote the projection onto the ith component. Clearly Φ_1 is an epimorphism onto $\mathrm{PMlt}(\mathcal{L})$ and Φ_2 and Φ_3 are epimorphisms onto $\mathrm{LMlt}(\mathcal{L})$.

In the Bol loop context, the subloop we call the radical made its first appearance in the work of Funk and Nagy [**17**, p. 67, Theorem 1]. The following is the algebraic version of their geometric result.

Theorem 5.14. *Let $\mathcal{L}$ be a Bol loop, and let $\Phi_3 : \mathrm{Btp}\,(\mathcal{L}) \to \mathrm{LMlt}(\mathcal{L})$ be the projection onto the third factor. Then $\ker\,(\Phi_3) \cong \mathcal{L}'$.*

Proof. $(f, g, 1) \in \ker\,(\Phi_3)$ if and only if g can be written as $g = L(x_1^{-1}) \cdots L(x_n^{-1})$ for some $x_i \in \mathcal{L}$, $i = 1, \ldots, n$, such that $L(x_1) \cdots L(x_n) = I$. Thus $(f, g, 1) \in \ker(\Phi_3)$ if and only if $g \in L(\mathcal{L})'$, and so the restriction of Φ_2 to $\ker\,(\Phi_3)$ is an isomorphism onto $L(\mathcal{L})'$. □

Remark 5.15. In particular, if $\mathcal{L}$ is a radical-free Bol loop, then the group $\mathrm{Btp}\,(\mathcal{L})$ simultaneously encodes both the graph of the Aschbacher automorphism $\tau \in \mathrm{Aut}\,(\mathrm{LMlt}(\mathcal{L}))$ and the graph of the homomorphism $\mathrm{LMlt}(\mathcal{L}) \to \mathrm{PMlt}(\mathcal{L})$ described in Theorem 5.13. These are given by, respectively, $f_3 \mapsto f_2$ and $f_3 \mapsto f_1$ for $(f_1, f_2, f_3) \in \mathrm{Btp}\,(\mathcal{L})$.

Lemma 5.16. *Let $\mathcal{L}$ be a loop and set $G := \mathrm{LMlt}(\mathcal{L})$. Then $Z(G) = G \cap \{R(x) : x \in \mathrm{Nuc}_r(\mathcal{L})\}$. Therefore the set $\mathcal{M} := \{x \in \mathrm{Nuc}_r(\mathcal{L}) : R(x) \in G\}$ is an abelian group.*

Proof. An element $a \in \mathcal{L}$ is in $\mathrm{Nuc}_r(\mathcal{L})$ if and only if $R(a)$ centralizes G in the full multiplication group $\mathrm{Mlt}(\mathcal{L})$. So if some such $R(a) \in G$, then $R(a) \in Z(G)$. Conversely, if $g \in Z(G)$, then setting $a = g1$, we have $x \cdot a = L(x)g1 = gL(x)1 = gx$, and so $g = R(a)$ and $a \in \mathrm{Nuc}_r(\mathcal{L})$. The rest follows because the mapping $R : \mathcal{M} \to Z(G);\ x \mapsto R(x)$ is an anti-isomorphism. □

Theorem 5.17. *Let $\mathcal{L}$ be a Bol loop, let $G = \mathrm{LMlt}(\mathcal{L})$, and let $\Phi_1 : \mathrm{Btp}\,(\mathcal{L}) \to \mathrm{PMlt}(\mathcal{L})$ be the projection onto the first factor. Then* $\ker(\Phi_1) \cong Z(G) \cap \{g : g = L(x_1) \cdots L(x_n) = L(x_1^{-1}) \cdots L(x_n^{-1}), x_i \in \mathcal{L},\ i = 1, \ldots, n\}$*. If $\mathcal{L}$ is radical-free, then* $\ker(\Phi_1) \cong Z(G) \cap C_G(\tau)$ *where $\tau \in \mathrm{Aut}\,(G)$ is the Aschbacher automorphism.*

