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GENERALIZED RADON TRANSFORM

R. ROOPKUMAR

ABSTRACT. We extend the Radon transform to the context of Boehmians consistent with the Radon transform on the space of tempered distributions and on the space of distributions by constructing suitable Boehmian spaces. We also prove that the generalized Radon transforms are continuous, linear, bijections and their inverses are also continuous.

1. Introduction. Starting from the work [**13**] of Radon, the theory of Radon transform has been developed on various testing function spaces and distribution spaces, and their properties have been discussed. See [**1**–**6**]. Let $\mathcal{S}(\mathbf{R}^N)$ denote the space of smooth rapidly decreasing functions on $\mathbf{R}^N$, $N > 1$, and $\mathcal{D}(\mathbf{R}^N)$ denote the space of smooth functions with compact supports. Their dual spaces $\mathcal{S}'(\mathbf{R}^N)$, $\mathcal{D}'(\mathbf{R}^N)$ are called the space of tempered distributions and the space of distributions respectively. We also denote by $\mathcal{I}(\mathbf{R}^N)$, the space of all slowly increasing continuous functions on $\mathbf{R}^N$ with the following notion of convergence: $\eta_n \to \eta$ as $n \to \infty$ if there exists $m \in \mathbf{N}$ and a sequence (C_n) in $\mathbf{R}$ converging to zero, such that $|\eta_n(x) - \eta(x)| \leq C_n(1 + \|x\|^m)$, for all $x \in \mathbf{R}^N$, where $\|x\|$ is the Euclidean norm of $x \in \mathbf{R}^N$.

The space $R\mathcal{S}$ consists of all functions $f\colon \mathbf{R} \times S^{N-1} \to \mathbf{C}$ satisfying the following conditions:

$$\alpha : \|f\|_m = \sup_{\substack{(s,w)\in\mathbf{R}\times S^{N-1}\\ 0\leq k\leq m}} (1+s^2)^m \left|\frac{\partial^k}{\partial s^k}\, f(s,w)\right| < \infty.$$

$\beta : f(s,w) = f(-s,-w)$ for all $s \in \mathbf{R}$ and $w \in S^{N-1}$.

$\gamma : \displaystyle\int_{-\infty}^{\infty} f(s,w)s^k\, ds$ is a polynomial of degree $\leq k$ in w, $\forall\, k \in N_0$.

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Let K be an operator defined, see (1.10) in [**6**], on $\mathcal{S}(\mathbf{R} \times S^{N-1})$ by

$$(1.1)\quad (Kf)(s,w) = \begin{cases} \dfrac{1}{2(2\pi)^{N-1}} \left(\dfrac{\partial}{i\partial s}\right)^{N-1} f(s,w) & \text{if } N \text{ is odd,} \\ \dfrac{1}{2(2\pi)^{N-1}}\, i\,\mathcal{H} \left(\dfrac{\partial}{i\partial s}\right)^{N-1} f(s,w) & \text{if } N \text{ is even} \end{cases}$$

where $\mathcal{H}g(t,w) = 1/\pi \ P\int_{-\infty}^{\infty} (g(s,w)/t-s)\, ds$, the Hilbert transform of $g(\ \cdot\ ,w)$. Let $KR\mathcal{S}$ be the image of $R\mathcal{S}$ under the operator K. Clearly, $KR\mathcal{S} \subset \mathcal{E}(R \times S^{N-1})$, the space of all smooth functions on $R \times S^{N-1}$. The space $KR\mathcal{S}$ is given a suitable topology which makes K as a continuous map from $R\mathcal{S}$ into $KR\mathcal{S}$.

In the literature, the Radon transform is introduced on the space of integrable Boehmians, and rapidly decreasing Boehmians by Mikusiński and Zayed [**11**]. Though this theory is generalizing the Radon transform on $\mathcal{L}^1(\mathbf{R}^N)$ and on $\mathcal{S}(\mathbf{R}^N)$, it is not generalizing the Radon transform on distribution spaces $\mathcal{D}'(\mathbf{R}^N)$, $\mathcal{S}'(\mathbf{R}^N)$. Our objective of the paper is to find some Boehmian spaces which are larger than $\mathcal{S}'(\mathbf{R}^N)$, $\mathcal{D}'(\mathbf{R}^N)$, rapidly decreasing Boehmians, integrable Boehmians and to extend the Radon transform as a continuous, linear, bijection with its inverse also continuous.

We know that the space $\mathcal{B}(\mathcal{I}(\mathbf{R}^N), (\mathcal{D}(\mathbf{R}^N), *), *, \Delta)$ of tempered Boehmian space is larger than $\mathcal{S}'(\mathbf{R}^N)$, rapidly decreasing Boehmians and the space $\mathcal{B}(C^\infty(\mathbf{R}^N)$, $(\mathcal{D}(\mathbf{R}^N), *), *, \Delta)$ of C^∞-Boehmians contains $\mathcal{D}'(\mathbf{R}^N)$, integrable Boehmians. Since the classical Radon transform is not applicable on $\mathcal{I}(\mathbf{R}^N)$ or $C^\infty(\mathbf{R}^N)$, we are forced to use the distributional Radon transform and hence the range Boehmians are constructed with distribution spaces. We also alter the space of tempered Boehmians and C^∞-Boehmians by using $\mathcal{S}'(\mathbf{R}^N)$ and $\mathcal{D}'(\mathbf{R}^N)$ instead of $\mathcal{I}(\mathbf{R}^N)$ and $C^\infty(\mathbf{R}^N)$ respectively, since the notions of convergence on these function spaces are not suitable to prove the continuity of the inverse Radon transform. This alteration may enlarge the collection of representatives of each Boehmian and does not change the spaces of Boehmians as vector spaces.

