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POSITIVE INTEGERS WHOSE EULER FUNCTION IS A POWER OF THEIR KERNEL FUNCTION

J.-M. DE KONINCK, F. LUCA AND A. SANKARANARAYANAN

1. Introduction. For a positive integer n, let $\gamma(n) := \prod_{p|n} p$. The function $\gamma(n)$ is sometimes referred to as either the *algebraic radical* of n, or the *squarefree kernel* of n. Let $\phi(n)$, $\sigma(n)$ and $\omega(n)$ denote the Euler function of n, the sum of the positive divisors of n and the number of distinct prime factors of n, respectively. We also write $P(n)$ for the largest prime factor of n (with the convention that $P(1) = 1$), and $\mu(n)$ for the Möbius function of n.

Jean-Marie De Koninck, see [**3**], asked for all the positive integers n which are solutions of the equation

$$f(n) = \gamma(n)^2, \tag{1}$$

where $f \in \{\phi, \sigma\}$. With $f = \phi$, the above equation has precisely six solutions, and all these are listed in the last section of this paper. With $f = \sigma$, it is conjectured that $n = 1, 1782$ are the only solutions of the above equation, but we do not even know if this equation admits finitely many or infinitely many solutions n. In [**4**], it is shown, among other things, that every positive integer n satisfying equation (1) with $f = \sigma$ can be bounded above by a function depending on $\omega(n)$. In particular, if one puts an upper bound on the number of distinct prime factors of the positive integer n satisfying equation (1) with $f = \sigma$, then one can bound the positive integer n.

In this paper, we let k be any positive integer, and we let E_k be the set of positive integer solutions n for the equation

$$\phi(n) = \gamma(n)^k. \tag{2}$$

We also set $N_k := |E_k|$. It is easy to see that $E_1 = \{1, 2^2, 2 \cdot 3^2\}$. Moreover, for $k \geq 2$, each one of the numbers $1,\ 2^{k+1},\ 2^k \cdot 3^{k+1},\ 2^{k-1} \cdot$

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5^{k+1} is in E_k, and therefore $N_k \geq 4$ for all $k \geq 2$. Note further that if $n \in E_k$, the $\phi(n\gamma(n)) = \phi(n)\gamma(n) = \gamma(n)^{k+1}$, and therefore $n\gamma(n) \in E_{k+1}$. Since the map $n \mapsto n\gamma(n)$ is injective, we conclude that $N_{k+1} \geq N_k$.

In this paper, we give upper and lower bounds on N_k and we also give an upper bound on the largest possible member of E_k.

Theorem. *There exist positive computable constants c_1 and c_2 such that the inequality*

$$\exp(c_1 k \log k) < N_k < \exp(c_2 k^2) \tag{3}$$

holds for all positive integers k. Moreover, if $n \in E_k$, then

$$n < 3^{(k+1)^{k+2}}. \tag{4}$$

In particular, from the above theorem, we see that N_k tends to infinity with k.

In Section 2, we prove our Theorem. In Section 3, we compute E_k for $k = 1, 2, 3, 4$.

2. The proof of the theorem.

Proof. We start with the upper bound on N_k. Since $N_1 = 3$, $N_2 = 6$ and $N_3 = 16$, see Section 3, it follows that the upper bound (3) holds for $k = 1, 2, 3$ and with any $c_2 > \log 3$. Assume now that $k \geq 4$ and that $n > 2$ is in E_k. Since $\phi(n) > 1$, it follows that $\phi(n)$ is even, so that $2|n$. Let $n = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $2 = p_1 < p_2 < \cdots < p_l$ are prime numbers and α_i are positive integers for $i = 1, \ldots, l$. Since 2 divides $p_i - 1$ for all $i = 2, \ldots, l$ we see that $2^{l-1}|\phi(n)$. Since $\phi(n) = \gamma(n)^k$, it follows that $2^k|\phi(n)$, and therefore that $l - 1 \leq k$. When $l = 1$, it follows that $n = 2^{\alpha_1}$, so that $2^{\alpha_1 - 1} = \phi(n) = 2^k$, in which case $n = 2^{k+1}$. From now on, we shall assume that $l \geq 2$. Fix an integer l in the interval $[2, k+1]$. The equation

$$\phi(n) = \gamma(n)^k$$

can be rewritten as

$$\prod_{i=1}^{l}(p_i - 1)\prod_{i=1}^{l} p_i^{\alpha_i - 1} = \prod_{i=1}^{l} p_i^k,$$

which is equivalent to

$$\prod_{i=1}^{l}(p_i - 1) = \prod_{i=1}^{l} p_i^{\beta_i}, \tag{5}$$

where $\beta_i := k - \alpha_i + 1$. Note that the numbers β_i are nonnegative integers in the interval $[0, k]$ and that $\beta_l = 0$, so that $\alpha_l = k + 1$. Conversely, every solution $(p_1, \dots, p_l, \beta_1, \dots, \beta_l)$ in prime numbers $2 = p_1 < \cdots < p_l$ and nonnegative integers $\beta_1, \dots, \beta_l$ in the interval $[0, k]$ of equation (5) leads to a solution $n = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ of the equation $\phi(n) = \gamma(n)^k$ simply by setting $\alpha_i := k - \beta_i + 1$, and by unique factorization. It follows that distinct solutions of equation (5) lead to distinct elements of E_k. Thus, it suffices to find an upper bound for the number of solutions of (5).

