THE HURWITZ ZETA FUNCTION
AS A CONVERGENT SERIES

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ABSTRACT. New series for the Hurwitz zeta function which converge on the whole plane, except $s = 1$, are developed. This is applied to obtain a remarkably simple evaluation of some special values of the function.

1. Introduction. Classically the Riemann zeta function, or more generally, the Hurwitz zeta function, is defined on a half plane using a series and then it is analytically extended, with respect to s, to the whole plane except for a simple pole at $s = 1$ with residue 1,

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad \text{for} \quad \Re s > 1 \quad \text{and} \quad 0 < x \le 1,$$

however in many calculations x can be taken any positive number. The Riemann zeta function is obtained from the Hurwitz function by setting $x = 1$. In this paper we define the Hurwitz zeta function by a series which converges on the whole plane except for $s = 1$. In fact we define a family of series parameterized by certain easily constructible sequences of natural numbers $\{g_n\}_{n=0}^{\infty}$. Our constructions and proofs are elementary and they require only the basic properties of Bernoulli numbers (for basic properties of Bernoulli numbers and L-functions we refer the reader to [**3**] and [**23**]) and complex analysis of one variable, see, e.g., [**46**]. The new series leads to a very simple and natural evaluation of L-functions at negative integers. One example of our series is the following:

Let B_n be Bernoulli numbers, and let g_n be a sequence defined by

$$g_n = [\![\sqrt{\ln(n+3)} \,]\!] \quad \text{for} \quad n = 0, 1, \dots,$$

where $[\![\cdot]\!]$ is the integer part of a real number. Then the Hurwitz zeta function $\zeta = \zeta(s,x)$, where $s \in \mathbf{C} \setminus \{1\}$ and $0 < x \le 1$, can be represented as follows:

$$
\begin{aligned}
\zeta(s,x) = {}& \frac{1}{x^s} + \frac{1}{s-1}\left[x^{-s+1} + B_1(s-1)\,x^{-s} + B_2\,\frac{(s-1)s}{2!}\,x^{-s-1} \right] \\
& + \sum_{n=1}^{\infty}\left[\sum_{\beta=2g_{n-1}+2}^{2g_n} B_\beta\,\frac{s\dots(s+\beta-2)}{\beta!}\,\frac{1}{(n+x)^{s+\beta-1}} \right] \\
& - \sum_{n=1}^{\infty}\left\{ \sum_{\beta=2g_{n-1}+2}^{\infty}\left[s\dots(s+\beta-1) \right.\right. \\
& \qquad\qquad\qquad \left.\left. \times \sum_{j=0}^{2g_{n-1}} B_j\,\frac{1}{j!(\beta-j+1)!} \right]\frac{1}{(n+x)^{\beta+s}} \right\}.
\end{aligned}
$$

This representation can be used as a very simple evaluation of some values of the Hurwitz zeta function, and consequently, of L-series.

In Section 2 we mention a few historical remarks. In Section 3 we introduce the key H-function which is closely related to Bernoulli's polynomials and we establish the basic, rather interesting properties, of our H-function. Section 4 is the main part of our paper and in particular we formulate and prove Theorem 4.2 where a family of series convergent on the whole plane except $s = 1$ is described. As a corollary we obtain the well-known values of Hurwitz's zeta function at negative integers by merely plugging our negative integers into our series. Finally, in Section 5, as an application of the previous results, we obtain a simple evaluation of Dirichlet L-functions. In this section we were also influenced by Stark's beautiful derivation of Dirichlet's class-number formula quoted in the references.

2. Historical remarks. In 1731 the summation of the series $\zeta(n)$, when n is an integer ≥ 2, had been a classical problem. In the years 1731–1735, Euler discovered, independently of MacLaurin, the Euler-MacLaurin summation formula and applied it to the calculation $\zeta(n)$.

Following a breakthrough in 1735, Euler discovered the remarkable fact that $\zeta(2n)/\pi^{2n}$, $n \in \mathbf{N}$, are all rational numbers. Euler also found a key connection, in the case of real s, between the zeta function and prime numbers, which is today the well-known product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \Re(s) > 1,$$

where the product runs over all prime numbers. This formula implies that $\sum_p (1/p)$ is a divergent series and in particular that there are infinitely many primes. (See [**53**, par. 17 to 20].)

