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ROBUST GLOBAL EXPONENTIAL STABILITY OF LINEAR IMPULSIVE SYSTEMS WITH TIME-VARYING DELAY AND UNCERTAINTY

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ABSTRACT. This paper studies linear impulsive systems with varying time-delay and uncertainty. By using the method of Lyapunov functions and matrix inequalities, robust global exponential stability criteria are established in terms of fairly simple algebraic conditions. Estimate of the decay rate of the solutions of such systems are also derived. Some examples are given to illustrate the main results.

1. Introduction. Many real world systems display both continuous and discrete characteristics. For example, evolutionary processes such as biological neural networks, bursting rhythm models in pathology, optimal control models in economics, frequency-modulated signal processing systems and flying object motions, etc., are characterized by abrupt changes of states at certain time instants. Those sudden and sharp changes are often of very short duration and are thus assumed to occur instantaneously in the form of impulses. Such impulses may be represented by discrete maps. Systems undergoing abrupt changes may not be well described by using purely continuous or purely discrete models. However, they can be appropriately modeled by impulsive systems. It is now recognized that the theory of impulsive systems provides a natural framework for mathematical modeling of many real world phenomena. Significant progress has been made in the theory of impulsive systems in recent years, see [**1, 4, 5, 7–11**] and references therein. However, the corresponding theory for impulsive systems with uncertainty has not been fully developed. Recently, some robust stability results for impulsive systems with uncertainty have been established in

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[**9, 10**]. In this paper, we shall study linear impulsive systems with uncertainty and time-varying delay. By using Lyapunov functions, matrix inequalities and the theory of M-matrix, we shall establish some criteria on robust global exponential stability and provide some estimate of the decay rate for such systems. Some examples are also worked out to demonstrate the main results.

2. Preliminaries. Let R^n denote the n-dimensional real vector space and $||A||$ the norm of a matrix A induced by the Euclidean norm, i.e., $||A|| = [\lambda_{\max}(A^T A)]^{1/2}$. Let N denote the set of positive integers, i.e., $N = \{1, 2, \dots, \}$ and $R^+ = [0, +\infty)$. Let $\Delta t_k = t_{k+1} - t_k, S_\rho = \{x \in R^n : ||x|| \le \rho\}$, and $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) the maximum (minimum) eigenvalue of a symmetric matrix X.

Consider the following linear impulsive system with time-varying delay and uncertainty

$$
\begin{cases} \dot{x}(t) = \widetilde{A}x(t) + \widetilde{B}x(t - h(t)) & t \in [t_{k-1}, t_k), \\ \Delta x(t) = \widetilde{C}_k x(t^-) + \widetilde{D}_k x(t^- - h(t)) & t = t_k,\, k \in N, \end{cases} \tag{1}
$$

where $h(t)$ is a continuous function on R^+ satisfying $0 \le h(t) \le \tau$, for some positive constant $\tau > 0$, and $\widetilde{A}, \widetilde{B}, \widetilde{C_k}, \widetilde{D_k} \in R^{n\times n}$ are interval matrices satisfying

$$
\begin{aligned}
\widetilde{A} \in N[A^{(1)}, A^{(2)}] &= \{\widetilde{A} = (\tilde{a}_{ij})_{n\times n} : a_{ij}{}^{(1)} \le \tilde{a}_{ij} \le a_{ij}{}^{(2)}\}, \\
\widetilde{B} \in N[B^{(1)}, B^{(2)}] &= \{\widetilde{B} = (\tilde{b}_{ij})_{n\times n} : b_{ij}{}^{(1)} \le \tilde{b}_{ij} \le b_{ij}{}^{(2)}\}, \\
\widetilde{C}_k \in N[C_k^{(1)}, C_k^{(2)}] &= \{\widetilde{C}_k = (\tilde{c}_{ij_k})_{n\times n} : c_{ij_k}{}^{(1)} \le \tilde{c}_{ij_k} \le c_{ij_k}{}^{(2)}\}, \\
\widetilde{D}_k \in N[D_k^{(1)}, D_k^{(2)}] &= \{\widetilde{D}_k = (\tilde{d}_{ij_k})_{n\times n} : d_{ij_k}{}^{(1)} \le \tilde{d}_{ij_k} \le d_{ij_k}{}^{(2)}\},
\end{aligned}
$$

where $A^{(l)} = (a_{ij}{}^{(l)})_{n\times n}$, $B^{(l)} = (b_{ij}{}^{(l)})_{n\times n}$, $C_k^{(l)} = (c_{ij_k}^{(l)})_{n\times n}$ and $D_k^{(l)} = (d_{ij_k}^{(l)})_{n\times n} \in R^{n\times n}$, $l = 1, 2$, $k \in N$, are known matrices. The following definition is adopted from [**2**].

Definition 1. System (1) is said to be robustly globally exponentially stable with decay rate $\lambda > 0$ if for every $\widetilde{A} \in N[A^{(1)}, A^{(2)}]$, $\widetilde{B} \in N[B^{(1)}, B^{(2)}]$, $\widetilde{C}_k \in N[C_k^{(1)}, C_k^{(2)}]$, $\widetilde{D}_k \in N[D_k^{(1)}, D_k^{(2)}]$ the trivial solution of system (1) is globally exponentially stable with decay rate $\lambda > 0$.

The following lemma is taken from [**12**].

Lemma 1. *An interval matrix $\widetilde{X} \in N[X^{(1)}, X^{(2)}]$ can be described as*

$$\widetilde{X} = X + E_X \Sigma_X F_X, \tag{2}$$

where $X = (X^{(1)} + X^{(2)})/2$, $H = (h_{ij})_{n\times n} = (X^{(2)} - X^{(1)})/2$, $\Sigma_X \in \Sigma^* = \{\Sigma \in R^{n^2\times n^2} : \Sigma = \operatorname{diag}\{\varepsilon_{11}, \dots, \varepsilon_{n^2n^2}\}, |\varepsilon_{ij}| \le 1;\ i,j = 1,2,\dots,n.\}$, $E_X \cdot E_X^T = \operatorname{diag}\{\sum_{j=1}^n h_{1j}, \sum_{j=1}^n h_{2j}, \dots, \sum_{j=1}^n h_{nj}\} \in R^{n\times n}$, $F_X^T \cdot F_X = \operatorname{diag}\{\sum_{j=1}^n h_{j1}, \sum_{j=1}^n h_{j2}, \dots, \sum_{j=1}^n h_{jn}\} \in R^{n\times n}$.

By Lemma 1, we can rewrite system (1) as

$$\begin{cases} \dot{x}(t) = (A + E_A\Sigma_A F_A)x(t) + (B + E_B\Sigma_B F_B)x(t - h(t)) \\ \hfill t \in [t_{k-1}, t_k), \\ \Delta x(t) = (C_k + E_{C_k}\Sigma_{C_k}F_{C_k})x(t^-) + (D_k + E_{D_k}\Sigma_{D_k}F_{D_k})x(t^- - h(t)) \\ \hfill t = t_k. \end{cases} \tag{3}$$

where $\Sigma_A, \Sigma_B, \Sigma_{C_k}, \Sigma_{D_k} \in \Sigma^*$, $k \in N$.