Proof. A triple $(1, f_2, f_3)$ of permutations is an autotopism if and only if $f_2 = f_3 = R(a)$ where $a = f_2(1) \in \mathrm{Nuc}_r(\mathcal{L})$. As in the proof of Lemma 5.16, this holds if and only if $R(a)$ centralizes $\mathrm{LMlt}(\mathcal{L})$ in $\mathrm{Mlt}(\mathcal{L})$, and so $(1, R(a), R(a)) \in \mathrm{Btp}\,(\mathcal{L})$ if and only if $R(a) \in Z(\mathrm{LMlt}(\mathcal{L}))$ and $R(a) = L(x_1) \cdots L(x_n) = L(x_1^{-1}) \cdots L(x_n^{-1})$ for some $x_i \in \mathcal{L},\ i = 1, \ldots, n\}$. The remaining assertion follows immediately. □

Corollary 5.18. *Let $\mathcal{L}$ be a radical-free Bol loop, let $G = \mathrm{LMlt}(\mathcal{L})$, and let $\tau \in \mathrm{Aut}\,(G)$ be the Aschbacher automorphism. If $Z(G) \cap C_G(\tau) = \langle 1 \rangle$, then* $\mathrm{Btp}\,(\mathcal{L}) \cong \mathrm{LMlt}(\mathcal{L}) \cong \mathrm{PMlt}(\mathcal{L})$.

Remark 5.19. Before proceeding on to 2-divisible Bol loops, it is probably worthwhile to insert a few historical remarks. The concept of twisted subgroup (though obviously not the terminology we have adopted), and its relationship with quasi-group and loop theory, has been around for some time, and is not limited to the connection with Bol loops. For example, a *Fischer group* is a group G and a subset $T \subseteq G$ of involutions which generate G such that for all $x, y \in T$, $(xy)^3 = 1$, and $xyx \in T$. If $1 \in T$, then T is a twisted subgroup. Fischer groups arise in the study of distributive, symmetric quasi-groups and commutative Moufang loops of exponent 3 [**14**]; [**6**, p. 133]. In a different, but related direction, if we give a twisted subgroup T of a group G the binary operation $x \star y := xy^{-1}x$, $x, y \in T$, then $(T, \star)$ is a left quasi-group which is balanced ($x \star y = y$ if and only if $y \star x = x$), left distributive ($x \star (y \star z) = (x \star y) \star (x \star z)$), left key ($x \star (x \star y) = y$), and idempotent ($x \star x = x$). (Other subsets of groups can also be given this structure, such as conjugacy class with the operation $(x, y) \mapsto xyx^{-1}$.) If $T = G$, $(T, \star)$ is called the "core" of G (this is not the same usage as in group theory), and the same properties hold even if G is a Moufang loop [**7**]. Studies of these structures, with twisted subgroups as a principal example, can be found in the work of Nobusawa and his collaborators, see [**30** and the references therein], who were in turn influenced by the work of Loos [**26**] in symmetric spaces. See also Pierce [**33, 34**] and Umaya [**38**]. Doro [**10**] used these structures in his study of simple Moufang loops. Nowadays the structure $(T, \star)$ is known as an *involutory quandle*, thanks largely to Joyce's applications of the idea to knot theory [**21**]. As far as we have been able to determine, Aschbacher's paper [**1**], which was motivated by work of Feder and Vardi [**12**], seems to be the first in which twisted subgroups are used for a purpose other than the study of quasi-groups and loops.

6. Bol loops of odd order. We saw from Example 4.2 that a twisted subgroup of odd order need not be 2-divisible. However, a twisted subgroup of odd order which has a compatible Bol loop structure is indeed 2-divisible. This is, in fact, a well-known consequence of the left power-alternative property for Bol loops.

Proposition 6.1 e.g., [**22**]. *The order of any element of a finite Bol loop divides the order of the loop.*

In particular, instead of stating results for finite, 2-divisible Bol loops, we may simply state them for Bol loops of odd order.