In 1937 Dirichlet used Euler's idea to show that in each arithmetic sequence

$$a, \ a + d, \ a + 2d, \ \dots,$$

in which a and d are natural numbers with no common factor, there are infinitely many primes. In order to prove this result, Dirichlet defined an L-series attached to Dirichlet's characters. (See [**13**] and also [**14**].)

Now it was Riemann who in his eight-page paper, see [**43**], made a number of crucial discoveries and key conjectures related to the zeta function and to the distribution of prime numbers. In particular, Riemann showed in his paper that $\zeta(s)$ can be analytically extended to a whole plane and $\zeta(s) - 1/(s - 1)$ is an entire function.

In 1882 Hurwitz realized that one can analytically extend the L-functions attached to a quadratic character by introducing an auxiliary function: today called the Hurwitz zeta function, see [**22**]. (Hurwitz himself only considered the case when a is rational, see [**39**, Section 4.2].)

Today the Hurwitz zeta function and its related functions are in the center of mathematical investigations. The distribution of primes and their relation to Hurwitz's zeta function has remained one of the key problems in current number theory, with the Riemann hypothesis being the last unsolved problem contained in [**43**]. There is an important connection with probability theory (see [**30**] and [**31**], for example) and with some special functions. Not only is $\zeta(s, x)$ a special function itself, but $\zeta(s, x)$ has numerous relations with other special functions including gamma functions, theta functions, and hypergeometric functions, see [**37**, Chapter 1,] and [**17**]. It also has relations with the evaluation

of integrals (see [**9, 15, 16**]), with cosmology (see [**24**, p. 134] and [**34–36**]), and with the convexity and monotonicity of zeta functions (see [**5**]). Rather recently the fascinating connection with noncommutative geometry and spectral interpretation of the critical zeros of the Riemann zeta function as an absorbtion spectrum was discovered, see [**10–12**]. There is also an established link between Riemann's hypothesis and random matrix theory which originated in physics, see [**25, 26, 47**]. A delightful mix of modern and classical topics related to the Hurwitz zeta function may be found in [**32**].

Of course, this is just an extremely quick glimpse at the vast recent research related to the Hurwitz zeta function.

It is remarkable, after all of these years and with the growing interest in zeta functions by mathematicians and physicists from different areas, how relevant and how modern are the ideas of Euler on the zeta function. We still seem to be very far from establishing Euler's dream of determining the nature of the values $\zeta(2n+1)$, $n \in \mathbf{N}$. However, in 1979 Apéry showed that $\zeta(3)$ is irrational, see [**1, 2, 51**]. In 2000 Rivoal showed that infinitely many values $\zeta(2n+1)$, $n \in \mathbf{N}$ are irrational, see also [**4, 44, 45, 54**]. Zagier proposed a very interesting conjecture about the values of the Dedekind zeta function $\zeta_F(m)$ attached to an algebraic number field F, which relates $\zeta_F(m)$ with polylogarithms and K-theory, see [**54, 55**]. In [**18**] Goncharov proved Zagier's conjecture for $m = 3$. (See also [**19**] for a nice, short survey of related topics). The paper [**20**] contains a detailed exposition of a number of fundamental discoveries related to trilogarithms, K-theory, and $\zeta_F(3)$. See [**6–8**] for important, far-reaching Beilinson conjectures about the values of motivic L-functions, Deligne cohomology and regulators (also [**40**] contains a very nice, informative exposition of these topics). For a fascinating exposition related to periods defined as integrals of n-forms on algebraic varieties over an algebraic closure of $\mathbf{Q}$ and special values of L-functions, see [**28**].

Very recently some of the key conjectures related to the K-theory of fields were settled by Rost and Voevodsky. These developments led to descriptions of K-groups of rings of algebraic integers. This in turn is related to the values of the Dedekind zeta function at odd negative integers. (See [**52**] for details.)