2. Preliminaries.

Definition 2.1. The Radon transform R on $\mathcal{S}(\mathbf{R}^N)$ is defined by

$$(R\phi)(s,w) = \int_{x\cdot w=s} \phi(x)\,dm(x) \tag{2.1}$$

where dm is the Euclidean measure on the hyperplane $x \cdot w = s$.

The Radon transform is a continuous, one-to-one map from $\mathcal{S}(\mathbf{R}^N)$ into the space $\mathcal{S}(\mathbf{R} \times S^{N-1})$ of all rapidly decreasing functions on $R \times S^{N-1}$ whose range $R\mathcal{S}$ is characterized in [**6**, Theorem 2.1].

Definition 2.2. The adjoint Radon transform $R^t\colon \mathcal{E}(\mathbf{R} \times S^{N-1}) \to \mathcal{E}(\mathbf{R}^N)$ is defined by

$$(R^t f)(x) = \int_{S^{N-1}} \phi(x \cdot w, w)\,dw. \tag{2.2}$$

It is proved that $R^t\colon \mathcal{E}(\mathbf{R}\times S^{N-1}) \to \mathcal{E}(\mathbf{R}^N)$ is a bicontinuous, bilinear map [**3**].

Definition 2.3. The Radon transform of a tempered distribution Λ is defined as

$$\langle R\Lambda, f\rangle = \Lambda(R^t f), \quad f \in KR\mathcal{S}. \tag{2.3}$$

The Radon transform $R\colon \mathcal{S}'(\mathbf{R}^N) \to (KR\mathcal{S})'$ is a linear homeomorphism. See [**6**].

Given a continuous function f on $R \times S^{N-1}$, and $\phi \in \mathcal{D}(\mathbf{R}^N)$, define

$$(f\times\phi)(s,w) = \int_{\mathbf{R}^n} f((s-y\cdot w), w)\phi(y)\,dy, \quad (s,w) \in \mathbf{R}\times S^{N-1}. \tag{2.4}$$

Lemma 2.5. *If $f \in R\mathcal{S}$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then $f \times \phi \in R\mathcal{S}$.*

Proof. First we show that

$$\frac{\partial}{\partial s}(f \times \phi)(s,w) = \left(\frac{\partial f}{\partial s} \times \phi\right)(s,w). \tag{2.5}$$

Consider

$$\lim_{t\to s} \int_{\mathbf{R}^N} \frac{f((t-y\cdot w),w) - f((s-y\cdot w),w)}{t-s}\,\phi(y)\,dy,$$

since the integrand is dominated by $\|f\|_1|\phi(y)|$ for all t, and $\phi \in \mathcal{D}(\mathbf{R}^N)$, using dominated convergence theorem we can take limit into the integral and hence (2.5) follows. To prove $\|f\times\phi\|_m < \infty$, consider

$$\begin{aligned}
(1+s^2)^m&\left|\left(\frac{\partial}{\partial s} f\times\phi\right)(s,w)\right| \\
&\le (1+s^2)^m \int_{\mathbf{R}^N} \left|\frac{\partial}{\partial s} f((s-y\cdot w),w)\phi(y)\right| dy \\
&\le \int_{\mathbf{R}^N} (1+(|s-y\cdot w|+|y\cdot w|)^2)^m \\
&\qquad\qquad\qquad \times \left|\frac{\partial}{\partial s} f((s-y\cdot w),w)\phi(y)\right| dy \\
&\le \int_{\mathbf{R}^N} (1+(|s-y\cdot w|+C)^2)^m \\
&\qquad \times \left|\frac{\partial}{\partial s} f((s-y\cdot w),w)\phi(y)\right| dy \quad (\text{for some } C>0) \\
&\le C'\|f\|_m \int_{\mathbf{R}^N} |\phi(y)|\,dy \quad (\text{for some } C'>0).
\end{aligned}$$

Since $f \in R\mathcal{S}$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, it follows that $\|f\times\phi\|_m < \infty$.

Let $(s,w) \in \mathbf{R}\times S^{N-1}$ be arbitrary. Then

$$\begin{aligned}
(f\times\phi)(-s,-w) &= \int_{\mathbf{R}^N} f((-s+y\cdot w),-w)\phi(y)\,dy \\
&= \int_{\mathbf{R}^N} f((s-y\cdot w),w)\phi(y)\,dy \\
&= (f\times\phi)(s,w).
\end{aligned}$$

For $k \in \mathbf{N}_0$ and $(s,w) \in \mathbf{R} \times S^{N-1}$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} &(f \times \phi)(s,w) s^k \, ds \\
&= \int_{-\infty}^{\infty} \int_{\mathbf{R}^N} f((s - y \cdot w), w) \phi(y) \, dy \, s^k \, ds \\
&= \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} f((s - y \cdot w), w) s^k \, ds \, \phi(y) \, dy \\
&= \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} f((s, w), w)(s + y \cdot w)^k \, ds \, \phi(y) \, dy \\
&= \sum_{j=0}^{k} \binom{k}{j} \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} f((s,w),w) s^j \, ds \, (y \cdot w)^{k-j} \phi(y) \, dy
\end{aligned}$$

which is a polynomial of degree $\leq k$ since each term in the above sum is a polynomial of degree at most k. $\square$