Notice also that every solution of (5) leads to a solution of the system of equations

$$p_i - 1 = \prod_{j<i} p_j^{\gamma_{ij}}, \quad \text{for } i = 2, \dots, l, \tag{6}$$

where γ_{ij} are nonnegative integers such that

$$\sum_{j<i\le l} \gamma_{ij} = \beta_j \quad \text{holds for all} \quad j = 2, \dots, l-1. \tag{7}$$

For a fixed $j = 1, \dots, l-1$, the number of $l-j+1$-tuples of nonnegative integers $(\gamma_{j+1,j}, \dots, \gamma_{l,j}, \beta_j)$ satisfying equation (7), with $\beta_j \le k$, is

$$\binom{k+l-j}{l-j}.$$

Thus, the total number of solutions of (5) with a fixed value of l is at most

$$\begin{aligned}\prod_{j=1}^{l-1}\binom{k+l-j}{l-j} &= \prod_{j=1}^{l-1}\binom{k+j}{j} \le \prod_{j=1}^{k}\binom{2k}{j}\\ &\le \left(\frac{1}{k}\sum_{j=1}^{k}\binom{2k}{j}\right)^k < \left(\frac{2^{2k}}{k}\right)^k,\end{aligned} \tag{8}$$

where we used the AGM-inequality. Summing up (8) over all $l \in [2, k+1]$, and accounting also for the numbers $n = 1,\ 2^{k+1}$ in E_k, we get

$$N_k \le 2 + \frac{2^{2k^2}}{k^{k-1}} < 2^{2k^2} \quad \text{for} \quad k \ge 2. \tag{9}$$

Thus, inequality (3) holds with $c_2 = 2\log 2 > \log 3$ and for all values of the positive integer k.

We now prove inequality (4). From the computation of E_k for $k = 1, 2$, one sees that inequality (4) holds for these two values of k. Assume now that $k \ge 3$, and let $n \in E_k$ be a number with $\omega(n) = l$, where $l \in [2, k+1]$. Then, with the previous notations, we have

$$\begin{aligned} p_2 + 1 &\le 2^k + 2 < (2+1)^k = 3^k < 3^{k+1}, \\ p_3 + 1 &\le 2^k p_2^k + 2 < (2+1)^k (p_2+1)^k < 3^{k^2+k} < 3^{(k+1)^2}, \end{aligned}$$

and, by induction, one shows that the inequality

$$p_j + 1 < 3^{(k+1)^{j-1}}$$

holds for all values of $j = 2, 3, \ldots, l$. Indeed, assuming that the above inequality holds for the index $j < l$ and all indices $i \le j$, we get that

$$\begin{aligned} p_{j+1} + 1 &\le 2^k p_2^k \cdots p_j^k + 1 < (2+1)^k (p_2+1)^k \cdots (p_j+1)^k \\ &< 3^{k \sum_{i=0}^{j-1} (k+1)^i} \\ &= 3^{(k+1)^j - 1} < 3^{(k+1)^j}, \end{aligned}$$

where in the above inequality we used the identity

$$\sum_{i=0}^{j-1} (k+1)^i = \frac{(k+1)^j - 1}{k}.$$

Thus,

$$\gamma(n) \le 2 \cdot 3^{\sum_{j=2}^{l} (k+1)^{j-1}} < 3^{(k+1)^l} \le 3^{(k+1)^{k+1}},$$

and since $n | \gamma(n)^{k+1}$ whenever $n \in E_k$, we get $n < 3^{(k+1)^{k+2}}$, which is precisely inequality (4).

We now turn our attention to the lower bound on N_k. Here, we employ the following observation: Assume that $\mathcal{P}$ is a set of prime numbers containing the number 2 and such that

$$\prod_{p\in\mathcal{P}}(p-1) = 2^k \prod_{\substack{p\in\mathcal{P}\\ p\neq 2}} p^{\alpha_p} \tag{10}$$

holds with some integers α_p in the interval $[0, k]$. Then

$$n = 2\prod_{\substack{p\in\mathcal{P}\\ p\neq 2}} p^{\beta_p}$$

belongs to E_k, where $\beta_p = k-\alpha_p+1$. Moreover, by unique factorization, it follows that distinct sets of integers $\mathcal{P}$ satisfying equation (10) with some α_p's will lead to distinct solutions $n \in E_k$ (simply because $\gamma(n) = \prod_{p\in\mathcal{P}} p$).

To construct such sets $\mathcal{P}$, we start by taking a large real number x and by writing

$$Q(x) = \prod_{p\le x}(p-1). \tag{11}$$

For any positive integer m and any prime number p, we let $\mu_p(m)$ be the order at which p appears in the factorization of m. For any coprime positive integers a and d and any positive real number y we write $\pi(y; d, a)$ for the number of primes $p \le y$ such that $p \equiv a \pmod{d}$. We also write $\pi(y)$ for the total number of primes $p \le y$. We now consider the factorization of $Q(x)$. Let $q \le x/2$ be an arbitrary fixed prime. Then,

(12)

$$\mu_q(Q(x)) = \sum_{r\ge 1}\pi(x; q^r, 1) = \sum_{\substack{r\ge 1\\ q^r\le x^{1/3}}}\pi(x; q^r, 1) + \sum_{\substack{r\ge 1\\ q^r> x^{1/3}}}\pi(x; q^r, 1).$$