Lemma 2. *Let $\Sigma \in \Sigma^*$. Then, for any positive constant λ and any $\xi \in R^{n^2}$, $\eta \in R^{n^2}$, the following inequality holds.*

$$2\xi^T \Sigma \eta \le \lambda^{-1}\xi^T\xi + \lambda \eta^T \eta. \tag{4}$$

Proof. It follows by using the Schwartz inequality and the fact that $\Sigma \cdot \Sigma^T = \Sigma^T \cdot \Sigma \le I_{n^2}$ where I_{n^2} is the $n^2 \times n^2$ identity matrix. $\square$

Lemma 3. *Let $X \in R^{n\times n}$ be a positive definite matrix and $Q \in R^{n\times n}$ a symmetric matrix. Then, for any $x \in R^n$, the following inequality holds.*

$$\lambda_{\min}(X^{-1}Q) \cdot x^T X x \le x^T Q x \le \lambda_{\max}(X^{-1}Q) \cdot x^T X x. \tag{5}$$

Proof. Since X is a positive definite matrix, there exists a nonsingular matrix C such that $X = C^T C$. Let $J = x^T Qx/x^T Xx$ and $z = Cx$. Then,

$$J = \frac{z^T(C^{-1})^T Q(C^{-1})z}{z^T z} \le \lambda_{\max}((C^{-1})^T Q(C^{-1})). \tag{6}$$

Since

$$(C^{-1})^T Q(C^{-1}) = C \cdot C^{-1} \cdot (C^{-1})^T QC^{-1} = C \cdot X^{-1}Q \cdot C^{-1},$$

we get

$$\lambda_{\max}(X^{-1}Q) = \lambda_{\max}((C^{-1})^T Q(C^{-1})). \tag{7}$$

By (6) and (7), we get $x^T Qx \le \lambda_{\max}(X^{-1}Q) \cdot x^T Xx$. Similarly, we can get $\lambda_{\min}(X^{-1}Q) \cdot x^T Xx \le x^T Qx$. Hence, inequality (5) follows.

We shall use the following results [**6**].

Lemma 4 (Halanay lemma). *Let $m \in C[R, R_+]$ be a scalar positive function such that*

$$D^+ m(t) \le -a_1 m(t) + a_2 \bar{m}(t), \quad t \ge t_0, \tag{8}$$

where $a_1 > a_2 > 0$ and $\bar{m}(t) = \sup_{t-\tau \le s \le t}\{m(s)\}$.

Then, there exist constants $\gamma > 0$, $\alpha > 0$ such that, for all $t \ge t_0$,

$$m(t) \le \gamma \bar{m}(t_0) e^{-\alpha(t-t_0)}, \tag{9}$$

where $\alpha > 0$ satisfies $\alpha - a_1 + a_2 e^{\alpha\tau} = 0$.

Lemma 5. *Let $x_i \in C^1[R, R]$, $i = 1, \dots, n$, and assume that*

(i)

$$D^+|x_i(t)| \le \sum_{j=1}^n c_{ij}|x_j(t)| + \sum_{j=1}^n d_{ij}|\bar{x}_j(t)|, \quad i = 1, \dots, n \tag{10}$$

Here, $\bar{x}_j(t) = \sup_{t-\tau\le s\le t} x_j(s)$, $c_{ij} \ge 0$, $i \ne j$; $d_{ij} \ge 0$, $i,j = 1,2,\dots,n$.

(ii) $M = -(c_{ij}+d_{ij})_{n\times n}$ *is an* M*-matrix.*

Then, there exist constants $\gamma_i > 0$, $\alpha > 0$ *such that for all* $t \ge t_0$, $i = 1,2,\dots,n$,

$$|x_i(t)| \le \gamma_i \cdot \left\{ \sum_{j=1}^{n} |\bar{x}_j(t_0)| \right\} e^{-\alpha(t-t_0)}. \tag{11}$$

3. Main results. For any positive definite matrix $P \in R^{n\times n}$, and the matrices given in system (3) and constants $\mu_i > 0$, $i = 1,2,\dots,9$, we introduce the following notations for convenience.

$$Y(\mu_2,\mu_3) = \mu_3^{-1} I + \mu_2^{-1} F_B^T F_B;$$

$$\begin{aligned} Z(\mu_1,\mu_2,\mu_3;P) = {} & A^T P + PA + \mu_1 P E_A E_A^T P + \mu_2 P E_B E_B^T P \\ & + \mu_3 P B B^T P + \mu_1^{-1} F_A^T F_A; \end{aligned}$$

$$\begin{aligned} a(\mu_1,\mu_2,\mu_3;P) &= -\lambda_{\max}\{P^{-1}\cdot Z(\mu_1,\mu_2,\mu_3;P)\}; \\ b(\mu_2,\mu_3;P) &= \lambda_{\max}\{P^{-1}\cdot Y(\mu_2,\mu_3)\}; \end{aligned}$$

$$\begin{aligned} & W_k(\mu_4,\dots,\mu_9;P) \\ & = ||P||\cdot||E_{C_k}||^2\cdot||F_{C_k}||^2\cdot I + (\mu_5^{-1}+\mu_6^{-1}+\mu_9^{-1}) F_{C_k}^T F_{C_k} \\ & \quad + (I+C_k)^T \Big[P + P(\mu_4 D_k D_k^T + \mu_5 E_{C_k} E_{C_k}^T + \mu_8 E_{D_k} E_{D_k}^T) P \Big] (I+C_k); \end{aligned}$$

$$\begin{aligned} & X_k(\mu_4,\dots,\mu_9;P) \\ & = (\mu_4^{-1} + ||P||\cdot||E_{D_k}||^2\cdot||F_{D_k}||^2)\cdot I + (\mu_7^{-1}+\mu_8^{-1}+\mu_9||E_{C_k}^T P E_{D_k}||^2) \\ & \quad \cdot F_{D_k}^T F_{D_k} + D_k^T \Big[P + P(\mu_6 E_{C_k} E_{C_k}^T + \mu_7 E_{D_k} E_{D_k}^T) P \Big] D_k; \end{aligned}$$

$$\alpha_k(\mu_4, \cdots, \mu_9; P) = \lambda_{\max}\{P^{-1} \cdot W_k(\mu_4, \cdots, \mu_9; P)\};$$
$$\beta_k(\mu_4, \cdots, \mu_9; P) = \lambda_{\max}\{P^{-1} \cdot X_k(\mu_4, \cdots, \mu_9; P)\}.$$

Theorem 1. *Assume that there exist a positive definite matrix P and constants $\mu_i > 0$, $i = 1, 2, \ldots, 9$, $\delta > 1$ and $M \geq 1$ such that the following inequalities hold.*

(i) $0 < b(\mu_2, \mu_3; P) < -a(\mu_1, \mu_2, \mu_3; P)$;

(ii) $\delta\tau \leq \sigma = \inf_{k \in N}\{t_k - t_{k-1}\}$;