Renewed interest in the values $\zeta(1 - k)$ has its origin in the fact that the values $(1 - p^{k-1})\zeta(1 - k)$ can be p-adically interpolated. In fact, these values can be extended to a p-adic zeta function ζ_p such that $\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k)$ for each $k \in \mathbf{N}$. (See [**27**, Chapter 2].) In [**33**] it was observed that $\zeta(-a)$, $a \in \mathbf{N}\cup\{0\}$, can be directly connected with the partial sum

$$S_a(M) := \sum_{n=1}^{M-1} n^a; \quad \text{in fact} \quad \zeta(-a) = \int_0^1 S_a(x)\,dx.$$

Also in [**33**] and independently in [**38**], it was observed that using a well-known formula for $\zeta(s)$ expressed by the values $\zeta(s + q)$, $q \in \mathbf{N}$, see [**29**, p. 147] and also [**42**], one can derive the formula for $\zeta(1 - k)$, $k \in \mathbf{N}$, in a simple way. For a very nice survey of these results see [**49**].

These considerations are related to some ideas, somewhat in the original spirit of Euler, that one could modify each term $1/n^s$, $n \in \mathbf{N}$, with some function of n and s which behaves like a difference of values of Bernoulli polynomials, and which will make our series convergent on the entire plane with the exception of the point $s = 1$, without altering the values of $\zeta(s)$ if $\Re s > 1$. The additional idea is to apply this procedure conveniently in some intervals in $\mathbf{N}$ in which we make our summation. The goal of this paper is to show that this idea works when carefully executed in the more general case of Hurwitz's zeta function.

3. H-function.

3.1 Definition of the H-function. One way to define Bernoulli polynomials $B_n(x)$ and numbers $B_n = B_n(0)$, see [**3**, p. 264], is the following expansion

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n.$$

The property of the Bernoulli polynomials, see [**3**, p. 265], which we need later on is

$$(3.1) \qquad B_n(x + 1) = B_n(x) + nx^{n-1} \quad \text{for} \quad n \geq 1$$

and of Bernoulli numbers

$$(3.2) \qquad \sum_{j=0}^{m-1} B_j \binom{m}{j} = 0.$$

We define the function $H(k, s, x)$, where $k = 1, 2, \ldots$, and $s \in \mathbf{C}$, $x \in \mathbf{R}$, $x > 0$,

$$(3.3) \qquad H(k, s, x) = \sum_{j=0}^{2k} B_j \binom{s + j - 2}{j} x^{-s-j+1}.$$

Here the complex powers are meant as principal values.

Taking into account that $B_1 = -1/2$ and that the odd Bernoulli numbers are zero, except for B_1, another equivalent version of the H function, is

$$(3.4) \qquad H(k, s, x) = \sum_{j=0}^{2k} B_j \binom{-(s-1)}{j} x^{-s-j+1} + (1-s)x^{-s}.$$

In order to motivate $H(k, s, x)$, we see that these functions are actually Bernoulli polynomials when s is a negative integer. Indeed suppose that $a \in \mathbf{Z}$ and $0 \le a < 2k$. Then, using (3.4),

$$
\begin{aligned}
H(k, -a, x) &= \sum_{j=0}^{2k} B_j \binom{a+1}{j} x^{a+1-j} + (a+1)x^a \\
&= \sum_{j=0}^{a+1} B_j \binom{a+1}{j} x^{a+1-j} + (a+1)x^a \\
&= B_{a+1}(x+1),
\end{aligned}
$$

(3.5)

where $B_{a+1}(x+1)$ is the $(a+1)$th Bernoulli polynomial.

3.2 Formulation of properties of the H-function.

Lemma 3.1. *We have the identity*

$$\frac{\partial H(k, s, x)}{\partial x} = (1 - s)H(k, s + 1, x).$$

Proof. The lemma can be proved by a simple direct computation. Since

$$\binom{s+j-2}{j}(-s-j+1) = (1-s)\binom{s+j-1}{j}$$

we have

$$\frac{\partial H(k,s,x)}{\partial x} = \sum_{j=0}^{2k} B_j \binom{s+j-2}{j}(-s-j+1)x^{-s-j}$$

$$= (1-s)\left[\sum_{j=0}^{2k} B_j \binom{s+j-1}{j} x^{-s-j}\right]$$

$$= (1-s)H(k,s+1,x),$$

which completes the proof.   □

The following proposition gives an explicit formula for the difference $H(k,s,x) - H(k,s,x-1)$ as a power series of $1/x$.