For the first sum in (12) above, we use the Bombieri-Vinogradov theorem (see page 262 in [**5**]) to conclude that

(13)

$$\sum_{\substack{r\geq 1\\ q^r\leq x^{1/3}}} \pi(x;q^r,1) = \sum_{\substack{r\geq 1\\ q^r\leq x^{1/3}}} \frac{\pi(x)}{\phi(q^r)} + O\left(\frac{x}{\log^2 x}\right)$$

$$= \pi(x)\left(\sum_{r\geq 1}\frac{1}{\phi(q^r)} - \sum_{\substack{r\geq 1\\ q^r> x^{1/3}}}\frac{1}{\phi(q^r)}\right) + O\left(\frac{x}{\log^2 x}\right).$$

Clearly,

$$\sum_{\substack{r\geq 1\\ q^r> x^{1/3}}}\frac{1}{\phi(q^r)} \ll \sum_{\substack{r\geq 1\\ q^r> x^{1/3}}}\frac{1}{q^r} < \frac{1}{x^{1/3}}\sum_{s\geq 0}\frac{1}{q^s} \ll \frac{1}{x^{1/3}}. \tag{14}$$

With (14), we get from (13) that

$$\begin{aligned}\sum_{\substack{r\geq 1\\ q^r\leq x^{1/3}}} \pi(x;q^r,1) &= \pi(x)\sum_{r\geq 1}\frac{1}{\phi(q^r)} + O\left(\frac{x}{\log^2 x}\right)\\ &= \frac{q\pi(x)}{(q-1)^2} + O\left(\frac{x}{\log^2 x}\right).\end{aligned} \tag{15}$$

For the second sum in (12), we simply use the fact that, when $x^{1/3} < q^r \leq x$, we have

$$\pi(x;q^r,1) \leq \frac{x}{q^r} < x^{2/3}.$$

Now, since the number of such numbers r satisfying $q^r \leq x$ is $\leq \log x/\log q \leq \log x/\log 2$, we get that the second sum in (12) can be bounded above as

$$\sum_{\substack{r\geq 1\\ q^r> x^{1/3}}} \pi(x;q^r,1) \ll x^{2/3}\log x = o\left(\frac{x}{\log^2 x}\right). \tag{16}$$

With (15) and (16), we get that (12) becomes

$$\mu_q(Q(x)) = \frac{q\pi(x)}{(q-1)^2} + O\left(\frac{x}{\log^2 x}\right). \tag{17}$$

We use formula (17) together with the prime number theorem to get that the estimates

$$\mu_2(Q(x)) = 2\pi(x)(1+o(1)) > \pi(x) \tag{18}$$

and

$$\mu_q(Q(x)) < \frac{3}{4}\pi(x)(1+o(1)) < \pi(x)$$

hold for all sufficiently large values of x, and uniformly for primes $q \geq 3$. In particular, if we write

$$Q(x) = 2^{\alpha_2(x)} \prod_{2<q\leq x/2} q^{\alpha_q(x)}, \tag{19}$$

then the inequality

$$\alpha_2(x) > \alpha_q(x) \tag{20}$$

holds for all sufficiently large values of x and all odd primes q.

We now let $y \leq x$ and write $\pi(x;y)$ for the number of prime numbers $p \leq x$ such that the largest prime factor of $p-1$, written $P(p-1)$, is less than or equal to y. A long time ago, Erdős, see [**1**], showed that there exists a number $\rho_0 > 0$ such that the inequality $\pi(x^{1+\rho};x) > c(\rho)\pi(x^{1+\rho})$ holds with some positive constant $c(\rho)$ depending on ρ for all $\rho \in (0,\rho_0)$, provided x is sufficiently large. The best (largest) value of ρ_0 for which the above inequality is known to hold with some positive constant $c(\rho)$ for all $\rho \in (0,\rho_0)$ is $2\sqrt{e}-1$ and is due to Friedlander, see [**2**]. It is conjectured that such a positive constant $c(\rho)$ exists for all values of ρ. Actually, Erdős proved even more, namely, that there exists an absolute constant $c_3 > 0$ such that the inequality

$$\pi(x^{1+\rho};x) > (1-c_3\rho)\pi(x^{1+\rho}) \tag{21}$$

holds for all sufficiently large values of x and for all positive numbers ρ such that $1-c_3\rho > 0$. In particular, one can choose $\rho_0 = 1/c_3$. Inequality (21) above follows from the argument on pages 212–213 of [**1**].

Writing $\mathcal{N}(x) := \{x < p \leq x^{1+\rho} \mid P(p-1) \leq x,\ \mu(p-1) \neq 0\}$, we can show that

$$|\mathcal{N}(x)| > \left(\frac{1}{10} - c_3\rho\right)\pi(x^{1+\rho}), \tag{22}$$

provided x is sufficiently large. Indeed, note that

$$|\mathcal{N}(x)| \geq \pi(x^{1+\rho}; x) - \pi(x) - |\mathcal{N}_1(x)|, \tag{23}$$

where

$$\mathcal{N}_1(x) := \{p \leq x^{1+\rho} \mid \mu(p-1) = 0\}.$$

It is obvious that

$$\begin{aligned} |\mathcal{N}_1(x)| \leq \sum_{q \geq 2} \pi(x^{1+\rho}; q^2, 1) \leq & \sum_{q \leq x^{1/6}} \pi(x^{1+\rho}; q^2, 1) \\ & + \sum_{q > x^{1/6}} \pi(x^{1+\rho}; q^2, 1). \end{aligned} \tag{24}$$