(iii) $\max\{e^{\lambda\tau}, \alpha_k(\mu_4, \ldots, \mu_9; P) + \beta_k(\mu_4, \ldots, \mu_9; P)e^{\lambda\tau}\} \leq M < e^{\lambda\delta\tau}$, $k \in N$, *where $\lambda > 0$ satisfying equation*

$$\lambda + a(\mu_1, \mu_2, \mu_3; P) + b(\mu_2, \mu_3; P)e^{\lambda\tau} = 0. \tag{12}$$

Then, system (1) *is robustly globally exponentially stable with decay rate $\{\lambda - (\ln M/\delta\tau)\}/2$.*

Proof. Let $V(x) = x^T P x$. Then we have

$$\lambda_{\min}(P) \cdot ||x||^2 \leq V(x) \leq \lambda_{\max}(P) \cdot ||x||^2.$$

Firstly, we shall show for all $t_{k-1} \leq t < t_k$, $k \in N$, there exists a constant $\gamma \geq 1$ such that

$$V(x(t)) \leq \gamma \cdot \bar{V}(x(t_{k-1}))e^{-\lambda(t - t_{k-1})}, \tag{13}$$

where $\bar{V}(x(t)) = \sup_{t-\tau \leq s \leq t}\{V(x(s))\}$.

By Lemma 2 and Lemma 3 we get, for $t_{k-1} \leq t < t_k$, $k \in N$,

(14)
$$\begin{aligned}
D^+V(x(t)) &= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T(A^T P + PA)x + 2x^T PBx(t - h(t)) \\
&\quad + 2x^T PE_A \Sigma_A F_A x + 2x^T PE_B \Sigma_B F_B x(t - h(t)) \\
&\leq x^T\{A^T P + PA + \mu_1 PE_A E_A^T P + \mu_2 PE_B E_B^T P \\
&\quad + \mu_3 PBB^T P + \mu_1^{-1} F_A^T F_A\}x
\end{aligned}$$

$$
\begin{aligned}
&+ x^T(t-h(t))\{\mu_2^{-1}F_B^TF_B + \mu_3^{-1}I\}x(t-h(t)) \\
&= x^T Z(\mu_1,\mu_2,\mu_3;P)x + x^T(t-h(t))Y(\mu_2,\mu_3)x(t-h(t)) \\
&\le \lambda_{\max}\{P^{-1}Z(\mu_1,\mu_2,\mu_3;P)\} \\
&\quad \cdot V(x(t)) + \lambda_{\max}\{P^{-1}Y(\mu_2,\mu_3)\}\cdot V(x(t-h(t))) \\
&= -a(\mu_1,\mu_2,\mu_3;P)\cdot V(x(t)) + b(\mu_2,\mu_3;P) \\
&\quad \cdot V(x(t-h(t))).
\end{aligned}
$$

From Lemma 4, condition (i) and (14), we see that there exist constants $\gamma > 0, \lambda > 0$ such that

$$
V(x(t)) \le \gamma \cdot \overline{V}(x(t_{k-1}))e^{-\lambda(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k). \tag{15}
$$

where $\overline{V}(x(t_{k-1})) = \sup_{t_{k-1}-\tau \le s \le t_{k-1}}\{V(x(s))\}$ and $\lambda > 0$ satisfies (12).

Thus, (13) is true.

Secondly, we shall show

$$
\begin{aligned}
V(x(t_k)) \le \alpha_k(\mu_4,\cdots,\mu_9;P)\cdot V(x(t_k^-)) + \beta_k(\mu_4,\cdots,\mu_9;P) \\
\cdot V(x(t_k - h(t_k))), \quad k \in N.
\end{aligned} \tag{16}
$$

Let $h_k = h(t_k)$. For system (3), we get, for $t = t_k$,

(17)

$$
\begin{aligned}
V(x(t_k)) &= V(x(t_k^-) + \Delta x(t_k)) \\
&= \Big\{(I+C_k)x(t_k^-) + E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-) \\
&\qquad + D_k x(t_k - h_k) + E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k - h_k)\Big\}^T P \\
&\quad \cdot \Big\{(I+C_k)x(t_k^-) + E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-) \\
&\qquad + D_k x(t_k - h_k) + E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k - h_k)\Big\} \\
&= x(t_k^-)^T(I+C_k)^T P(I+C_k)x(t_k^-) \\
&\quad + x(t_k - h_k)^T D_k^T P D_k x(t_k - h_k) \\
&\quad + 2x(t_k^-)^T(I+C_k)^T P D_k x(t_k - h_k) \\
&\quad + x(t_k^-)^T F_{C_k}^T \Sigma_{C_k}^T E_{C_k}^T P E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2x(t_k^-)(I+C_k)^T P E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-)\\
&+ 2x(t_k^-)F_{C_k}^T\Sigma_{C_k}^T E_{C_k}^T P D_k x(t_k-h_k)\\
&+ x(t_k-h_k)^T F_{D_k}^T\Sigma_{D_k}^T E_{D_k}^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k)\\
&+ 2x(t_k-h_k)^T D_k^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k)\\
&+ 2x(t_k^-)^T(I+C_k)^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k)\\
&+ 2x(t_k^-)^T F_{C_k}^T\Sigma_{C_k}^T E_{C_k}^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k).
\end{aligned}
$$

Using Lemma 2, we obtain the following inequalities:

$$
\begin{aligned}
(18)\quad & 2x(t_k^-)^T(I+C_k)^T P D_k x(t_k-h_k)\\
&\qquad \le \mu_4 x(t_k^-)^T(I+C_k)^T P D_k D_k^T P(I+C_k)x(t_k^-)\\
&\qquad\quad + \mu_4^{-1}x(t_k-h_k)^T x(t_k-h_k);
\end{aligned}
$$

$$
\begin{aligned}
(19)\quad & x(t_k^-)^T F_{C_k}^T\Sigma_{C_k}^T E_{C_k}^T P E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-)\\
&\qquad \le ||P||\cdot||E_{C_k}||^2\cdot||F_{C_k}||^2 x(t_k^-)^T x(t_k^-);
\end{aligned}
$$

$$
\begin{aligned}
(20)\quad & 2x(t_k^-)(I+C_k)^T P E_{C_k}\Sigma_{C_k}F_{C_k}x(t_k^-)\\
&\qquad \le \mu_5 x(t_k^-)^T(I+C_k)^T P E_{C_k}E_{C_k}^T P(I+C_k)x(t_k^-)\\
&\qquad\quad + \mu_5^{-1}x(t_k^-)^T F_{C_k}^T F_{C_k}x(t_k^-);
\end{aligned}
$$

$$
\begin{aligned}
(21)\quad & 2x(t_k^-)F_{C_k}^T\Sigma_{C_k}^T E_{C_k}^T P D_k x(t_k-h_k)\\
&\qquad \le \mu_6 x(t_k-h_k)^T D_k^T P E_{C_k}E_{C_k}^T P D_k x(t_k-h_k)\\
&\qquad\quad + \mu_6^{-1}x(t_k^-)^T F_{C_k}^T F_{C_k}x(t_k^-);
\end{aligned}
$$