Lemma 2.6. *If $f \in R\mathcal{S}$, and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then for each $(s,w) \in \mathbf{R} \times S^{N-1}$, $\mathcal{H}(f \times \phi)(s,w) = (\mathcal{H} f \times \phi)(s,w)$.*

Proof. Let $z \in \mathbf{C} \setminus \mathbf{R}$. Using Fubini's theorem we get

$$\begin{aligned}
\mathcal{H}(f \times \phi)(z,w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(f \times \phi)(s,w)}{z - s} \, ds \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{\mathbf{R}^N} \frac{f(s - y \cdot w, w)}{z - s} \, \phi(y) \, dy \, ds \\
&= \frac{1}{\pi} \int_{\mathbf{R}^N} \int_{\infty}^{\infty} \frac{f(s - y \cdot w, w)}{z - s} \, \phi(y) \, ds \, dy \\
&= \frac{1}{\pi} \int_{\mathbf{R}^N} \int_{-\infty}^{\infty} \frac{f(s,w)}{(z - y \cdot w) - s} \, \phi(y) \, ds \, dy \\
&= (\mathcal{H} f \times \phi)(z,w).
\end{aligned}$$

Let $s \in \mathbf{R}$, $t > 0$ and $w \in S^{N-1}$ be arbitrarily fixed. By the classical theory of Hilbert transform [**12**, pp. 170–171], we have the limits $\lim_{t \to 0^{\pm}} \mathcal{H}(f \times \phi)((s + it), w) = F^{\pm}(s,w)$ exist in $\mathcal{L}^p(\mathbf{R}^1)$, for any $p > 1$, and $\mathcal{H}(f \times \phi)(s,w) = (F^{+} - F^{-})(s,w)$. If $G^{\pm}(s,w) =$

$\lim_{t\to 0^{\pm}} \mathcal{H} f((s+it),w)$, then we claim that

$$\lim_{t\to 0^{\pm}} (\mathcal{H} f \times \phi)((s+it),w) = (G^{\pm} \times \phi)(s,w).$$

Using Jensen's inequality and Fubini's theorem, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \int_{\mathbf{R}^N} (\mathcal{H} f((s+it-y\cdot w),w) - G^{+}((s-y\cdot w),w)\phi(y)\,dy \right|^p ds \\
&\le \int_{-\infty}^{\infty} \int_{\mathbf{R}^N} |\mathcal{H} f((s+it-y\cdot w),w) - G^{+}((s-y\cdot w),w)|^p |\phi(y)|\,dy\,ds \\
&= \int_{\mathbf{R}^N} |\phi(y)|\,dy \int_{-\infty}^{\infty} |\mathcal{H} f((s+it-y\cdot w),w) - G^{+}((s-y\cdot w),w)|^p\,ds \\
&= \int_{\mathbf{R}^N} |\phi(y)|\,dy \int_{-\infty}^{\infty} |\mathcal{H} f((s+it),w) - G^{+}(s,w)|^p\,ds.
\end{aligned}$$

Therefore we get $\lim_{t\to 0^{+}} (\mathcal{H} f \times \phi)((s+it),w) = (G^{+} \times \phi)(s,w)$, and hence $F^{+}(s,w) = \lim_{t\to 0^{+}} \mathcal{H}(f\times\phi)(s+it) = \lim_{t\to 0^{+}} (\mathcal{H} f\times\phi)(s+it) = (G^{+}\times\phi)(s,w)$. Similarly we can show that $F^{-}(s,w) = (G^{-}\times\phi)(s,w)$. Combining these observations we conclude that $\mathcal{H}(f\times\phi)(s,w) = (F^{+} - F^{-})(s,w) = ((G^{+}\times\phi) - (G^{-}\times\phi))(s,w) = ((G^{+}-G^{-})\times\phi)(s,w) = (\mathcal{H} f\times\phi)(s,w)$. $\square$

Lemma 2.7. *If $f \in \mathcal{S}(\mathbf{R}\times S^{N-1})$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$ then $K(f\times\phi) = Kf\times\phi$.*

Proof. When N is odd, the lemma follows from equation (2.5). For the other case, equation (2.5) and Lemma 2.6 prove the lemma. $\square$

Lemma 2.8. *If $f \in KR\mathcal{S}$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then $f\times\phi \in KR\mathcal{S}$.*

The proof of the lemma follows from Lemma 2.5 and Lemma 2.7.

Lemma 2.9. *If $f \in \mathcal{E}(\mathbf{R}\times S^{N-1})$ and $\phi,\psi \in \mathcal{D}(\mathbf{R}^N)$ then $f\times(\phi*\psi) = (f\times\phi)\times\psi$.*

Proof. Let $s \in \mathbf{R}^N$ and $w \in S^{N-1}$ be arbitrary. Using Fubini's theorem, we get that

$$\begin{aligned}
(f \times (\phi * \psi))(x \cdot w, w) &= \int_{\mathbf{R}^N} f((x-y)\cdot w, w)(\phi * \psi)(y)\, dy \\
&= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \phi(y-z)\psi(z)\, dz f((x-y)\cdot w, w)\, dy \\
&= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} f((x-y)\cdot w, w)\phi(y-z)\, dy \psi(z)\, dz \\
&= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} f((x-z-y)\cdot w, w)\phi(y)\, dy \psi(z)\, dz \\
&= \int_{\mathbf{R}^N} (f \times \phi)((x-z)\cdot w, w)\psi(z)\, dz \\
&= ((f \times \phi) \times \psi)(x \cdot w, w).
\end{aligned}$$