Proposition 3.1 (Expansion). *We have the following properties of the H-functions*:

1. *The function* $(s,x) \to H(k,s,x)$ *is smooth for* $s \in \mathbf{C}$, $x \in \mathbf{R}$, $x > 0$, *and holomorphic with respect to* s.

2. *For* $k = 1,2,\ldots$, *the function* $x^s[H(k,s,x) - H(k,s,x-1)]$ *can be expanded into a power series with respect to* $x > 1$ *with polynomial coefficients in* s:

$$(3.6) \quad \frac{x^s}{s-1}\left[H(k,s,x) - H(k,s,x-1)\right] + 1$$

$$= -\sum_{\beta=2k+2}^{\infty} \underbrace{\frac{1}{\beta+1}\binom{s+\beta-1}{\beta}\sum_{j=0}^{2k} B_j \binom{\beta+1}{j}}_{a_\beta} x^{-\beta}.$$

Proposition 3.2 (Estimates). *Let $|s| \leq r$, where $r \geq 4$ is a natural number. Then we have the following estimates of the coefficients a_β and of the series $\sum a_\beta x^{-\beta}$:*

$$(3.7) \qquad\qquad |a_\beta| \leq (\beta + r)^{2k+r} \frac{1}{(2\pi)^{2k+1}}$$

$$(3.8) \qquad \sum_{\beta=2k+2}^{\infty} |a_\beta| x^{-\beta} \leq \frac{2\pi}{(2\pi x)^{2k+2}} (4k + 2r + 2)! \left(\frac{x}{x-1} \right)^{4k+2r+3}$$

$$\text{for} \quad x > 1,$$

which implies, for $x > 1$,

$$(3.9) \quad \left| \frac{x^s}{s-1} \left[H(k,s,x) - H(k,s,x-1) \right] + 1 \right|$$

$$\leq \frac{2\pi}{(2\pi x)^{2k+2}} (4k + 2r + 2)! \left(\frac{x}{x-1} \right)^{4k+2r+3}.$$

We observe that the formula (3.9) is a generalization of the well-known identity (3.1).

As an immediate consequence of Proposition 3.1, we have

Corollary 3.1. *For $n = 2, 3, \ldots$, we have*

$$(3.10) \quad \frac{1}{s-1} \left[H(k,s,n) - H(k,s,n-1) \right] + \frac{1}{n^s}$$

$$= - \sum_{\beta=2k+2}^{\infty} \frac{1}{\beta+1} \binom{s+\beta-1}{\beta} \left[\sum_{j=0}^{2k} B_j \binom{\beta+1}{j} \right] \frac{1}{n^{s+\beta+1}}.$$

Remark 3.1. The same *type* propositions can be formulated for the derivatives

$$\frac{\partial^{i+j}}{\partial s^i \partial x^j} \left[\frac{x^s}{s-1} \left[H(k,s,x) - H(k,s,x-1) \right] + 1 \right];$$

actually it is a repetition with small modifications of the proofs we are giving below for the function. In particular, it implies that the series 3.6 can be differentiated term by term with respect to s and x.

3.3. Proof of the "expansion" proposition. The first part of the proposition is obvious by using definition (3.3) of the function H.

The second part requires some calculations. From the definition of the function H, we obtain

$$x^s \left[H(k, s, x) - H(k, s, x - 1) \right]$$

$$= x^s H(k, s, x) - \frac{x^s}{(x-1)^s} (x-1)^s H(k, s, x - 1)$$

$$= \sum_{j=0}^{2k} B_j \binom{s+j-2}{j} x^{-j+1} - \frac{x^s}{(x-1)^s} \sum_{j=0}^{2k} B_j \binom{s+j-2}{j} (x-1)^{-j+1}.$$

To get positive exponents in the expressions in the above sums, we substitute $x = 1/t$ for $0 < t < 1$ since $x > 1$. Then we calculate the last two-line sums in the above expression.