For the first sum in (24), we use the Bombieri-Vinogradov theorem to conclude that

(25)

$$\begin{aligned} \sum_{q \leq x^{1/6}} \pi(x^{1+\rho}; q^2, 1) &= \pi(x^{1+\rho}) \sum_{q \leq x^{1/6}} \frac{1}{\phi(q^2)} + O\left(\frac{x^{1+\rho}}{\log^2 x}\right) \\ &< \pi(x^{1+\rho})\left(\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \sum_{n \geq 7} \frac{1}{(n-1)n}\right) \\ &\quad + o(\pi(x^{1+\rho})) \\ &= \pi(x^{1+\rho})\left(\frac{53}{60} + o(1)\right). \end{aligned}$$

For the second sum in (24), we simply use the fact that the inequality

$$\pi(x^{1+\rho}; q^2, 1) \leq \frac{x^{1+\rho}}{q^2}$$

holds for all $q > x^{1/6}$ to conclude that

$$\sum_{q > x^{1/6}} \pi(x^{1+\rho}; q^2, 1) \ll x^{1+\rho} \sum_{q > x^{1/6}} \frac{1}{q^2} \ll x^{5/6+\rho} = o(\pi(x^{1+\rho})). \tag{26}$$

From (24), (25) and (26), we get

$$|\mathcal{N}_1(x)| < \pi(x^{1+\rho})\left(\frac{53}{60} + o(1)\right). \tag{27}$$

Thus, from (21), (23) and (27), we get that

$$\begin{aligned}|\mathcal{N}(x)| &> \pi(x^{1+\rho})(1-c_3\rho) - \pi(x) - \pi(x^{1+\rho})\left(\frac{53}{60}+o(1)\right)\\ &= \pi(x^{1+\rho})\left(\frac{7}{60} - c_3\rho + o(1)\right) > \left(\frac{1}{10} - c_3\rho\right)\pi(x^{1+\rho}),\end{aligned}$$

which is precisely inequality (22).

We now let $\varepsilon \in (0, 1/10)$ be arbitrary and define ρ implicitly by $1/10 - c_3\rho = \varepsilon$. In particular, the inequality

$$|\mathcal{N}(x)| > \varepsilon\pi(x^{1+\rho}) \tag{28}$$

holds for all sufficiently large values of x.

We now return to our problem. Let $\lambda > 0$ be any small positive real number (less than 1). Moreover, let k be a large integer and write it as $k = l + \delta + \lfloor \lambda l \rfloor$ for some integer l and some $\delta \in \{0, 1\}$. It is clear that such a pair of integers l and δ always exists. In fact, if $\{u_l\}_{l\geq 0}$ denotes the sequence of integers defined by $u_l := l + \lfloor \lambda l \rfloor$, we then see that $u_{l+1} - u_l \in \{1, 2\}$ holds for all $l \geq 0$. In particular, every positive integer k can be represented as $k = u_l + \delta$ for some positive integer l and some $\delta \in \{0, 1\}$. Let $2 = p_1 < p_2 < \cdots$ be the sequence of all prime numbers and let $(m_j)_{j\geq 1}$ be the sequence of integers given by

$$m_j := \mu_2(Q(p_j)).$$

It is clear that $(m_j)_{j\geq 1}$ is an increasing sequence. Moreover, with the notation of (19), we have that

$$\begin{aligned}m_{j+1} - m_j &= \alpha_2(p_{j+1}) - \alpha_2(p_j) = \mu_2(p_{j+1} - 1)\\ &\leq \frac{\log(p_{j+1}-1)}{\log 2} \ll \log p_j.\end{aligned} \tag{29}$$

With the numbers l and δ that we have constructed, we let j be the largest positive integer such that $m_j \leq \lfloor \lambda l \rfloor$. In this case, $\lfloor \lambda l \rfloor = m_j + m$, where $m \ll \log p_j$, because of (29). Set $x = p_j$ and construct the set of prime numbers $\mathcal{P}$ as follows: $\mathcal{P}$ is the union of the set $\mathcal{Q} := \{p \leq p_j\}$ with a set of primes $\mathcal{R}$ of cardinality $R := m + \delta + l$ and which consists of prime numbers p in the set $\mathcal{N}(p_j)$. We first note that such a set $\mathcal{P}$ fulfills (10). Indeed, we clearly have that

$$\begin{aligned}\mu_2\big(\prod_{p\in\mathcal{P}}(p-1)\big) &= \mu_2(Q(p_j)) + \mu_2\big(\prod_{p\in\mathcal{R}}(p-1)\big)\\ &= m_j + R = m_j + m + \delta + l\\ &= l + \lfloor \lambda l\rfloor + \delta = k\end{aligned}$$

(because all the primes $p \in \mathcal{R}$ are congruent to 3 modulo 4), while the inequality

$$\begin{aligned}k = \mu_2\big(\prod_{p\in\mathcal{P}}(p-1)\big) &= m_j + R = \mu_2(Q(p_j)) + R > \mu_q(Q(p_j)) + R\\ &> \mu_q\big(\prod_{p\in\mathcal{P}}(p-1)\big)\end{aligned}$$

holds by (20) together with the fact that all primes $p \in \mathcal{R}$ have the property that $p-1$ is squarefree, that is, $2|p-1$ for all $p \in \mathcal{R}$ and there is no odd prime q such that $q^2|p-1$ for some $p \in \mathcal{R}$. Note also that only the primes $q \in \mathcal{Q}$ can appear in the factorization of $\prod_{p\in\mathcal{P}}(p-1)$, because $P(p-1) \le p_j$ for all $p \in \mathcal{R}$, and that these primes do indeed belong to $\mathcal{P}$.