Hence the lemma. $\square$

Lemma 2.10. *If $f \in \mathcal{E}(\mathbf{R}^N \times S^{-1})$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then $R^t(f \times \phi) = R^t f * \phi$.*

Proof. The equality

$$\int_{S^{N-1}} \int_{\mathbf{R}^N} f((x-y)\cdot w, w)\phi(y)\, dy\, dw = \int_{\mathbf{R}^N} \int_{S^{N-1}} f((x-y)\cdot w, w)\, dw \phi(y)\, dy$$

proves the lemma. $\square$

Definition 2.11. For $f \in \mathcal{E}(\mathbf{R} \times S^{N-1})$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$ define $\breve{f}(s,w) = f(-s,w)$, $(s,w) \in \mathbf{R} \times S^{N-1}$, $\check{\phi}(x) = \phi(-x)$, for all $x \in \mathbf{R}^N$.

Lemma 2.12. *Let $f \in \mathcal{E}(\mathbf{R} \times S^{N-1})$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$. Then*

(i) $(f \times \phi)\breve{} = \breve{f} \times \check{\phi}$;

(ii) $R^t \breve{f} = (R^t f)\breve{}$.

Proof. Let $(s,w) \in \mathbf{R} \times S^{N-1}$. Now

$$\begin{aligned}(\check{f} \times \check{\phi})(x \cdot w, w) &= \int_{\mathbf{R}^N} \check{f}((x-y)\cdot w, w)\check{\phi}(y)\, dy \\ &= \int_{\mathbf{R}^N} f((-x+y)\cdot w, w)\phi(-y)\, dy \\ &= \int_{\mathbf{R}^N} f((-x-y)\cdot w, w)\phi(y)\, dy \\ &= (f \times \phi)(-x \cdot w, w) \\ &= (f \times \phi)\check{}(x \cdot w, w)\end{aligned}$$

Hence (i) holds.

We prove (ii) by

$$\begin{aligned}R^t(\check{f})(x) &= \int_{S^{N-1}} \check{f}(x \cdot w, w)\, dw \\ &= \int_{S^{N-1}} f(-(x \cdot w), w)\, dw \\ &= \int_{S^{N-1}} f((-x)\cdot w, w)\, dw \\ &= (R^t f)(-x) = (R^t f)\check{}(x). \qquad \square\end{aligned}$$

Definition 2.13. For $u \in (KR\mathcal{S})'$, and $\phi \in \mathcal{D}(\mathbf{R}^N)$, define

$$\langle (u \otimes \phi), f\rangle = \langle u, (f \times \check{\phi})\rangle, \quad f \in KR\mathcal{S}, \tag{2.6}$$

where $\check{\phi}(x) = \phi(-x)$, for all $x \in \mathbf{R}^N$.

By Lemma 2.8, we see that the left-hand side of the above equation is meaningful. The linearity of $u \otimes \phi$ is obvious. The continuity of $u \otimes \phi$ follows by the fact that $f_n \times \phi \to f \times \phi$ as $n \to \infty$ in $R\mathcal{S}$ whenever $f_n \to f$ as $n \to \infty$ in $R\mathcal{S}$, K is continuous and by Lemma 2.7. Thus we can see that $u \otimes \phi \in (KR\mathcal{S})'$.

Lemma 2.14. *If* $u, v \in (KR\mathcal{S})'$, $\phi, \psi \in \mathcal{D}(\mathbf{R}^N)$ *and* $\alpha \in \mathbf{C}$, *then*

(i) $(u+v) \otimes \phi = (u \otimes \phi) + (v \otimes \phi)$.

(ii) $(\alpha u) \otimes \phi = \alpha(u \otimes \phi)$.

(iii) $u \otimes (\phi * \psi) = (u \otimes \phi) \otimes \phi$.

Proof. (i) and (ii) follow directly from the definition. To prove (iii), let $f \in (KR\mathcal{S})$. Now $(u\otimes(\phi*\psi))(f) = u(f\times(\psi*\phi)\check{}) = u((f\times\check{\psi})\times\check{\phi}) = (u\otimes\phi)(f\times\check{\psi}) = ((u\otimes\phi)\otimes\psi)(f)$. Here we have used the fact that the convolution $*$ is commutative on $\mathcal{D}(\mathbf{R}^N)\times\mathcal{D}(\mathbf{R}^N)$ and Lemma 2.9. □

Lemma 2.15. *If* $\Lambda \in \mathcal{S}'(\mathbf{R}^N)$*and* $\phi \in \mathcal{D}(\mathbf{R}^N)$, *then* $R(\Lambda * \phi) = R\Lambda \otimes \phi$ *in* $(KR\mathcal{S})'$.

Proof. Let $f \in KR\mathcal{S}(\mathbf{R}^N)$ be arbitrary. Using Lemma 2.10 we get,

$$\begin{aligned}\langle R(\Lambda*\phi), f\rangle = (\Lambda*\phi)(R^t f) \qquad &= (\Lambda*\phi*(R^t f)\check{})(0)\\ f = (\Lambda*(R^t f*\check{\phi})\check{})(0) \quad &= \Lambda(R^t f*\check{\phi})\\ = \Lambda(R^t(f\times\check{\phi})) \qquad &= \langle R\Lambda, (f\times\check{\phi})\rangle\\ = \langle (R\Lambda\otimes\phi), f\rangle \qquad & \end{aligned}$$

Thus the lemma follows. □

3. Boehmian spaces. A Boehmian space [**8**] is a special generalized function space consisting of convolution quotients with two notions of convergences. At this juncture it is helpful to recall the construction of an abstract Boehmian space.