$$
\begin{aligned}
(22)\quad & x(t_k-h_k)^T F_{D_k}^T\Sigma_{D_k}^T E_{D_k}^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k)\\
&\qquad \le ||P||\cdot||E_{D_k}||^2\cdot||F_{D_k}||^2 x(t_k-h_k)^T x(t_k-h_k);
\end{aligned}
$$

$$
\begin{aligned}
(23)\quad & 2x(t_k-h_k)^T D_k^T P E_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k)\\
&\qquad \le \mu_7 x(t_k-h_k)^T D_k^T P E_{D_k}E_{D_k}^T P D_k x(t_k-h_k)\\
&\qquad\quad + \mu_7^{-1}x(t_k-h_k)^T F_{D_k}^T F_{D_k}x(t_k-h_k);
\end{aligned}
$$

$$(24)\quad \begin{aligned} & 2x(t_k^-)^T(I+C_k)^T PE_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k) \\ & \quad \le \mu_8 x(t_k^-)^T(I+C_k)^T PE_{D_k}E_{D_k}^T P(I+C_k)x(t_k^-) \\ & \quad\quad + \mu_8^{-1}x(t_k-h_k)^T F_{D_k}^T F_{D_k}x(t_k-h_k); \end{aligned}$$

$$(25)\quad \begin{aligned} & 2x(t_k^-)^T F_{C_k}^T \Sigma_{C_k}^T E_{C_k}^T PE_{D_k}\Sigma_{D_k}F_{D_k}x(t_k-h_k) \\ & \quad \le \mu_9^{-1}x(t_k^-)^T F_{C_k}^T F_{C_k}x(t_k^-) + \mu_9 \\ & \quad\quad \cdot ||E_{C_k}^T PE_{D_k}||^2 x(t_k-h_k)^T F_{D_k}^T F_{D_k}x(t_k-h_k). \end{aligned}$$

Substituting (18)–(25) into (17), we get

(26)

$$\begin{aligned} V(x(t_k)) \le x(t_k^-)^T \cdot \Big\{ & ||P|| \cdot ||E_{C_k}||^2 \cdot ||F_{C_k}||^2 \cdot I \\ & + (\mu_5^{-1}+\mu_6^{-1}+\mu_9^{-1})F_{C_k}^T F_{C_k} \\ & + (I+C_k)^T \Big[P + P(\mu_4 D_k D_k^T + \mu_5 E_{C_k}E_{C_k}^T \\ & + \mu_8 E_{D_k}E_{D_k}^T)P \Big](I+C_k) \Big\} \cdot x(t_k^-) \\ + x(t_k-h_k)^T \Big\{ & (\mu_4^{-1} + ||P|| \cdot ||E_{D_k}||^2 \cdot ||F_{D_k}||^2) \cdot I \\ & + (\mu_7^{-1}+\mu_8^{-1}+\mu_9 ||E_{C_k}^T PE_{D_k}||^2)F_{D_k}^T F_{D_k} \\ + D_k^T \Big[P + P(\mu_6 & E_{C_k}E_{C_k}^T + \mu_7 E_{D_k}E_{D_k}^T)P \Big] D_k \Big\} \cdot x(t_k-h_k). \end{aligned}$$

Hence, (16) follows directly from Lemma 3.

Thirdly, we shall show by induction

(27)

$$V(x(t)) \le \gamma M^{k-1}\lambda_{\max}(P)||\phi||^2 e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k), \quad k \in N,$$

where $\lambda > 0$ is determined by (12).

When $k = 1$, since for all $t \in [t_0-\tau, t_0]$,

$$||x(t)|| = ||\phi(t)|| \le ||\phi|| = \sup_{t_0-\tau \le t \le t_0} ||\phi(t)||,$$

we get

$$V(x(t)) \le \lambda_{\max}(P) \cdot ||x(t)||^2 \le \lambda_{\max}(P) \cdot ||\phi||^2.$$

Hence, we have

$$\bar{V}(x(t_0)) \le \lambda_{\max}(P) \cdot ||\phi||^2. \tag{28}$$

By (13) and (28), we get

$$\begin{aligned} V(x(t)) &\le \gamma \overline{V}(x(t_0)) e^{-\lambda(t-t_0)} \le \gamma \lambda_{\max}(P) \cdot ||\phi||^2 e^{-\lambda(t-t_0)} \\ &= \gamma M^0 \lambda_{\max}(P) \cdot ||\phi||^2 e^{-\lambda(t-t_0)}, \quad t \in [t_0, t_1). \end{aligned}$$

Thus, (27) holds for $k = 1$.

Now assume (27) holds for $k \le m$, $m > 1$. Then we shall show that (27) holds for $k = m + 1$.

By (13), (16) and the induction assumption, we have

$$\begin{aligned} V(x(t_m)) \le\ & \alpha_m(\mu_4, \dots, \mu_9; P) V(x(t_m^-)) \\ & + \beta_m(\mu_4, \dots, \mu_9; P) V(x(t_m - h(t_m))) \\ \le\ & \gamma M^{m-1} \alpha_m(\mu_4, \dots, \mu_9; P) \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - t_0)} \\ & + \gamma M^{m-1} \beta_m(\mu_4, \dots, \mu_9; P) \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - \tau - t_0)} \\ =\ & \gamma M^{m-1} \lambda_{\max}(P) ||\phi||^2 \big\{ \alpha_m(\mu_4, \dots, \mu_9; P) \\ & \qquad + \beta_m(\mu_4, \dots, \mu_9; P) e^{\lambda\tau} \big\} e^{-\lambda(t_m - t_0)} \\ \le\ & \gamma M^m \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - t_0)}. \end{aligned} \tag{30}$$

Hence, by conditions (3), (13) and (30), for $k = m + 1$, $t \in [t_m, t_{m+1})$, we get

$$\begin{aligned} V(x(t)) &\le \gamma \cdot \overline{V}(x(t_m)) \cdot e^{-\lambda(t-t_m)} \\ &= \gamma \cdot \max_{t_m - \tau \le t \le t_m} \{V(x(t))\} \cdot e^{-\lambda(t-t_m)} \\ &= \gamma \cdot \max\Big\{ \sup_{t_m - \tau \le t < t_m} \{V(x(t))\}, V(x(t_m)) \Big\} e^{-\lambda(t-t_m)} \\ &\le \gamma \cdot \max\big\{ M^{m-1} \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - \tau - t_0)}, \\ &\qquad M^m \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - t_0)} \big\} e^{-\lambda(t-t_m)} \\ &= \gamma \cdot \max\big\{ M^{m-1} e^{\lambda\tau}, M^m \big\} \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t_m - t_0)} e^{-\lambda(t-t_m)} \\ &\le \gamma M^m \lambda_{\max}(P) ||\phi||^2 e^{-\lambda(t-t_0)}. \end{aligned} \tag{31}$$

Therefore, by induction principle, we see (27) holds for all $k \in N$.