We now note that the part $\mathcal{Q}$ of $\mathcal{P}$ is uniquely determined in terms of j, hence of k, while $\mathcal{R}$ is not. Thus, in order to prove our lower bound, we shall show that for large k, we can choose our set $\mathcal{R}$ in at least $\exp(c_1 k \log k)$ distinct ways, where c_1 is a positive constant.

In order to do so, we need some estimates concerning the size of $\mathcal{R}$. Clearly, by (18), we have

$$m_j = 2\pi(p_j)(1 + o(1)) = 2j(1 + o(1)),$$

and $m \ll \log p_j \ll \log j$. Thus,

$$\lambda l = m_j + m + O(1) = 2j(1 + o(1)) + O(\log j) = 2j(1 + o(1)), \tag{30}$$

and therefore

$$R = m + \delta + l = \frac{2j}{\lambda}(1 + o(1)) + O(\log j) = \frac{2j}{\lambda}(1 + o(1)). \tag{31}$$

Let

$$T := |\mathcal{N}(p_j)| > \varepsilon\pi(p_j^{1+\rho}) > c_4 j^{1+\rho}\log^\rho j, \tag{32}$$

where by (28) we can choose $c_4 := \varepsilon/2$ provided that j, hence k, is sufficiently large. Since $R = o(T)$, we may use Stirling's formula to approximate the factorial, in which case we see that the number of ways of choosing $\mathcal{R}$, hence N_k, is at least

$$\begin{aligned}\binom{T}{R} &> \exp\left(R\log\left(\frac{T}{R}\right)(1+o(1))\right)\\ &= \exp\left(\frac{2j}{\lambda}\log\left(\frac{c_4\lambda}{2}\,j^\rho\log^\rho j\right)(1+o(1))\right)\\ &= \exp\left(\frac{2\rho}{\lambda}\,(1+o(1))j\log j\right),\end{aligned} \tag{33}$$

where we used (31) and (32). Finally, since

$$k = l + \delta + \lfloor \lambda l\rfloor = (1+\lambda)l(1+o(1)),$$

we get that

$$l = \frac{k}{1+\lambda}\,(1+o(1)). \tag{34}$$

Hence, from (34) and (30), it follows that

$$j = \frac{\lambda k}{2(1+\lambda)}\,(1+o(1)). \tag{35}$$

Thus, putting (35) into (33), we get

$$\binom{T}{R} > \exp\left(\frac{\rho}{1+\lambda}\,(1+o(1))k\log k\right). \tag{36}$$

Therefore, if we choose c_1 to be any constant strictly smaller than ρ, and then choose $\lambda > 0$ such that the inequality

$$c_1 < \frac{\rho}{1+\lambda}$$

holds, we see, by (36), that the inequality

$$N_k > \exp(c_1 k \log k)$$

holds for all sufficiently large values of k.

The Theorem is therefore proved. □

3. Computational results. In this section, we compute E_k for $k = 1, 2, 3, 4$. Note first that $n = 1 \in E_k$ for all $k > 1$. Hence, from now on, we assume that $n > 1$. Note also that, if $n > 1$ is in E_k, it follows that $\phi(n) \geq \gamma(n) > 1$ and therefore that $\phi(n)$ is even. In particular, we get that $2|n$.

Suppose that $k = 1$ and $n > 2$. In this case, $2||\phi(n)$, in which case n can have at most one odd prime factor. If $n = 2^\alpha$ for some positive integer α, we then get $2^{\alpha-1} = \phi(n) = \gamma(n) = 2$, so that $\alpha = 2$ and $n = 4$. If $n = 2^\alpha p^\beta$ with some odd prime number p and some positive integers α and β, we then get $2^{\alpha-1}(p-1)p^{\beta-1} = 2p$. Since $p - 1$ is even and coprime to p, we get $\alpha = 1$, $p - 1 = 2$ and $\beta = 2$, so that $n = 2 \cdot 3^2$. Thus, $E_1 = \{1, 2^2, 2 \cdot 3^2\}$.

Suppose now that $k = 2$. Since $2^2||\phi(n)$, it follows that n can have at most two odd prime factors. If $n = 2^\alpha$, then $2^{\alpha-1} = \phi(n) = 2^2$, in which case $\alpha = 3$ and $n = 2^3$. If $n = 2^\alpha p^\beta$ with some odd prime number p and some positive integers α and β, we get $2^{\alpha-1}(p-1)p^{\beta-1} = 2^2p^2$, and since $p - 1$ is even and coprime to p, we get $\beta - 1 = 2$, so that either $\alpha - 1 = 0$, $p - 1 = 4$ or $\alpha - 1 = 1$, $p - 1 = 2$. Thus, we get the solutions $n = 2 \cdot 5^3$ and $n = 2^2 \cdot 3^3$. Finally, assume that $n = 2^\alpha p_1^{\beta_1} p_2^{\beta_2}$ with $p_1 < p_2$ odd prime numbers and positive integers α, β_1, β_2. In this case, we get

$$2^{\alpha-1}(p_1 - 1)(p_2 - 1)p_1^{\beta_1-1}p_2^{\beta_2-1} = 2^2p_1^2p_2^2. \tag{37}$$