For Bol loops of odd order, the Cauchy and Strong Lagrange theorems for twisted subgroups immediately transfer to the loop level.

Theorem 6.2 (Cauchy's theorem). *Let $\mathcal{L}$ be a Bol loop of odd order. For every prime p dividing $\mathcal{L}$, there exists $x \in \mathcal{L}$ of order p.*

Proof. By Corollary 4.22, there exists $L(x) \in L(\mathcal{L})$ of order p. Since $L(x^n) = L(x)^n$ for all n, x has order p. □

Remark 6.3. As mentioned in Remark 4.23 on the twisted subgroup level, Cauchy's theorem does not extend to all Bol loops, because the simple Moufang loop of order 180 does not have an element of order 5.

Theorem 6.4 (Strong Lagrange theorem). *Let $\mathcal{L}$ be a Bol loop of odd order. If $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{L}$ are subloops, then $|\mathcal{K}_1|$ divides $|\mathcal{K}_2|$.*

Proof. By Proposition 4.24, $|L(\mathcal{K}_1)|$ divides $|L(\mathcal{K}_2)|$. □

Problem 6.5. *Does the strong Lagrange property hold for all Bol loops?*

Remark 6.6. If a classification of finite, simple Bol loops were known, then it would be enough to verify the strong Lagrange property for such loops [**8**]. However, this observation merely reduces one hard problem to another.

Theorem 6.7. *Let π be a set of odd primes, and let $\mathcal{L}$ be a finite Bol π-loop. Then* $\mathrm{LMlt}(\mathcal{L})$ *is a π-group.*

Proof. This is Theorem 4.17 interpreted on the loop level. □

By the Feit-Thompson theorem [**13**], we conclude the following.

Corollary 6.8. *If $\mathcal{L}$ is a finite Bol loop of odd order, then* $\mathrm{LMlt}(\mathcal{L})$ *is a solvable group.*

The status of the Sylow and Hall theorems for Bol loops is unclear even for Bol loops of odd order. For Moufang loops we have the following results of Glauberman.

Proposition 6.9 (Hall's theorem, [**19**, p. 413, Theorem 16 and p. 409, Theorem 12]). *Let $\mathcal{L}$ be a Moufang loop of odd order and let π be a set of primes. Then $\mathcal{L}$ contains a Hall π-subloop.*

The Sylow theorem for Moufang loops of odd order follows immediately, although Glauberman also gave a separate proof, [**19**, p. 410, Theorem 13]. Glauberman also proved other Hall-like properties of π-subloops [**19**, p. 413, Theorem 16].

Problem 6.10.

(1) *For a given set π of primes, does every Bol loop of odd order have a Hall π-subloop?*

(2) *If the answer to* (1) *is no, then for a given odd prime p, does every Bol loop of odd order have a Sylow p-subloop?*

Finally, we turn to some preliminary investigations of simple Bol loops of odd order. This is motivated by Glauberman's Feit-Thompson theorems for B-loops and Moufang loops.

Proposition 6.11 [**19**, p. 412, Theorem 14 and p. 413, Theorem 16]. *Let $\mathcal{L}$ be a Bol loop of odd order. If $\mathcal{L}$ is a B-loop or a Moufang loop, then $\mathcal{L}$ is solvable.*

Corollary 6.12. *Let X be the class consisting of all Moufang loops of odd order and all finite B-loops. Let V be any variety of Bol loops of odd order such that every loop in V is an extension of two loops in X. Then every loop in V is solvable.*

A left loop is said to have the *A_l-property* if every left inner mapping is an automorphism [**22**, p. 35].

Corollary 6.13. *If $\mathcal{L}$ is an A_l Bol loop of odd order, then $\mathcal{L}$ is solvable.*

Proof. By [**16**, Theorem 4.11], $\mathcal{L}$ is an extension of a group by a B-loop. ◻

These considerations pave the way to the following problem. We will not give a complete answer, but we will present some results which we think will play a role in its solution.