(3.11)

$$\sum_{j=0}^{2k} B_j \binom{s + j - 2}{j} t^{j-1} - \sum_{j=0}^{2k} B_j \binom{s + j - 2}{j} t^{j-1} (1 - t)^{-s-j+1}$$

$$= \sum_{j=0}^{2k} B_j \binom{s + j - 2}{j} t^{j-1} \left[1 - (1 - t)^{-s-j+1} \right].$$

Next we use the binomial expansion:

$$t^{j-1} \left[1 - (1 - t)^{-s-j+1} \right] = t^{j-1} \left[1 - \sum_{\alpha=0}^{\infty} (-1)^\alpha \binom{-s - j + 1}{\alpha} t^\alpha \right]$$

$$= t^{j-1} \left[1 - \sum_{\alpha=0}^{\infty} \binom{s + j + \alpha - 2}{\alpha} t^\alpha \right]$$

$$= - \sum_{\alpha=1}^{\infty} \binom{s + j + \alpha - 2}{\alpha} t^{\alpha+j-1}.$$

The series in (3.11) becomes

$$-\sum_{j=0}^{2k} B_j \binom{s+j-2}{j} \sum_{\alpha=1}^{\infty} \binom{s+j+\alpha-2}{\alpha} t^{\alpha+j-1}$$

$$= -\sum_{\alpha=1}^{\infty} \sum_{j=0}^{2k} B_j \binom{s+j-2}{j} \binom{s+j+\alpha-2}{\alpha} t^{\alpha+j-1}.$$

After changing the summation index, the sum (3.11) can be written

$$\sum_{j=0}^{2k} B_j \binom{s+j-2}{j} t^{j-1} \left[1 - (1-t)^{-s-j+1}\right]$$

(3.12)
$$= -\sum_{\beta=0}^{\infty} \sum_{j=0}^{\min(\beta,2k)} B_j \binom{s+j-2}{j} \binom{s+\beta-1}{\beta-j+1} t^{\beta}$$

$$= -\sum_{\beta=0}^{\infty} \binom{s+\beta-1}{\beta+1} \sum_{j=0}^{\min(\beta,2k)} B_j \binom{\beta+1}{j} t^{\beta}.$$

But if $\beta = 0, 1, \dots, 2k+1$, then using (3.2), we obtain

$$\sum_{j=0}^{\min(\beta,2k)} B_j \binom{\beta+1}{j}$$

$$= \begin{cases} B_0 \binom{1}{0} = B_0 = 1 & \text{if } \beta = 0, \\ \sum_{j=0}^{\beta} B_j \binom{\beta+1}{j} = 0 & \text{if } \beta = 1, 2, \dots, 2k, \\ \sum_{j=0}^{2k} B_j \binom{2k+2}{j} & \\ = \sum_{j=0}^{2k+1} B_j \binom{2k+2}{j} = 0 & \text{if } \beta = 2k+1. \end{cases}$$

Combining (3.12) with the above formulas for coefficients we proved

$$\sum_{j=0}^{2k} B_j \binom{s+j-2}{j} t^{j-1} \left[1 - (1-t)^{-s-j+1}\right]$$

$$= -(s-1) - \sum_{\beta=2k+2}^{\infty} \binom{s+\beta-1}{\beta+1} \sum_{j=0}^{2k} B_j \binom{\beta+1}{j} t^{\beta}.$$

Finally, replacing t by $1/x$, we get

$$x^s \left[H(k, s, x) - H(k, s, x - 1) \right] + (s - 1)$$

$$= - \sum_{\beta=2k+2}^{\infty} \binom{s + \beta - 1}{\beta + 1} \sum_{j=0}^{2k} B_j \binom{\beta + 1}{j} x^{-\beta}$$

$$= -(s - 1) \sum_{\beta=2k+2}^{\infty} \frac{1}{\beta + 1} \binom{s + \beta - 1}{\beta} \sum_{j=0}^{2k} B_j \binom{\beta + 1}{j} x^{-\beta}$$

what was needed to be proved.