Since $(p_1 - 1)(p_2 - 1)$ is a multiple of 4 coprime to p_2, we get that $\alpha = 1$, $\beta_2 = 3$, $2||p_1 - 1$ and $2||p_2 - 1$. Since $p_1 - 1$ is coprime to p_1, we get that $p_1 - 1 = 2$, so that $p_1 = 3$. Equation (37) now becomes

$$3^{\beta_1-1}(p_2 - 1) = 2 \cdot 3^2.$$

Hence, either $\beta_1 = 1$ and $p_2 = 2 \cdot 3^2 + 1 = 19$, or $\beta_1 = 2$ and $p_2 = 2 \cdot 3 + 1 = 7$. We have thus obtained the solutions $n = 2 \cdot 3 \cdot 19^3$ and $n = 2 \cdot 3^2 \cdot 7^3$. It follows that $E_2 = \{1, 2^3, 2^2 \cdot 3^3, 2 \cdot 5^3, 2 \cdot 3^2 \cdot 7^3, 2 \cdot 3 \cdot 19^3\}$.

Suppose now that $k = 3$. In this case, $2^3 \| \phi(n)$. This implies that, if n is not a power of 2, it must have at most three odd prime factors. If $n = 2^\alpha$, we then get $2^{\alpha-1} = \phi(n) = 2^3$, in which case $\alpha = 4$ and $n = 2^4$. We now assume that n has at least one odd prime factor. We have to consider the three cases

(i) $n = 2^\alpha \cdot p_1^{\beta_1}$, $2 < p_1$;

(ii) $n = 2^\alpha \cdot p_1^{\beta_1} \cdot p_2^{\beta_2}$, $2 < p_1 < p_2$;

(iii) $n = 2^\alpha \cdot p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3}$, $2 < p_1 < p_2 < p_3$.

In case (i), we have

$$2^{\alpha-1} p_1^{\beta_1 - 1}(p_1 - 1) = 2^3 p_1^3,$$

so that

$$p_1 - 1 = 2^{4-\alpha} p_1^{4-\beta_1}. \tag{38}$$

Since $p_1 - 1$ is coprime to p_1, it follows that $\beta_1 = 4$ and $p_1 = 2^{4-\alpha} + 1$. The only possibilities are therefore $\alpha = 2,\ p_1 = 5$ and $\alpha = 3,\ p_1 = 3$, which yields the solutions $n = 2^2 \cdot 5^4,\ 2^3 \cdot 3^4$.

In case (ii), we have

$$(p_1 - 1)(p_2 - 1) = 2^{4-\alpha} p_1^{4-\beta_1} p_2^{4-\beta_2}.$$

Since $(p_1 - 1)(p_2 - 1)$ is a multiple of 4 which is coprime to p_2, it follows that $\beta_2 = 4$ and $\alpha = 1, 2$. If $\alpha = 2$, we get $(p_1 - 1)(p_2 - 1) = 4p_1^{4-\beta_1}$, and since $p_2 - 1$ is even and $p_1 - 1$ is even and coprime to p_1, we get that $p_1 - 1 = 2$, so that $p_1 = 3$. Thus, $p_2 - 1 = 2p_1^{4-\beta_1} = 2 \cdot 3^{4-\beta_1}$, and since $p_2 > 3$, we see that the only possibilities are $\beta_1 = 2,\ p_2 = 19$ and $\beta_1 = 3,\ p_2 = 7$. We have thus obtained the solutions $n = 2^2 \cdot 3^2 \cdot 19^4$ and $n = 2^2 \cdot 3^3 \cdot 7^4$. If $\alpha = 1$, we get that $(p_1 - 1)(p_2 - 1) = 2^3 p_1^{4-\beta_1}$. The only possibilities for p_1 are 3 and 5. If $p_1 = 3$, we get $p_2 - 1 = 2^2 \cdot p_1^{4-\beta_1} = 2^2 \cdot 3^{4-\beta_1}$, and the only possibilities are $\beta_1 = 1,\ p_2 = 109$; $\beta_1 = 2,\ p_2 = 37$; $\beta_1 = 3,\ p_2 = 13$; $\beta_1 = 4,\ p_2 = 5$. We have thus obtained the solutions $n = 2^1 \cdot 3^1 \cdot 109^4,\ 2^1 \cdot 3^2 \cdot 37^4,\ 2^1 \cdot 3^3 \cdot 13^4,\ 2^1 \cdot 3^4 \cdot 5^4$. If $p_1 = 5$, we then get $p_2 - 1 = 2 \cdot p_1^{4-\beta_1} = 2 \cdot 5^{4-\beta_1}$ and since $p_2 > 5$, the only possibilities are $\beta_1 = 1,\ p_2 = 251$; $\beta_1 = 3,\ p_2 = 11$. We have thus obtained the solutions $n = 2^1 \cdot 5^1 \cdot 251^4,\ 2^1 \cdot 5^3 \cdot 11^4$.