Lastly, we show

$$||x(t)|| \le K||\phi||e^{-\alpha(t-t_0)}, \quad t \ge t_0, \tag{32}$$

where, $\alpha = \{\lambda - (\ln M/\delta\tau)\}/2 > 0$, $K = \sqrt{\gamma \cdot (\lambda_{\max}(P)/\lambda_{\min}(P))}$.

Since $\delta\tau \le \sigma = \inf_{k\in N}\{t_k - t_{k-1}\}$, we have $k-1 \le (t_{k-1} - t_0/\delta\tau)$, which implies $M^{k-1} \le e^{(\ln M/\delta\tau)(t_{k-1}-t_0)}$. Thus, for $t \in [t_{k-1}, t_k)$, we get

$$\begin{aligned} ||x(t)||^2 &\le \frac{V(x(t))}{\lambda_{\min}(P)} \le \gamma \cdot \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\, ||\phi||^2\, M^{k-1} e^{-\lambda(t-t_0)} \\ &\le \gamma \cdot \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\, ||\phi||^2\, e^{\{-\lambda + (\ln M/\delta\tau)\}(t-t_0)}. \end{aligned} \tag{33}$$

Hence, (32) is true, which implies that system (1) is robustly globally exponentially stable with decay rate $\{\lambda - (\ln M/\delta\tau)\}/2$. The proof is thus complete. □

Theorem 2. *Assume that there exist a positive definite matrix P, and constants $a > b > 0$, $\mu_1 > 0, \dots, \mu_9 > 0$, $\delta > 1$, $c_k \ge 0$, $d_k \ge 0$, $k \in N$ and $M \ge 1$ such that the following inequalities hold.*

(i) $Z(\mu_1, \mu_2, \mu_3; P) + a \cdot P \le 0$;

(ii) $Y(\mu_2, \mu_3) - b \cdot P \le 0$;

(iii) $W_k(\mu_4, \dots, \mu_9; P) - c_k \cdot P \le 0, \quad k \in N$;

(iv) $X_k(\mu_4, \dots, \mu_9; P - d_k \cdot P \le 0$, $k \in N$;

(v) $\delta\tau \le \sigma = \inf_{k\in N}\{t_k - t_{k-1}\}$;

(vi) $\max\{e^{\lambda\tau},\ c_k + d_k e^{\lambda\tau}\} \le M < e^{\lambda\delta\tau}$, $k \in N$, *where* $\lambda > 0$ *satisfying equation*

$$\lambda - a + be^{\lambda\tau} = 0. \tag{34}$$

Then, system (1) *is robustly globally exponentially stable with decay rate* $\{\lambda - (\ln M/\delta\tau)\}/2$.

Proof. Let $V(x) = x^T Px$. From conditions (i)–(iv), we get, as in Theorem 1,

(35)
$$\begin{aligned} D^+V(x(t)) &\le x^T Z(\mu_1,\mu_2,\mu_3;P)x + x^T(t-h(t))Y(\mu_2,\mu_3)x(t-h(t)) \\ &\le -a\cdot V(x(t)) + b\cdot V(x(t-h(t))), \end{aligned}$$

$$\begin{aligned} V(x(t_k)) &\le x(t_k^-)^T W_k(\mu_4,\dots,\mu_9;P)x(t_k^-) \\ &\qquad + x(t_k-h(t_k))^T X_k(\mu_4,\dots,\mu_9;P)x(t_k-h(t_k)) \\ &\le c_k\cdot V(x(t_k^-)) + d_k\cdot V(x(t_k-h(t_k))), \quad k\in N. \end{aligned} \tag{36}$$

The rest of the proof is the same as in Theorem 1 and thus the details are omitted. $\square$

Next we shall make use of the M-matrix theory and derive some fairly simple algebraic criteria with which we do not need to solve matrix inequalities.

Let $\bar{a}_{ij} = \max\{|a_{ij}^{(1)}|, |a_{ij}^{(2)}|\}$, $\bar{b}_{ij} = \max\{|b_{ij}^{(1)}|, |b_{ij}^{(2)}|\}$, $\bar{c}_{ij_k} = \max\times \{|c_{ij_k}^{(1)}|, |c_{ij_k}^{(2)}|\}$, and $\bar{d}_{ij_k} = \max\{|d_{ij_k}^{(1)}|, |d_{ij_k}^{(2)}|\}$, $i,j=1,2,\dots,n$, $k\in N$.

Define $\alpha_k = \max_{1\le j\le n}\Big\{\sum_{i=1}^n \hat{c}_{ij_k}\Big\}$, $\beta_k = \max_{1\le j\le n}\Big\{\sum_{i=1}^n \bar{d}_{ij_k}\Big\}$, where

$$\hat{c}_{ij_k} = \begin{cases} \max\{|1+c_{ii_k}^{(1)}|, |1+c_{ii_k}^{(2)}|\} & \text{if } i=j=1,2,\dots,n, \\ \bar{c}_{ij_k} & \text{if } i\ne j=1,2,\dots,n. \end{cases}$$

Define $H = (h_{ij})_{n\times n}$ with

$$h_{ij} = \begin{cases} a_{ii}^{(2)} + \bar{b}_{ii} & \text{if } i=j=1,2,\dots,n, \\ \bar{a}_{ij} + \bar{b}_{ij} & \text{if } i\ne j,\ i,j=1,2,\dots,n. \end{cases} \tag{37}$$

Theorem 3. *Assume that $-H(h_{ij})_{n\times n}$ is an M-matrix, and suppose further that*

(i) $\delta\tau \le \sigma = \inf_{k\in N}\{t_k - t_{k-1}\}$, *for some* $\delta > 1$;

(ii) $\max\{e^{\alpha\tau}, \alpha_k + \beta_k e^{\alpha\tau}\} \le M < e^{\alpha\delta\tau}$, $k \in N$, *for some* $\alpha > 0$, $M \ge 1$.

Then, system (1) *is robustly globally exponentially stable with decay rate* $\alpha - (\ln M/\delta\tau)$.

Proof. Let $x(t)$ be any solution of system (1). Then, for $t \in [t_{k-1}, t_k)$, taking the time Dini derivative of $|x_i(t)|$ along system (1), we get

(38)
$$\begin{aligned} D^+|x_i(t)| &\le \tilde{a}_{ii}|x_i(t)| + \sum_{j=1, j\ne i}^{n} |\tilde{a}_{ij}|\,|x_j(t)| + \sum_{j=1}^{n} |\tilde{b}_{ij}|\,|x_j(t-h(t))| \\ &\le a_{ii}^{(2)}|x_i(t)| + \sum_{j=1, j\ne i}^{n} \bar{a}_{ij}|x_j(t)| + \sum_{j=1}^{n} \bar{b}_{ij}|\bar{x}_j(t)|. \end{aligned}$$

By Lemma 5, we see that there exist constants $\gamma_i > 0$, $\alpha > 0$ such that for all $i = 1, 2, \dots, n$,

$$(39) \qquad |x_i(t)| \le \gamma_i \cdots \left\{ \sum_{j=1}^{n} |\bar{x}_j(t_{k-1})| \right\} e^{-\alpha(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k).$$