3.4 Proof of the "estimates" proposition. Here we consider the series

(3.13)

$$\sum_{\beta=2k+2}^{\infty} a_\beta \, x^{-\beta} = \sum_{\beta=2k+2}^{\infty} \underbrace{\frac{1}{\beta + 1} \binom{s + \beta - 1}{\beta} \sum_{j=0}^{2k} B_j \binom{\beta + 1}{j}}_{a_\beta} x^{-\beta}$$

from the first proposition where we write a_β as

(3.14)
$$a_\beta = a_\beta(s, k) = s \ldots (s + \beta - 1) \sum_{j=0}^{2k} B_j \frac{1}{j!(\beta - j + 1)!},$$

$$\beta \geq 2k + 2.$$

3.4.1 *Estimate of a_β.* In our calculations we shall use the following inequalities for Bernoulli numbers, see [**41**, p. 17]:

(3.15) $$|B_j| \leq \frac{j!}{12(2\pi)^{j-2}} \leq \frac{4j!}{(2\pi)^j} \quad \Longrightarrow \quad \frac{|B_j|}{j!} \leq \frac{4}{(2\pi)^j}.$$

Using (3.15) and also $|s| \leq r$, we obtain

$$
\begin{aligned}
|a_\beta| &\leq |s \dots (s + \beta - 1)| \sum_{j=0}^{2k} \frac{4}{(2\pi)^j} \frac{1}{(\beta - j + 1)!} \\
&= 4 \frac{r}{1} \dots \frac{r + \beta - 1}{\beta} \sum_{j=0}^{2k} \frac{\beta!}{(2\pi)^j (\beta - j + 1)!} \\
&\leq 4 \binom{r + \beta - 1}{\beta} \sum_{j=0}^{2k} \left(\frac{\beta}{2\pi} \right)^j \\
&\leq 4 \binom{r + \beta - 1}{\beta} \left(\frac{\beta}{2\pi} \right)^{2k+1}.
\end{aligned}
$$

(3.16)

From (3.16) we get (assume that $r \geq 4$)

$$
\begin{aligned}
|a_\beta| &\leq 4 \frac{(\beta + r - 1) \dots (\beta + 1)}{(r - 1)!} \left(\frac{\beta}{2\pi} \right)^{2k+1} \\
&\leq (\beta + r - 1)^{r-1} \left(\frac{\beta}{2\pi} \right)^{2k+1} \\
&\leq (\beta + r)^{2k+r} \frac{1}{(2\pi)^{2k+1}}.
\end{aligned}
$$

3.4.2 *Estimate of the series.* From the estimate of α_β above we have

(3.17)

$$
\sum_{\beta=2k+2}^{\infty} |a_\beta| x^{-\beta} \leq \frac{1}{(2\pi)^{2k+1}} \sum_{\beta=2k+2}^{\infty} (\beta + r)^{2k+r} x^{-\beta} \quad \text{for} \quad x > 1.
$$

Now we estimate the right-hand side of (3.17).

First we need a formula for $|t| < 1$:

$$\left(\sum_{\beta=0}^{\infty} t^{\beta+m}\right)^{(m)} = [t^m(1-t)^{-1}]^{(m)}$$

$$= \sum_{\alpha=0}^{m} \binom{m}{\alpha} (t^m)^{(m-\alpha)}[(1-t)^{-1}]^{(\alpha)}$$

$$= \sum_{\alpha=0}^{m} \binom{m}{\alpha} m\ldots(\alpha+1)t^{\alpha}\alpha\ldots1(1-t)^{-1-\alpha}$$

$$= \frac{m!}{1-t}\sum_{\alpha=0}^{m}\binom{m}{\alpha}\left(\frac{t}{1-t}\right)^{\alpha}$$

$$= \frac{m!}{1-t}\left(1+\frac{t}{1-t}\right)^{m}$$

$$= \frac{m!}{(1-t)^{m+1}}.$$

So we obtained

$$(3.18) \qquad \left(\sum_{\beta=0}^{\infty} t^{\beta+m}\right)^{(m)} = \frac{m!}{(1-t)^{m+1}} \quad \text{for} \quad |t| < 1.$$