To construct a Boehmian space we need a topological vector space G, a commutative semi-group $(S, *)$, an operation $\star : G\times S \to G$ satisfying the conditions:

For all $\alpha, \beta \in G$ and $\zeta, \eta \in S$, (i) $\alpha \star (\zeta * \eta) = (\alpha \star \zeta) \star \eta$, (ii) $(\alpha+\beta)\star\zeta = \alpha\star\zeta+\beta\star\zeta$; and a collection Δ of sequences from S satisfying

(a) If $(\lambda_n), (\eta_n) \in \Delta$ then $(\lambda_n * \eta_n) \in \Delta$.

(b) If $\alpha, \beta \in G$ and $(\lambda_n) \in \Delta$ such that $\alpha\star\lambda_n = \beta\star\lambda_n$, for all $n \in \mathbf{N}$ then $\alpha = \beta$.

Let $\mathcal{A}$ denote the collection of all pairs of sequences $((\alpha_n), (\lambda_n))$ where $\alpha_n \in G$, for all $n \in \mathbf{N}$ and $(\lambda_n) \in \Delta$ satisfying the property

$$\alpha_n \star \lambda_m = \alpha_m \star \lambda_n, \quad \forall\, m, n \in \mathbf{N}. \tag{3.1}$$

Each element of $\mathcal{A}$ is called a quotient and is denoted by α_n/λ_n. Define a relation $\sim$ on $\mathcal{A}$ by

$$\alpha_n/\lambda_n \sim \beta_n/\eta_n \quad \text{if} \quad \alpha_n \star \eta_m = \beta_m \star \lambda_n, \quad \forall\, m, n \in \mathbf{N}. \tag{3.2}$$

It is easy to verify that $\sim$ is an equivalence relation on $\mathcal{A}$ and hence it decomposes $\mathcal{A}$ into disjoint equivalence classes. Each equivalence class is called a Boehmian and is denoted by $[\alpha_n/\lambda_n]$. The collection of all Boehmians is denoted by $\mathcal{B}$, more explicitly $\mathcal{B}(G, (S, *), \star, \Delta)$.

$\mathcal{B}$ is a vector space with addition and scalar multiplication defined as follows.

- $[\alpha_n/\lambda_n] + [\beta_n/\eta_n] = [(\alpha_n \star \eta_n + \beta_n \star \lambda_n)/(\lambda_n \star \eta_n)]$.
- $a\,[\alpha_n/\lambda_n] = [(a\alpha_n)/\lambda_n]$.

The operation $\star$ can be extended to $\mathcal{B} \times S$ by the following definition.

Definition 3.1. If $x = [\alpha_n/\lambda_n] \in \mathcal{B}$ and $\zeta \in S$, then

$$x \star \zeta = [(\alpha_n \star \zeta)/\lambda_n]\,.$$

Each $\alpha \in G$ can be uniquely identified with a member of $\mathcal{B}$ by $\alpha \mapsto [(\alpha \star \lambda_n)/\lambda_n]$ where $(\lambda_n) \in \Delta$ is arbitrary. We say that a Boehmian x belongs to G if x represents a member of G in $\mathcal{B}$.

Now we recall the existing two notions of convergences [**8**] on $\mathcal{B}$.

Definition 3.2. A sequence (x_n) in $\mathcal{B}$ is said to δ-converge to x in $\mathcal{B}$, denoted by $x_n \xrightarrow{\delta} x$ as $n \to \infty$ if there exists a delta sequence (λ_k) such that for each $n, k \in \mathbf{N}$, $x_n \star \lambda_k, x \star \lambda_k \in G$ and for each $k \in \mathbf{N}$, $x_n \star \lambda_k \to x \star \lambda_k$ as $n \to \infty$ in G.

Definition 3.3. $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ if there exist $\alpha_n \in G$ and $(\lambda_k) \in \Delta$ such that $(x_n - x) \star \lambda_n = [(\alpha_n \star \lambda_k)/\lambda_k]$ for all $n \in \mathbf{N}$ and $\alpha_n \to 0$ as $n \to \infty$ in G.

Theorem 3.4 [**8**]. *$x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}$ if and only if there exist $\alpha_{n,k}, \alpha_k \in G$ and $(\lambda_k) \in \Delta$ such that $x_n = [\alpha_{n,k}/\lambda_k]$, $x = [\alpha_k/\lambda_k]$ and for each $k \in \mathbf{N}$, $\alpha_{n,k} \to \alpha_k$ as $n \to \infty$ in G.*

We denote $\mathcal{B}_1$ by the Boehmian space $\mathcal{B}(\mathcal{S}'(\mathbf{R}^N), (\mathcal{D}(\mathbf{R}^N), *), *, \Delta)$ where $*$ is the convolution defined on $\mathcal{S}'(\mathbf{R}^N) \times \mathcal{D}(\mathbf{R}^N)$ by

$$(\Lambda * \phi)(\psi) = \Lambda(\psi * \check{\phi}), \quad \forall\, \psi \in \mathcal{S}(\mathbf{R}^N), \tag{3.3}$$

$$(\psi * \phi)(x) = \int_{\mathbf{R}^n} \psi(x-y)\phi(y)\, dy, \quad x \in \mathbf{R}^n, \tag{3.4}$$

and Δ is the collection of all sequences (ϕ_n) in $\mathcal{D}(\mathbf{R}^N)$ satisfying

(1) $\int_{\mathbf{R}^n} \phi_n(x)\, dx = 1$, for all $n \in \mathbf{N}$.