When $t = t_k$, from system (1), we get

(40)
$$\begin{aligned} |x_j(t_{k-1})| &= \left| x_j(t_{k-1}^-) + \Delta x_j(t_{k-1}) \right| \\ &= \Bigg| (1 + \tilde{c}_{jj_{k-1}}) x_j(t_{k-1}^-) + \sum_{\substack{l\ne j,\\ l=1}}^{n} \tilde{c}_{jl_{k-1}} x_l(t_{k-1}^-) \\ &\quad + \sum_{l=1}^{n} \tilde{d}_{jl_{k-1}} x_l(t_{k-1} - h(t_{k-1})) \Bigg| \\ &\le \sum_{l=1}^{n} \hat{c}_{jl_{k-1}} |x_l(t_{k-1}^-)| + \sum_{l=1}^{n} \bar{d}_{jl_{k-1}} |x_l(t_{k-1} - h(t_{k-1}))|, \\ &\qquad j = 1, 2, \dots, n. \end{aligned}$$

Defining $v(t) = \sum_{j=1}^{n} |x_j(t)|$ and $\gamma = \sum_{j=1}^{n} \gamma_j$, then by (39) for $t \in [t_{k-1}, t_k)$, we get

(41)
$$v(t) = \sum_{j=1}^{n} |x_j(t)| \le \sum_{j=1}^{n} \gamma_j \bar{v}(t_{k-1}) e^{-\alpha(t-t_{k-1})} = \gamma \bar{v}(t_{k-1}) e^{-\alpha(t-t_{k-1})}.$$

By (40), we have

$$\begin{aligned} v(t_{k-1}) &\le \sum_{j=1}^{n}\sum_{l=1}^{n} \hat{c}_{jl_{k-1}} |x_l(t_{k-1}^-)| \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{d}_{jl_{k-1}} |x_l(t_{k-1} - h(t_{k-1}))| \\ &= \sum_{l=1}^{n} \left\{ \sum_{j=1}^{n} \hat{c}_{jl_{k-1}} \right\} |x_l(t_{k-1}^-)| \\ &\quad + \sum_{l=1}^{n} \left\{ \sum_{j=1}^{n} \bar{d}_{jl_{k-1}} \right\} |x_l(t_{k-1} - h(t_{k-1}))| \\ &\le \alpha_{k-1} \cdot v(t_{k-1}^-) + \beta_{k-1} \cdot v(t_{k-1} - h(t_{k-1})). \end{aligned} \tag{42}$$

Then, by a similar argument to that used in the proof inequality (27) in Theorem 1, we obtain

$$v(t) \le \gamma M^{k-1} \left\{ \sum_{j=1}^{n} |\phi_j| \right\} e^{-\alpha(t-t_0)}, \quad t \in [t_{k-1}, t_k), \quad k \in N, \tag{43}$$

Thus, by (43) and conditions (i)–(ii), we get

$$\begin{gathered} \max_{1 \le i \le n} \{|x_i(t)|\} \le v(t) \le \gamma \left\{ \sum_{j=1}^{n} |\phi_j| \right\} e^{-\{\alpha - (\ln M/\delta\tau)\}(t-t_0)}, \\ t \in [t_{k-1}, t_k), \quad k \in N. \end{gathered} \tag{44}$$

Hence, system (1) is robustly globally exponentially stable with the decay rate $\alpha - (\ln M/\delta\tau)$. $\square$

Remark 2. The positive constant $\alpha > 0$ determined by Lemma 5 is difficult to get. In the following, we give an estimate of such a decay rate.

Theorem 4. *Assume that there exists a positive constant* $\lambda > 0$ *such that* $-\lambda I - H$ *is an* M*-matrix and conditions* (i)–(ii) *of Theorem* 3 *hold. Then system* (1) *is robustly exponentially stable with at least decay rate* λ.

Proof. Let $y_i(t) = x_i(t)e^{\lambda t}$, $i = 1, 2, \dots, n$, then

(45)

$$\begin{aligned} D^+|y_i(t)| & \\ &\le e^{\lambda t} D^+|x_i(t)| + \lambda e^{\lambda t}|x_i(t)| \\ &\le e^{\lambda t}\left\{(\tilde{a}_{ii} + \lambda)|x_i(t)| + \sum_{j=1, j\neq i}^{n} |\tilde{a}_{ij}||x_j(t)| + \sum_{j=1}^{n} |\tilde{b}_{ij}||x_j(t - h(t))|\right\} \\ &\le e^{\lambda t}\left\{(a_{ii}^{(2)} + \lambda)|x_i(t)| + \sum_{j=1, j\neq i}^{n} \bar{a}_{ij}|x_j(t)| + \sum_{j=1}^{n} \bar{b}_{ij}|\bar{x}_j(t)|\right\} \\ &= (a_{ii}^{(2)} + \lambda)|y_i(t)| + \sum_{j=1, j\neq i}^{n} \bar{a}_{ij}|y_j(t)| + \sum_{j=1}^{n} \bar{b}_{ij}|\bar{y}_j(t)|, \\ &\qquad\qquad t \in [t_{k-1}, t_k). \end{aligned}$$

By Lemma 5, there exist constants $\gamma_i > 0$, $\alpha > 0$ such that for all $t \in [t_{k-1}, t_k)$, $k \in N$,

$$|y_i(t)| \le \gamma_i \cdot \left\{\sum_{j=1}^{n} |\bar{y}_j(t_0)|\right\} e^{-\alpha(t - t_0)}, \quad i = 1, 2, \dots, n. \tag{46}$$

Hence, for $i = 1, 2, \dots, n$,

$$\begin{aligned} |x_i(t)| &\le \bar{\gamma}_i \cdot \left\{\sum_{j=1}^{n} |\bar{x}_j(t_0)|\right\} e^{-(\alpha+\lambda)(t - t_0)} \\ &\le \bar{\gamma}_i \cdot \left\{\sum_{j=1}^{n} |\bar{x}_j(t_0)|\right\} e^{-\lambda(t - t_0)}, \quad t \in [t_{k-1}, t_k), \quad k \in N, \end{aligned} \tag{47}$$

here $\bar{\gamma}_i = \gamma_i e^{\lambda t_0}$.

The rest of the proof is the same as that in Theorem 3 and is thus omitted. □

Remark 3. It can be seen that the matrix A in Theorems 1–4 has to be stable. In the following, we shall relax this condition on A. It turns out that the time delay may help to stabilize the system.