We apply (3.18) in the following estimates for $0 < t < 1$:

$$\sum_{\beta=2k+2}^{\infty} (\beta+r)^{2k+r}t^{\beta} = t^{2k+2}\sum_{\alpha=0}^{\infty}(\alpha+2k+r+2)^{2k+r}\,t^{\alpha}$$

$$(3.19) \qquad\qquad\qquad \leq t^{2k+2}\left[\sum_{\alpha=0}^{\infty}t^{\alpha+4k+2r+2}\right]^{(4k+2r+2)}$$

$$\leq t^{2k+2}\frac{(4k+2r+2)!}{(1-t)^{4k+2r+3}}.$$

Switching back to x by taking $t = 1/x$ in (3.19) and using (3.17), we get

$$(3.20) \quad \sum_{\beta=2k+2}^{\infty} |a_\beta| \, x^{-\beta} \leq \frac{2\pi}{(2\pi x)^{2k+2}} \, (4k + 2r + 2)! \left(\frac{x}{x-1} \right)^{4k+2r+3},$$

and the proposition is proved.

4. Series representations of the Hurwitz zeta function.

4.1 Formulation of results. In this section we establish two representations of the Hurwitz zeta function $\zeta = \zeta(s, x)$. First we define the following families of formal series. Later we specify for what arguments these series converge. Let

(4.1)

$$\mathcal{H}(k, s, x) = \frac{1}{x^s} + \frac{1}{s-1} \, H(k, s, x)$$
$$+ \sum_{n=1}^{\infty} \left[\frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(k, s, n+x) - H(k, s, n+x-1) \right] \right],$$

where $s \in \mathbf{C} \setminus \{1\}$, $x \in \mathbf{R}$, $x > 0$ and $k = 1, 2, \ldots$.

Before we define the second family of series, let $g(n) = g_n$ be an integer-valued nondecreasing sequence such that $g(0) = 1$. Define

(4.2)

$$\mathcal{H}^g(s, x) = \frac{1}{x^s} + \frac{1}{s-1} \, H(1, s, x)$$
$$+ \sum_{n=1}^{\infty} \left[\frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(g_n, s, n+x) - H(g_{n-1}, s, n+x-1) \right] \right],$$

where $s \in \mathbf{C} \setminus \{1\}$ and $x \in \mathbf{R}$, $x > 0$.

Theorem 4.1. *With the above notation, we have*:

(a) *The series in the second line of* (4.1) *converges for $x > 0$ and $\Re s > -2k - 1$;*

(b) *The function $\mathcal{H}(k, s, x)$ is smooth with respect to the variables s and x, and holomorphic with respect to s for $\Re s > -2k - 1$ and $s \neq 1$;*

(c) *For $\Re s > -2k - 1$ and $s \neq 1$ the function $\mathcal{H}(k, s, x)$ can be written as*:

(4.3)

$$\mathcal{H}(k, s, x) = \zeta(s, x) = \frac{1}{x^s} + \frac{1}{s-1} H(k, s, x)$$
$$- \sum_{n=1}^{\infty} \sum_{\beta=2k+2}^{\infty} \frac{1}{\beta+1} \binom{s+\beta-1}{\beta} \sum_{j=0}^{2k} B_j \binom{\beta+1}{j} \frac{1}{(n+x)^{\beta+s}}.$$

Theorem 4.2. *There is a nondecreasing integer-valued sequence $\{g_n\}_{n=0}^{\infty}$, $g_0 = 1$, such that we have:*

(a) *The series in (4.2) converges for $x > 0$ and $s \in \mathbf{C} \setminus \{1\}$;*

(b) *The function $\mathcal{H}^g(s, x)$ is smooth with respect to the variables $s \in \mathbf{C} \setminus \{1\}$ and $x > 0$ and holomorphic with respect to s;*

(c) *For all $s \in \mathbf{C} \setminus \{1\}$ and $x > 0$, we have*

(4.4)

$$\mathcal{H}^g(s, x) = \zeta(s, x) = \frac{1}{x^s} + \frac{1}{s-1} H(1, s, x)$$
$$+ \sum_{n=1}^{\infty} \sum_{\beta=2g_{n-1}+2}^{2g_n} B_\beta \frac{1}{\beta} \binom{s+\beta-2}{\beta-1} \frac{1}{(n+x)^{s+\beta-1}}$$
$$- \sum_{n=1}^{\infty} \sum_{\beta=2g_{n-1}+2}^{\infty} \frac{1}{\beta+1} \binom{s+\beta-1}{\beta} \sum_{j=0}^{2g_{n-1}} B_j \binom{\beta+1}{j} \frac{1}{(n+x)^{\beta+s}};$$