In case (iii), we get the equation

$$(p_1-1)(p_2-1)(p_3-1) = 2^{4-\alpha} p_1^{4-\beta_1} p_2^{4-\beta_2} p_3^{4-\beta_3}. \tag{39}$$

Since $(p_1-1)(p_2-1)(p_3-1)$ is a multiple of 8 coprime to p_3, we get $\beta_3 = 4$, $\alpha = 1$ and $2||(p_i - 1)$ for $i = 1, 2, 3$. We now easily see that $p_1 = 3$ and therefore that equation (39) becomes

$$(p_2-1)(p_3-1) = 2^2 \cdot 3^{4-\beta_1} \cdot p_2^{4-\beta_2}.$$

In particular, $p_2 - 1 = 2 \cdot 3^i$ for some $i = 1, 2, 3, 4$. Note that if $i = 4$, then $\beta_1 = 4$ and $p_2 = 2p_1^{4-\beta_2} + 1$. But this is impossible; indeed, since $p_1 \equiv 1 \pmod 3$, it follows that $2p_1^{4-\beta_1} + 1$ is always a multiple of 3, and therefore that it cannot be a prime number larger than 3. Thus, the only possibilities are $i = 1, 2, 3$. Since $2\cdot 3^3+1 = 55$ is not a prime number, we are left only with $i = 1$, $p_2 = 7$ and $i = 2$, $p_2 = 19$. Assume first that $i = 1$, $p_2 = 7$. In this case, we get $p_3 - 1 = 2 \cdot 3^{3-\beta_1} \cdot p_2^{4-\beta_2} = 2 \cdot 3^{3-\beta_1} \cdot 7^{4-\beta_2}$. When $\beta_1 = 2$, we get $p_3 = 2 \cdot 3 \cdot 7^{4-\beta_2} + 1$. But this last expression is a prime number larger than 7 only when $\beta_2 = 3$ and $p_3 = 43$. This yields the solution $n = 2^1 \cdot 3^2 \cdot 7^3 \cdot 43^4$. The argument modulo 3 used above shows that $\beta_1 \neq 3$, which means that we only have to consider the instance $\beta_1 = 1$, in which case $p_3 = 2 \cdot 3^2 \cdot 7^{4-\beta_2} + 1$, with $\beta_2 = 1, 2, 3, 4$. The only possibilities are $\beta_2 = 2$, $p_3 = 883$; $\beta_2 = 3$, $p_3 = 127$; $\beta_2 = 4$, $p_3 = 19$, leading to the solutions $n = 2^1 \cdot 3^1 \cdot 7^2 \cdot 883^4$, $2^1 \cdot 3^1 \cdot 7^3 \cdot 127^4$, $2^1 \cdot 3^1 \cdot 7^4 \cdot 19^4$. Assume now that $i = 2$, $p_2 = 19$. In this case, $p_3 - 1 = 2 \cdot 3^{2-\beta_1} \cdot 19^{4-\beta_2} + 1$. The argument modulo 3 used above shows that $\beta_1 \neq 2$ and therefore that $\beta_1 = 1$. Since $p_3 > 19$, we also get that $\beta_2 \neq 4$. Thus, $\beta_2 = 1, 2, 3$, but none of the numbers $2 \cdot 3 \cdot 19^{4-\beta_2} + 1$ is a prime number for these values of β_2.

Hence, $N_3 = 16$ and $E_3 = \{1, 2^4, 2^2 \cdot 5^4, 2^3 \cdot 3^4, 2^2 \cdot 3^2 \cdot 19^4, 2^2 \cdot 3^3 \cdot 7^4, 2^1 \cdot 3^1 \cdot 109^4, 2^1 \cdot 3^2 \cdot 37^4, 2^1 \cdot 3^3 \cdot 13^4, 2^1 \cdot 3^4 \cdot 5^4, 2^1 \cdot 5^1 \cdot 251^4, 2^1 \cdot 5^3 \cdot 11^4, 2^1 \cdot 3^2 \cdot 7^3 \cdot 43^4, 2^1 \cdot 3^1 \cdot 7^2 \cdot 883^4, 2^1 \cdot 3^1 \cdot 7^3 \cdot 127^4, 2^1 \cdot 3^1 \cdot 7^4 \cdot 19^4\}$.

Similar arguments can be used to find E_4. We found it more appropriate to use MATHEMATICA to generate all 85 numbers belonging to E_4; these are listed below.