Clearly,

$$\widetilde{A} + \widetilde{B} \in N\left[A^{(1)} + B^{(1)}, A^{(2)} + B^{(2)}\right].$$

Hence, by Lemma 1, we have

$$\widetilde{A} + \widetilde{B} = (A+B) + E_{A+B}\Sigma_{A+B}F_{A+B}. \tag{48}$$

Theorem 5. *Assume that conditions* (ii) *and* (iii) *of Theorem* 1 *hold. Suppose further that*

(i*)

$$0 < b(\mu_2, \mu_3; P) < -a(\mu_1, \mu_2, \mu_3; P), \tag{49}$$

where for some positive definite matrix P and constants $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$,

$$\begin{aligned}
a(\mu_1, \mu_2, \mu_3; P) &= -\lambda_{\max}\{P^{-1} \cdot \widetilde{Z}(\mu_1, \mu_2, \mu_3; P)\},\\
b(\mu_2, \mu_3; P) &= \frac{\widetilde{Y}(\mu_2, \mu_3)}{\lambda_{\min}(P)},\\
\widetilde{Z}(\mu_1, \mu_2, \mu_3; P) &= (A+B)^T P + P(A+B) + \mu_1 P E_{A+B}E_{A+B}^T P\\
&\quad + \mu_2 P E_B E_B^T P + \mu_3 P B B^T P + \mu_1^{-1} F_{A+B}^T F_{A+B},\\
\widetilde{Y}(\mu_2, \mu_3) &= (\mu_3^{-1} + \mu_2^{-1}\|F_B\|^2)\\
&\quad \times \min\Big\{2, 3\Big[\tau^2\big(\|A\| + \|B\| + \|E_A\|\|F_A\| + \|E_B\|\|F_B\|\big)^2\\
&\qquad + \big(\|C_k\| + \|D_k\| + \|E_{C_k}\|\|F_{C_k}\| + \|E_{D_k}\|\|F_{D_k}\|\big)^2\Big]\Big\}.
\end{aligned}$$

Then, system (1) *is robustly globally exponentially stable with decay rate* $\{\lambda - (\ln M/\delta\tau)\}/2$.

Proof. Let $V(x) = x^T Px$. Then, for $t \in [t_k, t_{k+1})$, $k \in N$,

$$\begin{aligned} D^+V(x(t)) &= 2x^T P\big[\widetilde{A}x(t) + \widetilde{B}x(t-h(t))\big] \\ &= 2x^T P\big[\big(\widetilde{A}+\widetilde{B}\big)x(t) + \widetilde{B}(x(t-h(t)) - x(t))\big] \\ &= x^T\big[P(\widetilde{A}+\widetilde{B}) + (\widetilde{A}+\widetilde{B})^T P\big]x \\ &\quad + 2x^T P\widetilde{B}(x(t-h(t)) - x(t)). \end{aligned} \tag{50}$$

By Lemma 2 and (48), we have

$$\begin{aligned} &2x^T P\big(P(\widetilde{A}+\widetilde{B}) + (\widetilde{A}+\widetilde{B})^T P\big)x \\ &\quad = x^T\big(P(A+B)+(A+B)^T P\big)x + 2x^T PE_{A+B}\Sigma_{A+B}F_{A+B}x \\ &\quad \le x^T\big(P(A+B) + (A+B)^T P + \mu_1 PE_{A+B}E_{A+B}^T P \\ &\qquad + \mu_1^{-1}F_{A+B}^T F_{A+B}\big)x, \end{aligned} \tag{51}$$

and

$$\begin{aligned} &2x^T P\widetilde{B}(x(t-h(t)) - x(t)) \\ &\quad = 2x^T PB(x(t-h(t))-x(t))+2x^T PE_B\Sigma_B F_B(x(t-h(t))-x(t)) \\ &\quad \le x^T\big(\mu_3 PBB^T P + \mu_2 PE_B E_B^T P\big)x + (\mu_3^{-1} + \mu_2^{-1}\|F_B\|^2) \\ &\qquad \cdot \|x(t-h(t)) - x(t)\|^2. \end{aligned} \tag{52}$$

In order to estimate $\|x(t-h(t)) - x(t)\|^2$, we consider system (1) as the initial data defined on $[-\tau, \tau]$. By the Hölder inequality, for $t \ge \tau$, we obtain

Case a). $t_k \le t - \tau \le t_{k+1}$ for some $k \in N$:

$$\begin{aligned} \|x(t-h(t))-x(t)\|^2 &\le \left\| \int_{t-h(t)}^{t} \dot{x}(s)\,ds \right\|^2 \\ &= \left\| \int_{t-h(t)}^{t} \big(\widetilde{A}x(s) + \widetilde{B}x(s-h(s))\big)\,ds \right\|^2 \\ &\le \tau \int_{-\tau}^{0} \big\|\widetilde{A}x(s+t) + \widetilde{B}x(s+t-h(s))\big\|^2\,ds \\ &\le \big(\|A\| + \|B\| + \|E_A\|\|F_A\| + \|E_B\|\|F_B\|\big)^2\tau^2 \\ &\quad \times \max_{-2\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\}. \end{aligned} \tag{53}$$

Case b). $t-\tau < t_k \le t$:

$$\begin{aligned}
\|x(t-h(t))-x(t)\|^2 &= \left\| \int_{t-h(t)}^{t_k^-} \dot{x}(s)\,ds + \Delta x(t_k) + \int_{t_k}^{t} \dot{x}(s)\,ds \right\|^2 \\
&\le 3\left\{ \left\| \int_{t-h(t)}^{t_k^-} \dot{x}(s)\,ds \right\|^2 + \|\Delta x(t_k)\|^2 + \left\| \int_{t_k}^{t} \dot{x}(s)ds \right\|^2 \right\} \\
&\le 3\Big\{ \tau^2\big(\|A\|+\|B\|+\|E_A\|\|F_A\|+\|E_B\|\|F_B\|\big)^2 \\
&\qquad + \big(\|C_k\|+\|D_k\|+\|E_{C_k}\|\|F_{C_k}\|+\|E_{D_k}\|\|F_{D_k}\|\big)^2 \Big\} \\
&\quad \times \max_{-2\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\}.
\end{aligned} \tag{54}$$

On the other hand, we see that

$$\begin{aligned}
\|x(t-h(t))-x(t)\|^2 &\le 2 \max_{-\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\} \\
&\le 2 \max_{-2\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\}.
\end{aligned} \tag{55}$$

Hence, by (53)–(55), we have

$$\begin{aligned}
&\|x(t-\tau)-x(t)\|^2 \\
&\le \min\Big\{2, 3\Big[\tau^2\big(\|A\|+\|B\|+\|E_A\|\|F_A\|+\|E_B\|\|F_B\|\big)^2 \\
&\qquad + \big(\|C_k\|+\|D_k\|+\|E_{C_k}\|\|F_{C_k}\|+\|E_{D_k}\|\|F_{D_k}\|\big)^2\Big]\Big\} \\
&\quad \times \max_{-2\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\}.
\end{aligned} \tag{56}$$

Substituting (56) into (52), and (51)–(52) into (50), yields

$$\begin{aligned}
&D^+V(x(t)) \\
&\le x^T\widetilde{Z}(\mu_1,\mu_2,\mu_3;P)x + \widetilde{Y}(\mu_2,\mu_3) \max_{-2\tau\le\theta\le 0}\{\|x(t+\theta)\|^2\} \\
&\le \lambda_{\max}\big\{P^{-1}\widetilde{Z}(\mu_1,\mu_2,\mu_3;P)\big\}\cdot V(x(t)) + \frac{\widetilde{Y}(\mu_2,\mu_3)}{\lambda_{\min}(P)} \\
&\quad \times \max_{-2\tau\le\theta\le 0}\{V(x(t+\theta))\} \\
&= -a(\mu_1,\mu_2,\mu_3;P)V(x(t)) + b(\mu_2,\mu_3;P) \max_{-2\tau\le\theta\le 0}\{V(x(t+\theta))\}.
\end{aligned} \tag{57}$$