(2) $\int_{\mathbf{R}^n} |\phi_n(x)|\, dx \le M$, for all $n \in \mathbf{N}$, for some $M > 0$.

(3) $s(\phi_n) \to 0$ as $n \to \infty$ where $s(\phi_n) = \sup\{\|x\| : x \in \mathbf{R}^N, \phi_n(x) \neq 0\}$.

To construct the Boehmian space $\mathcal{B}_2$, we first prove the following lemmas.

Lemma 3.5. *If $f \in R\mathcal{S}$ and $(\phi_n) \in \Delta$, then $f \times \phi_n \to f$ as $n \to \infty$ in $R\mathcal{S}$.*

Proof. For $m \in \mathbf{N}_0$, $(s,w) \in \mathbf{R} \times S^{N-1}$ and $0 \le k \le m$, using the properties of delta sequence and mean-value theorem we get

$$\begin{aligned}
&(1+s^2)^m \left| \int_{\mathbf{R}^n} \frac{\partial^k}{\partial s^k} [f((s - y\cdot w), w) - f(s,w)]\phi_n(y)\, dy \right| \\
&\quad \le (1+s^2)^m \int_{\mathbf{R}^n} \left| \frac{\partial^k}{\partial s^k} [f((s - y\cdot w), w) - f(s,w)] \right| |\phi_n(y)|\, dy \\
&\quad \le (1+s^2)^m \int_{\mathbf{R}^n} |y\cdot w| \left| \frac{\partial^{k+1} f}{\partial s^{k+1}} ((s - h(y\cdot w)), w) \right| \\
&\qquad\qquad \times |\phi_n(y)|\, dy, \quad \text{for some} \quad h \in (0,1) \\
&\quad \le s(\phi_n) \int_{\mathbf{R}^n} (1 + (|s - h(y\cdot w)| + |h(y\cdot w)|)^2)^m \\
&\qquad\qquad \times \left| \frac{\partial^{k+1} f}{\partial s^{k+1}} ((s - h(y\cdot w)), w) \right| |\phi_n(y)|\, dy
\end{aligned}$$

$$\leq s(\phi_n)\int_{\mathbf{R}^n}(1+(|s-h(y\cdot w)|+s(\phi_n))^2)^m \times \left|\frac{\partial^{k+1}f}{\partial s^{k+1}}((s-h(y\cdot w)),w)\right| |\phi_n(y)|dy$$
$$\leq C\,M\|f\|_{m+1}s(\phi_n) \quad \text{for some} \quad C>0.$$

Thus we get $\|f\times\phi_n - f\|_m \leq C's(\phi_n)\to 0$ as $n\to\infty$. $\square$

Lemma 3.6. *If $u\in(KR\mathcal{S})'$, and $(\phi_n)\in\Delta$, then $u\otimes\phi_n\to u$ as $n\to\infty$ in $(KR\mathcal{S})'$.*

The proof of the lemma follows by using Lemmas 2.7, 3.5 and the fact that K is continuous on $R\mathcal{S}$. $\square$

Using Lemma 2.14 and Lemma 3.6, we can construct the Boehmian space $\mathcal{B}_2=\mathcal{B}((KR\mathcal{S})',\mathcal{D}(\mathbf{R}^N,*),\otimes,\Delta)$.

4. Radon transform.

Definition 4.1. Define the Radon transform $\mathcal{R}:\mathcal{B}_1\to\mathcal{B}_2$ by

$$\mathcal{R}\left[\frac{\Lambda_k}{\phi_k}\right]=\left[\frac{R\Lambda_k}{\phi_k}\right]. \tag{4.1}$$

Using Lemma 2.15, we can verify that the above definition is well defined.

Lemma 4.2. *The generalized Radon transform $\mathcal{R}:\mathcal{B}_1\to\mathcal{B}_2$ is consistent with the Radon transform on $R:\mathcal{S}'(\mathbf{R}^N)\to(KR\mathcal{S})'$.*

Proof. We know that each $\Lambda\in\mathcal{S}'(\mathbf{R}^N)$ is represented by $[\Lambda*\phi_k/\phi_k]$ in $\mathcal{B}_1$. Now $\mathcal{R}[\Lambda*\phi_k/\phi_k]=[R(\Lambda*\phi_k)/\phi_k]=[R\Lambda\otimes\phi_k/\phi_k]$ which is nothing but the identification of $R\Lambda$ in $\mathcal{B}_2$. This completes the proof of the Lemma. $\square$

Theorem 4.3. *The Radon transform $\mathcal{R}:\mathcal{B}_1\to\mathcal{B}_2$ is a linear map.*

The proof of this theorem can be obtained by using Lemma 2.15 and the linearity of $R: \mathcal{S}'(\mathbf{R}^N) \to (KR\mathcal{S})'$.