Problem 6.14. *Do there exist any finite simple Bol loops of odd order? That is, is every finite Bol loop of odd order solvable?*

Let $\mathcal{L}$ be a 2-divisible Bol loop. Since $\mathcal{L}$ is left power-alternative, that is, $L(x^n) = L(x)^n$ for all $x \in \mathcal{L}$, we may use the 2-divisible twisted subgroup $L(\mathcal{L})$ to define a B-loop operation on $\mathcal{L}$. For $x, y \in \mathcal{L}$, we have $L(x) \odot L(y) = (L(x)L(y)^2 L(x))^{1/2} = L((x \cdot (y \cdot x))^{1/2})$. This leads us to the following.

Definition 6.15. Let $\mathcal{L}$ be a 2-divisible Bol loop. The *B-loop associated to* $\mathcal{L}$ is $(\mathcal{L}, \odot)$ with the binary operation $\odot : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ given by $x \odot y = (x \cdot ((y \cdot y) \cdot x))^{1/2}$. We will denote the B-loop $(\mathcal{L}, \odot)$ by $\mathcal{L}(1/2)$, and we follow a similar convention for subloops. Left multiplication maps for $\mathcal{L}(1/2)$ will be denoted by $M(x)y := x \odot y$.

Remark 6.16. In case $\mathcal{L}$ is already a B-loop, $(\mathcal{L}, \odot)$ is just $\mathcal{L}$ itself. This is because every Bruck loop satisfies the identity $(x \cdot y)^2 = x \cdot (y^2 \cdot x)$, [**22**, p. 73, 6.8(1)].

The B-loop associated to a Moufang loop of odd order was the key component in Glauberman's proofs of the Hall, Sylow, and Feit-Thompson theorems in [**19**]. The idea was to "pull back" the results from the associated B-loop to the Moufang loop. Since arbitrary Bol loops are not as structured as Moufang loops, one cannot expect this idea to work quite so well. Nevertheless, we can make some progress.

Lemma 6.17. *Let $\mathcal{L}$ be a 2-divisible Bol loop. Then the squaring map $s : \mathcal{L} \to \mathcal{L}; x \mapsto x \cdot x$ conjugates* $\mathrm{LMlt}(\mathcal{L}(1/2))$ *to* $\mathrm{PMlt}(\mathcal{L})$ *in* $\mathcal{L}!$.

Proof. For each $x \in \mathcal{L}$, $M(x) = s^{-1}P(x)s$. □

Theorem 6.18. *Let $\mathcal{L}$ be a 2-divisible Bol loop, and let $G = \mathrm{LMlt}(\mathcal{L})$. The mapping $L(\mathcal{L}) \to M(\mathcal{L}(1/2))$; $L(x) \mapsto M(x)$ extends to a homomorphism from G onto $\mathrm{LMlt}(\mathcal{L}(1/2))$ if and only if $\mathcal{L}' = \{1\}$. The kernel of the homomorphism is $Z(G) \cap C_G(\tau)$ where $\tau \in \mathrm{Aut}\,(G)$ is the Aschbacher automorphism.*

Proof. This follows from Lemma 6.17 and Theorem 5.13. □

Lemma 6.19. *A subloop $\mathcal{K}$ of a 2-divisible Bol loop $\mathcal{L}$ is normal if and only if, for all $x, y \in \mathcal{L}$, $x \cdot (\mathcal{K} \cdot y) = \mathcal{K} \cdot (x \cdot y)$.*

Proof. Referring to the conditions in (2.1), we see that only one direction requires proof. Thus assume $x \cdot (\mathcal{K} \cdot y) = \mathcal{K} \cdot (x \cdot y)$ for all $x, y \in \mathcal{L}$. Fix $x \in \mathcal{L}$ and set $u = x^{1/2}$. Using LAP and the left Bol identity,

$$\begin{aligned} x \cdot (\mathcal{K} \cdot y) &= u \cdot (u \cdot (\mathcal{K} \cdot y)) = u \cdot (\mathcal{K} \cdot (u \cdot y)) \\ &= (u \cdot (\mathcal{K} \cdot u)) \cdot y = (u \cdot (u \cdot \mathcal{K})) \cdot y \\ &= (x \cdot \mathcal{K}) \cdot y. \end{aligned}$$

Thus the other condition of (2.1) holds, and so $\mathcal{K}$ is normal. □

Our next result is inspired by Aschbacher's normality condition for subloops [**2**, Condition (NC)]. It enables us to express normality directly in terms of the left multiplication group.