The rest of the proof is similar to that used in the proof of Theorem 1 and thus is omitted. The proof is thus complete. $\square$

4. Examples. To illustrate the results obtained in Section 3, we shall discuss three examples.

Example 1. Consider system (1) in the form of system (3), where we let

$$A^{(1)} = \begin{pmatrix} -4.1 & 7.8 \\ -16 & -4.2 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -3.9 & 8.2 \\ -16 & -3.8 \end{pmatrix};$$
$$B^{(1)} = \begin{pmatrix} 0.9 & -1.2 \\ 0.9 & 1 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 1.1 & -0.8 \\ 1.1 & 1 \end{pmatrix};$$
$$C_k^{(1)} = C_k^{(2)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix};$$
$$D_k^{(1)} = D_k^{(2)} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$
$$k \in N.$$

Let $P = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu_1 = \mu_2 = \mu_3 = 1, \mu_4 = 1/2, \mu_5 = \cdots = \mu_9 = 1$, then $Z(\mu_1, \mu_2, \mu_3; P) = \begin{pmatrix} -13.5 & -4 \\ -4 & -7.3 \end{pmatrix}$, $Y(\mu_2, \mu_3) = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.4 \end{pmatrix}$; $a(\mu_1, \mu_2, \mu_3; P) = -4.18$, $b(\mu_2, \mu_3; P) = 1.4$; $W_k(\mu_4, \dots, \mu_9; P) = \begin{pmatrix} 5/2 & 0 \\ 0 & 9/8 \end{pmatrix}$, $X_k(\mu_4, \dots, \mu_9; P) = \begin{pmatrix} 5/2 & 0 \\ 0 & 9/4 \end{pmatrix}$; $\alpha_k(\mu_4, \dots, \mu_9; P) = (5/4)$, $\beta_k(\mu_4, \dots, \mu_9; P) = 9/4$.

Let $\tau = 1$, then by equation $\lambda + a(\mu_1, \mu_2, \mu_3; P) + b(\mu_2, \mu_3; P)e^{\lambda\tau} = 0$, we get $\lambda = 0.8643$ and $M = \alpha_k(\mu_4, \dots, \mu_9; P) + \beta_k(\mu_4, \dots, \mu_9; P)e^{\lambda\tau} = 6.6$. Hence, if $\sigma = \inf_{k\in N}\{t_k - t_{k-1}\} > (\ln M/\lambda) = 2.2$, then by Theorem 1 we conclude that the system is robustly globally exponentially stable with decay rate 0.0033.

Example 2. Consider system (3) with

$$A^{(1)} = \begin{pmatrix} -13.1 & 7.8 \\ -16 & -14.2 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -12.4 & 8.2 \\ -16 & -13.8 \end{pmatrix};$$
$$B^{(1)} = \begin{pmatrix} 0.9 & -1.2 \\ 0.9 & 1 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 1.1 & -0.8 \\ 1.1 & 1 \end{pmatrix};$$

$$C_k^{(1)} = \begin{pmatrix} -2-(1/k) & 0 \\ 0 & -2-(2/k) \end{pmatrix},$$

$$C_k^{(2)} = \begin{pmatrix} -2+(1/k) & 0 \\ 0 & -2+(2/k) \end{pmatrix};$$

$$D_k^{(1)} = \begin{pmatrix} 0.5-(1/k^2) & 0 \\ 0 & 0.5-(1/k) \end{pmatrix},$$

$$D_k^{(2)} = \begin{pmatrix} 0.5+(1/k^2) & 0 \\ 0 & 0.5+(1/k) \end{pmatrix},$$

$$k \in N.$$

Then, $-H = (h_{ij})_{n\times n} = \left(\begin{smallmatrix} 12.4 & -9.4 \\ -17.1 & 13.8 \end{smallmatrix}\right)$, which is an M-matrix, $\alpha_k = \max_{1\le j\le n}\left\{\sum_{i=1}^n \hat{c}_{ij_k}\right\} = 1+(2/k)$ and $\beta_k = \max_{1\le j\le n}\left\{\sum_{i=1}^n \bar{d}_{ij_k}\right\} = 0.5+(1/k)$.

Let $\tau = 1$. Since $-0.4\cdot I - H$ is still an M-matrix, we let $M = 5.24 \ge \alpha_k + \beta_k e^{0.4\tau}$, $k \in N$. Hence, if $\sigma = \inf_{k\in N}\{t_k - t_{k-1}\} > (\ln M/0.4) = 4.14$, then, by Theorem 4, we conclude that the system is robustly globally exponentially stable and at least with decay rate $\lambda = 0.4$.

Example 3. Consider system (3) with

$$A^{(1)} = A^{(2)} = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.5 \end{pmatrix}; \quad B^{(1)} = B^{(2)} = \begin{pmatrix} -1.5 & 0 \\ 1 & -0.5 \end{pmatrix};$$

$$C_k^{(1)} = C_k^{(2)} = \begin{pmatrix} 1/((k+1)^2+1) & 0 \\ 0 & 1/((k+1)^2+1) \end{pmatrix};$$

$$D_k^{(1)} = D_k^{(2)} = 0,$$

$$k \in N.$$

Obviously, $A = A^{(1)} = A^{(2)}$ is not a stable matrix. Hence, Theorems 1–4 are not applicable. Yet, by Matlab, with $\tau = 0.1$, $P = I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 1/2$, $\mu_4 = \cdots = \mu_9 = 1$, we get $a(\mu_1,\mu_2,\mu_3;P) = -0.7714$, $b(\mu_2,\mu_3;P) = 0.4800$, $\alpha_k(\mu_4,\dots,\mu_9;P) = 1.2$, and $\beta_k(\mu_4,\dots,\mu_9;P) = 1$.

By solving equation $\lambda + a(\mu_1,\mu_2,\mu_3;P) + b(\mu_2,\mu_3;P)e^{\lambda\tau} = 0$, we get $\lambda = 0.2779$. Hence, if $\delta = 30$, $M = 2.2282 \ge \alpha_k(\mu_4,\dots,\mu_9;P) + \beta_k(\mu_4,\dots,\mu_9;P)e^{\lambda\tau}$ and $\sigma = \inf_{k\in N}\{t_k - t_{k-1}\} > \delta\tau = 3.0$. Thus, by

Theorem 5, this system is robustly globally exponentially stable with decay rate 0.0054.

5. Conclusions. In this paper, by employing the method of Lyapunov functions, matrix inequalities, and the theory of M-matrix, we have established some global exponential stability criteria for linear impulsive system with time-varying delay and uncertainty. We have also estimated decay rates. Those criteria may be verified by solving matrix inequalities by Matlab or by checking the conditions for an M-matrix. Some examples have been worked out to demonstrate the main results.

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