Theorem 4.4. *The Radon transform $\mathcal{R}: \mathcal{B}_1 \to \mathcal{B}_2$ is a bijection.*

Proof. Let $F = [\Lambda_k/\phi_k] \in \mathcal{B}_1$ be such that $\mathcal{R}F = 0$ in $\mathcal{B}_2$. Then $R\Lambda_k = 0$, $k \in \mathbf{N}$. By the injectivity of $R: \mathcal{S}'(\mathbf{R}^N) \to (K\mathcal{S}R)'$ we get $\Lambda_k = 0$, $k \in \mathbf{N}$, and hence $F = 0$ in $\mathcal{B}_1$. Thus $\mathcal{R}$ is injective.

Let $[u_k/\phi_k] \in \mathcal{B}_2$ be arbitrary. Since $R: \mathcal{S}'(\mathbf{R}^N) \to (K\mathcal{S}R)'$ is surjective for each $k \in \mathbf{N}$, there exists $\Lambda_k \in \mathcal{S}'(\mathbf{R}^N)$ such that $R\Lambda_k = u_k$. Using

$$(R\Lambda_k) \otimes \phi_j = (R\Lambda_j) \otimes \phi_k, \quad j, k \in \mathbf{N}$$

and Lemma 2.15, we get

$$R(\Lambda_k * \phi_j) = R(\Lambda_j * \phi_k), \quad j, k \in \mathbf{N}.$$

Again by using the injectivity of R we get $((\Lambda_k), (\phi_k))$ is a quotient and hence $[\Lambda_k/\phi_k] \in \mathcal{B}_1$. It is straightforward to verify that this is the pre-image of the given Boehmian. $\square$

Theorem 4.5. *The Radon transform $\mathcal{R}: \mathcal{B}_1 \to \mathcal{B}_2$ is continuous with respect to δ-convergence and Δ-convergence.*

Proof. Let $F_n \xrightarrow{\delta} F$ as $n \to \infty$ in $\mathcal{B}_1$. Then there exist representatives $\Lambda_{n,k}/\phi_k$ and Λ_n/ϕ_n of F_n and F respectively such that, for each $k \in \mathbf{N}$, $\Lambda_{n,k} \to \Lambda_k$ as $n \to \infty$ in $\mathcal{S}'(\mathbf{R}^N)$. Using the continuity of the classical Radon transform on $\mathcal{S}'(\mathbf{R}^N)$ we get $R\Lambda_{n,k} \to R\Lambda_k$ as $n \to \infty$ in $(KR\mathcal{S})'$. Since $\mathcal{R}F_n = [R\Lambda_{n,k}/\phi_n]$ and $\mathcal{R}F = [R\Lambda_k/\phi_k]$, we get $\mathcal{R}F_n \xrightarrow{\delta} \mathcal{R}F$ as $n \to \infty$ in $\mathcal{B}_2$.

If $F_n \xrightarrow{\Delta} F$ as $n \to \infty$ in $\mathcal{B}_1$, then there exist $\Lambda_n \in \mathcal{S}'(\mathbf{R}^N)$ such that $(F_n - F) * \phi_n = [\Lambda_n/\phi_n]$ for some suitable $(\phi_n) \in \Delta$ and $\Lambda_n \to 0$ as $n \to \infty$ in $\mathcal{S}'(\mathbf{R}^N)$. Again, by using the continuity of the classical Radon transform, we complete the proof of the theorem. $\square$

Theorem 4.6. $\mathcal{R}^{-1} : \mathcal{B}_2 \to \mathcal{B}_1$ *is continuous with respect to both δ-convergence and Δ-convergence.*

Proof. Let $H_n \xrightarrow{\delta} H$ as $n \to \infty$ in $\mathcal{B}_2$. Then there exist $u_{n,k}, u_k \in (KR\mathcal{S})'$ and $(\phi_k) \in \Delta$ such that $X_n = [u_{n,k}/\phi_k]$, $X = [u_k/\phi_k]$ and for each $k \in \mathbf{N}$, $u_{n,k} \to u_k$ as $n \to \infty$ in $(KR\mathcal{S})'$. If $R\Lambda_{n,k} = u_{n,k}$ and $R\Lambda_k = u_k$, then $\mathcal{R}^{-1}H_n = [\Lambda_{n,k}/\phi_k]$ and $\mathcal{R}^{-1}H_n = [\Lambda_k/\phi_k]$. Since the inverse Radon transform $R^{-1}: (KR\mathcal{S})' \to \mathcal{S}'(\mathbf{R}^N)$ is continuous we get for each $k \in \mathbf{N}$, $\Lambda_{n,k} \to \Lambda_k$ as $n \to \infty$ in $\mathcal{S}'(\mathbf{R}^N)$. In other words we get $\mathcal{R}^{-1}H_n \xrightarrow{\delta} \mathcal{R}^{-1}H$ as $n \to \infty$ in $\mathcal{B}_1$.