Lemma 6.20. *A subloop $\mathcal{K}$ of a 2-divisible Bol loop $\mathcal{L}$ is normal if and only if, for each $x \in \mathcal{L}$, $y \in \mathcal{K}$, $g \in \mathrm{LMlt}(\mathcal{L})$,*

$$L(x)L(y)L(x^{-1}) = L(z)ghg^{-1} \tag{6.1}$$

for some $z \in \mathcal{K}$, $h \in \mathrm{LMlt}_1(\mathcal{L})$.

Proof. Set $G = \mathrm{LMlt}(\mathcal{L})$, $H = \mathrm{LMlt}_1(\mathcal{L})$. Fix $x \in \mathcal{L}$, $y \in \mathcal{K}$. Since $L(\mathcal{L})$ is a transversal of each conjugate gHg^{-1}, $g \in G$, we have $L(x)L(y)L(x^{-1}) = L(z)ghg^{-1}$ for some $z \in \mathcal{L}$, $h \in H$. Applying both sides to $w = g1$, we have $x \cdot (y \cdot (x^{-1} \cdot w)) = z \cdot w$. Now if $\mathcal{K}$ is normal, then by Lemma 6.19, or just (2.1), $x \cdot (y \cdot (x^{-1} \cdot w)) = u \cdot w$ for some $u \in \mathcal{K}$. Thus $z = u$ and so $z \in \mathcal{K}$ so that (6.1) holds. Conversely, if (6.1) holds, then fix $v \in \mathcal{L}$ and set $g = L(v)$. Let $z \in \mathcal{K}$, which depends on g, be given as in (6.1). Apply both sides of (6.1) to z to get $x \cdot (y \cdot (x^{-1} \cdot v)) = z \cdot v$. By Lemma 6.19, $\mathcal{K}$ is normal. □

In the Moufang case, the following result is [**19**, p. 401, Lemma 7(b)]. The proof in the general case is essentially the same, but with care in the parenthesization.

Lemma 6.21. *Let $\mathcal{L}$ be a 2-divisible Bol loop and suppose $\mathcal{K}$ is a subloop of $\mathcal{L}(1/2)$. Then $\mathcal{K}$ is a subloop of $\mathcal{L}$ if and only if $x^{-1} \cdot (\mathcal{K} \cdot x) = \mathcal{K}$ for all $x \in \mathcal{K}$.*

Proof. The "only if" is obvious, so assume $x^{-1} \cdot (\mathcal{K} \cdot x) = \mathcal{K}$ for all $x \in \mathcal{K}$. If $x \in \mathcal{K}$, then $\langle x \rangle \subset \mathcal{K}$, because powers in $\mathcal{L}$ agree with powers in $\mathcal{L}(1/2)$. Now fix $x, y \in \mathcal{K}$ and set $u = x^{1/2}$, $v = y^{1/2}$. Then using the definition of $\odot$, the Bol identity, and LAP, $\mathcal{K}$ contains $u \cdot ((u \odot v)^2 \cdot u^{-1}) = u \cdot ((u \cdot (v^2 \cdot u)) \cdot u^{-1}) = u^2 \cdot v^2 = x \cdot y$. A subset of a Bol loop closed under inversion and multiplication is a subloop, see, e.g., [**22**, p. 50, 3.10 (4)]. □