$1, 2^5, 2\cdot 17^5, 2^3\cdot 5^5, 2^4\cdot 3^5, 2\cdot 3^3\cdot 73^5, 2^2\cdot 3^2\cdot 109^5, 2^2\cdot 3^3\cdot 37^5, 2^2\cdot 3^4\cdot 13^5, 2^3\cdot 3\cdot 163^5, 2^3\cdot 3^3\cdot 19^5, 2^3\cdot 3^4\cdot 7^5, 2\cdot 5^3\cdot 101^5, 2^2\cdot 5^2\cdot 251^5, 2^2\cdot 5^4\cdot 11^5, 2\cdot 3^4\cdot 5^5\cdot 7^5, 2\cdot 3^5\cdot 5^4\cdot 11^5, 2\cdot 3^3\cdot 7^5\cdot 13^5, 2\cdot 3^3\cdot 5^5\cdot 19^5, 2^2\cdot 3^2\cdot 7^5\cdot 19^5, 2\cdot 3^4\cdot 5^4\cdot 31^5, 2\cdot 3^4\cdot 7^4\cdot 29^5, 2\cdot 3^2\cdot 13^5\cdot 19^5, 2\cdot 3^2\cdot 7^5\cdot 37^5, 2^2\cdot 3^3\cdot 7^4\cdot 43^5, 2\cdot 5^4\cdot 11^4\cdot 23^5, 2\cdot 3\cdot 19^5\cdot 37^5, 2\cdot 3\cdot 7^5\cdot 109^5, 2\cdot 3^4\cdot 5^3\cdot 151^5, 2\cdot 3\cdot 5^5\cdot 163^5, 2^2\cdot 3^2\cdot 7^4\cdot 127^5, 2\cdot 3^3\cdot 13^4\cdot 79^5, 2\cdot 3^5\cdot 5^2\cdot 251^5, 2\cdot 3^2\cdot 5^4\cdot 271^5, 2\cdot 3^4\cdot 7^3\cdot 197^5, 2^2\cdot 3\cdot 7^4\cdot 379^5, 2\cdot 3^4\cdot 5^2\cdot 751^5, 2\cdot 3\cdot 5^4\cdot 811^5, 2\cdot 3^2\cdot 19^4\cdot 229^5, 2\cdot 5\cdot 11^5\cdot 251^5, 2\cdot 3\cdot 7^4\cdot 757^5, 2^2\cdot 3^2\cdot 7^3\cdot 883^5, 2\cdot 3^2\cdot 37^4\cdot 223^5, 2\cdot 3^4\cdot 7^2\cdot 1373^5, 2\cdot 3^3\cdot 5^2\cdot 2251^5, 2^2\cdot 3\cdot 7^3\cdot 2647^5, 2\cdot 3\cdot 5^3\cdot 4051^5, 2\cdot 5^4\cdot 11^2\cdot 2663^5, 2\cdot 3^3\cdot 5\cdot 11251^5, 2^2\cdot 3^3\cdot 7\cdot 14407^5, 2\cdot 3\cdot 163^4\cdot 653^5, 2\cdot 3\cdot 13^3\cdot 9127^5, 2^2\cdot 3\cdot 7^2\cdot 18523^5, 2\cdot 3^3\cdot 13^2\cdot 13183^5, 2\cdot 3^2\cdot 5\cdot 33751^5, 2\cdot 5^2\cdot 251^4\cdot 503^5, 2\cdot 3^3\cdot 7\cdot 28813^5, 2\cdot 3^3\cdot 19^2\cdot 27437^5, 2\cdot 3\cdot 7\cdot 259309^5, 2\cdot 3\cdot 109^3\cdot 71287^5, 2\cdot 3\cdot 163^3\cdot 106277^5, 2\cdot 3^2\cdot 37\cdot 11244967^5, 2\cdot 3\cdot 7^4\cdot 19^5\cdot 43^5, 2\cdot 3\cdot 7^3\cdot 43^5\cdot 127^5, 2\cdot 3\cdot 7^2\cdot 43^5\cdot 883^5, 2\cdot 3\cdot 7^3\cdot 43^4\cdot 5419^5, 2\cdot 3\cdot 7\cdot 19^5\cdot 14407^5, 2\cdot 3\cdot 7^2\cdot 19^4\cdot 39103^5, 2\cdot 3^2\cdot 7^3\cdot 43^3\cdot 77659^5, 2\cdot 3\cdot 7^2\cdot 127^4\cdot 37339^5, 2\cdot 3\cdot 7\cdot 43^4\cdot 265483^5, 2\cdot 3^2\cdot 7^2\cdot 43^3\cdot 543607^5, 2\cdot 3\cdot 7^2\cdot 883^4\cdot 37087^5, 2\cdot 3\cdot 7^4\cdot 43^2\cdot 1431127^5, 2\cdot 3\cdot 7\cdot 19^3\cdot 5200567^5, 2\cdot 3\cdot 7^4\cdot 19\cdot 5473483^5, 2\cdot 3\cdot 7\cdot 883^4\cdot 259603^5, 2\cdot 3\cdot 7^2\cdot 19^2\cdot 14115823^5, 2\cdot 3^2\cdot 7^2\cdot 43^2\cdot 23375059^5, 2\cdot 3\cdot 7\cdot 43^2\cdot 490876219^5, 2\cdot 3\cdot 7^2\cdot 883^3\cdot 32746939^5, 2\cdot 3\cdot 7\cdot 19\cdot 1877404327^5, 2\cdot 3\cdot 7^2\cdot 127^2\cdot 602224603^5, 2\cdot 3\cdot 7^2\cdot 43\cdot 3015382483^5, 2\cdot 3\cdot 7^3\cdot 127\cdot 10926074923^5.$

Using MATHEMATICA, we also computed E_5. We will refrain from listing here all the members of E_5; let us simply mention that $N_5 = 969$. More precisely, if we let $E_{k,r}$ stand for the set of those $n \in E_k$ such that $\omega(n) = r$ and if we let $N_{k,r}$ stand for the cardinality of $E_{k,r}$, we obtained that $N_{5,0} = N_{5,1} = 1$, $N_{5,2} = 3$, $N_{5,3} = 17$, $N_{5,4} = 130$, $N_{5,5} = 672$ and $N_{5,6} = 145$ for a total of $N_5 = 969$.

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Département de Mathématiques, Université Laval, Québec G1K 7P4, CANADA
E-mail address: jmdk@mat.ulaval.ca

Mathematical Institute, UNAM, Ap. Postal 61-3 (Xangari), CP 58 089, Morelia, Michoacán, MEXICO
E-mail address: fluca@matmor.unam.mx

School of Mathematics, TIFR, Mumbai-400 005, INDIA
E-mail address: sank@math.tifr.res.in