Let $H_n \xrightarrow{\Delta} H$ as $n \to \infty$ in $\mathcal{B}_2$. Then there exist $u_n \in (KR\mathcal{S})'$ and $(\phi_k) \in \Delta$ such that $(H_n - H) \otimes \phi_n = [u_n \otimes \phi_k/\phi_k]$ and $u_n \to 0$ as $n \to \infty$ in $(KR\mathcal{S})'$. If $R\Lambda_n = u_n$, then one can verify that $(\mathcal{R}^{-1}H_n - \mathcal{R}^{-1}H) * \phi_n = [\Lambda_n * \phi_k/\phi_k]$, and hence by the continuity of $R^{-1}: (KR\mathcal{S})' \to \mathcal{S}'(\mathbf{R}^N)$ we get $\mathcal{R}^{-1}H_n \xrightarrow{\Delta} \mathcal{R}^{-1}H$ as $n \to \infty$ in $\mathcal{B}_1$. $\square$

Now we alter the space of C^∞-space as $\mathcal{B}_3 = \mathcal{B}(\mathcal{D}'(\mathbf{R}^N), (\mathcal{D}(\mathbf{R}^N), *), *, \Delta)$, where $* : \mathcal{D}'(\mathbf{R}^N) \times \mathcal{D}(\mathbf{R}^N) \to \mathcal{D}'(\mathbf{R}^N)$ by $(\Lambda * \phi)(\psi) = \Lambda(\psi * \check{\phi})$, for all $\psi \in \mathcal{D}(\mathbf{R}^N)$ and $*$, Δ are as defined earlier.

Let $R\mathcal{D}$ denote the image of $\mathcal{D}(\mathbf{R}^N)$ under the classical Radon transform. We know that $g \in R\mathcal{D}$ if and only if $g \in R\mathcal{S}$ and g has compact support. First we prove the following lemma which is necessary to construct the range Boehmian space.

Lemma 4.7. *If $g \in R\mathcal{D}$ and $\phi \in \mathcal{D}(\mathbf{R}^N)$, then $g \times \phi \in R\mathcal{D}$.*

Proof. The lemma follows if we prove $f \times \phi$ has compact support, by Lemma 2.5. Let support of g be contained in $[-r_1, r_1] \times S^{N-1}$ and for each x in the support of ϕ, $\|x\| \leq r_2$. We claim that the support of $g \times \phi$ is contained in $K = [-(r_1 + r_2), r_1 + r_2] \times S^{N-1}$. For $(s, w) \in K^c$, $(g \times \phi)(s, w) = \int_{\mathbf{R}^n} g(s - y.w, w)\phi(y)\, dy = 0$ since $(s - y \cdot w, w) \notin [-r_1, r_1] \times S^{N-1}$ for each y in the support of ϕ. $\square$

To construct the range Boehmian we take G as $(KR\mathcal{D})'$ and $\star$ as $\otimes : (KR\mathcal{D})' \times \mathcal{D}(\mathbf{R}^N)$ defined by

$$\langle (u \otimes \phi), g \rangle = \langle u, (g \times \check{\phi}) \rangle, \quad \forall\, g \in KR\mathcal{D}. \tag{4.2}$$

Now we can define $\mathcal{B}_4 = \mathcal{B}((KR\mathcal{D})', (\mathcal{D}(\mathbf{R}^N), *), \otimes, \Delta)$ and extend the generalized Radon transform $\mathcal{R}: \mathcal{B}_3 \to \mathcal{B}_4$ defined by

$$\mathcal{R}\left(\left[\frac{\Lambda_k}{\phi_k}\right]\right) = \left[\frac{R\Lambda_k}{\phi_k}\right] \tag{4.3}$$

where $R\Lambda_k$ is the distributional transform $R: \mathcal{D}'(\mathbf{R}^N) \to (KR\mathcal{D})'$, which is a continuous isomorphism [**6**, Section 4].

As we have done earlier, we can prove that $\mathcal{R}: \mathcal{B}_3 \to \mathcal{B}_4$ is a linear, bijection and $\mathcal{R}$ and $\mathcal{R}^{-1}$ are continuous with respect to δ-convergence as well as Δ-convergence, where $\mathcal{R}^{-1}: \mathcal{B}_4 \to \mathcal{B}_3$ is defined by

$$\mathcal{R}^{-1}\left(\left[\frac{u_k}{\phi_k}\right]\right) = \left[\frac{\Lambda_k * \phi_k}{\phi_k * \phi_k}\right]. \tag{4.4}$$

where $\Lambda_k \in \mathcal{D}'(\mathbf{R}^N)$ such that $R\Lambda_k = u_k$. We can also prove that $\mathcal{R}: \mathcal{B}_3 \to \mathcal{B}_4$ extends the classical Radon transform $R: \mathcal{D}'(\mathbf{R}^N) \to (KR\mathcal{D})'$.

Remark. We know that $\mathcal{B}_1 \subset \mathcal{B}_3$, and hence the theory of Radon transform on $\mathcal{B}_3$ is the most general one, in the context of Boehmians.

Though $\mathcal{B}_1$ and $\mathcal{B}_3$ are constructed by using distribution spaces, for each Boehmian $[\Lambda_k/\phi_k] \in \mathcal{B}_1$ (or $\mathcal{B}_3$), for any $(\psi_k) \in \Delta$, $[\Lambda_k * \psi_k/\phi_k * \psi_k]$ is another representative of the same Boehmian with $\Lambda * \psi_k$ as functions, where $(\Lambda * \psi_k)(x) = \Lambda(\tau_x \check{\psi}_k)$ and $(\tau_x \psi)(t) = \psi(t - x)$, for all $x, t \in \mathbf{R}^N$. It is a known fact that this convolution agrees with (3.3). Thus in this case, each Boehmian in $\mathcal{B}_1$ or in $\mathcal{B}_3$ can be approximated by sequences of functions. We conclude by posing the following interesting question:

Is it possible to approximate each $H \in \mathcal{B}_2$ (or B_4) by functions?

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Department of Mathematics, Alagappa University, Karaikudi - 630 003, India
E-mail address: roopkumarr@rediffmail.com