Lemma 6.22. *Let $\mathcal{L}$ be a radical-free, 2-divisible Bol loop, let $G = \mathrm{LMlt}(\mathcal{L})$, let $\tau \in \mathrm{Aut}\,(G)$ be the Aschbacher automorphism, and assume that $Z(G) \cap C_G(\tau) = \langle 1 \rangle$. If $\mathcal{K}(1/2)$ is a normal subloop of $\mathcal{L}(1/2)$, then $\mathcal{K}$ is a normal subloop of $\mathcal{L}$.*

Proof. Fix $x \in \mathcal{L}$, $y \in \mathcal{K}$, $g \in G$. As in the proof of Lemma 6.20, there exists $z \in \mathcal{L}$, $h \in \mathrm{LMlt}_1(\mathcal{L})$ such that

$$(*) \qquad L(x)L(y)L(x^{-1}) = L(z)ghg^{-1}$$

The hypotheses and Theorem 6.18 imply $\mathrm{LMlt}(\mathcal{L}(1/2)) \cong G$, and that on generators, the isomorphism sends each $L(x)$ to $M(x)$. Applying this to (∗), we have $M(x)M(y)M(x^{-1}) = M(z)\tilde{g}\tilde{h}\tilde{g}^{-1}$ for some $\tilde{g} \in \mathrm{LMlt}(\mathcal{L}(1/2))$, $\tilde{h} \in \mathrm{LMlt}_1(\mathcal{L}(1/2))$. If $\mathcal{K}(1/2)$ is normal, then by Lemma 6.20, $z \in \mathcal{K}$. Thus by (∗), the condition of Lemma 6.20 is satisfied for the sub*set* $\mathcal{K}$ of $\mathcal{L}$, and all that remains is to show that $\mathcal{K}$ is a subloop. Taking $x \in \mathcal{K}$ and $g = 1$ in (∗), and applying both sides to 1, we have that for each $x, y \in \mathcal{K}$, there exists $z \in \mathcal{K}$ such that $x \cdot (y \cdot x^{-1}) = z$. Now apply Lemma 6.21. □

Theorem 6.23. *Let $\mathcal{L}$ be a simple Bol loop of odd order, let $G = \mathrm{LMlt}(\mathcal{L})$, and, since $\mathcal{L}$ is radical-free, let $\tau \in \mathrm{Aut}\,(G)$ denote the Aschbacher automorphism. Then $Z(G) \cap C_G(\tau) \neq \langle 1 \rangle$.*

Proof. By Proposition 6.11, $\mathcal{L}(1/2)$ has a nontrivial normal subloop $\mathcal{K}(1/2)$. If $Z(G) \cap C_G(\tau) = \langle 1 \rangle$, then Lemma 6.22 implies that $\mathcal{K}$ is a nontrivial normal subloop of $\mathcal{L}$. □

Corollary 6.24. *Let $\mathcal{L}$ be a simple Bol loop of odd order, and let $G = \mathrm{LMlt}(\mathcal{L})$. Then G has nontrivial center, and $\mathrm{Nuc}_r(\mathcal{L})$ contains an abelian subgroup $\mathcal{M} = \{x \in \mathrm{Nuc}_r(\mathcal{L}) : R(x) \in G\} \neq \langle 1 \rangle$.*

Proof. This follows from Theorem 6.23 and Lemma 5.16. □

Remark 6.25. As a corollary, we obtain a new proof of the Moufang part of Proposition 6.11. In a Moufang loop, the three nuclei agree and the nucleus is normal, [**31**, p. 90, Corollary IV.1.5]. Thus a simple Moufang loop has trivial nucleus, and so by Corollary 6.24, cannot have odd order. □

In general, the right nucleus of a left Bol loop need not be a normal subloop. This follows from a construction of D. Robinson and K. Robinson [**37**], translated from right Bol loops to left Bol loops. However, their construction gives a Bol loop of even order. Thus the following problem still seems to be open, even for B-loops.

Problem 6.26. *Does there exist a Bol loop of odd order with nonnormal right nucleus?*

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