(d) *The function $\mathcal{H}^g(s, x)$ does not depend upon g, i.e., $\zeta(s, x) = \mathcal{H}^g(s, x)$ for all $s \in \mathbf{C} \setminus \{1\}$ and $0 < x \leq 1$.*

Corollary 4.1. *For $x = 1$ and $s \neq 1$, the Riemann zeta function $\zeta(s)$ has the following representations*

(1) *(4.1) for $\Re(s) > -2k - 1$,*

(2) *(4.2) for all $s \in \mathbf{C} \setminus \{1\}$,*

(3) (4.3) *for* $\Re(s) > -2k - 1$,

(4) (4.4) *for all* $s \in \mathbf{C} \setminus \{1\}$.

The following corollary is well known. (See [**3**, Theorem 12.13].) We obtain this corollary by simply substituting the nonnegative integer a in our series.

Corollary 4.2. *For any nonnegative integer a we have the following identity for the Hurwitz zeta function*

$$(4.5) \qquad\qquad \zeta(-a, x) = -\frac{B_{a+1}(x)}{a+1}.$$

Proof of Corollary 4.2.   Using Theorem 4.1(c) in the case when $a < 2k + 1$, we see that if $s = -a$, the formula (4.3) for the function $\mathcal{H}(k, -a, x)$ is reduced to few terms

$$\zeta(-a, x) = \mathcal{H}(k, -a, x) = x^a - \frac{1}{a+1} H(k, -a, x)$$
$$= \frac{(a+1)x^a - H(k, -a, x)}{a+1}.$$

Now using (3.5) we get

$$\zeta(-a, x) = \frac{(a+1)x^a - B_{a+1}(x+1)}{a+1} = -\frac{B_{a+1}(x)}{a+1},$$

which gives (4.5).

Before we approach the proofs of the theorems above, we present one more representation of the Hurwitz zeta function. This representation immediately follows from the formula (3.1) for $\zeta(s, x)$. Introduce the function

$$a(s) = \max(\llbracket -\Re s \rrbracket, 1)$$

where $\llbracket \cdot \rrbracket$ is the greatest integer function, i.e., taking integer values and $\llbracket y \rrbracket \le y < \llbracket y \rrbracket + 1$ for $y \in \mathbf{R}$. (In fact, instead of $a(s)$ we could take any positive integer-valued function which satisfies the condition $\Re s > -2a(s) - 1$.) Then we have the following corollary of Theorem 4.1.

Corollary 4.3. *We have*

(4.6)

$$\zeta(s,x) = \frac{1}{x^s} + \frac{1}{s-1} H(a(s),s,x)$$

$$+ \sum_{n=1}^{\infty} \left[\frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(a(s),s,n+x) - H(a(s),s,n+x-1) \right] \right]$$

for $s \in \mathbf{C} \setminus \{1\}$ *and* $0 < x \le 1$.

4.2 Proof of Theorem 4.1. To prove (a) we use Proposition 3.2, and we get the estimate (the second inequality below holds for $n > 4k + 2r + 3$):

(4.7)
$$\left| \frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(k,s,n+x) - H(k,s,n+x-1) \right] \right|$$

$$\le \frac{1}{(2\pi)^{2k+1}} \frac{1}{|(n+x)^{2k+2+s}|} (4k+2r+2)! \left(\frac{n+x}{n+x-1} \right)^{4k+2r+3}$$

$$\le \frac{e}{(2\pi)^{2k+1}} \frac{1}{|(n+x)^{2k+2+s}|} (4k+2r+2)!$$

and the series in (4.1) converges if $\Re(2k+2+s) > 1$ i.e., $\Re s > -2k-1$.

Part (b) of the theorem follows from the fact that each term of the series (4.1) is smooth with respect to x and s, holomorphic with respect to s and by using Remark 3.1 and estimates of the same kind as in (4.7).

To prove part (c), take $s \in \mathbf{C}$, $\Re s > 1$. Then the partial sums $\mathcal{H}_m(k,s,x)$ of $\mathcal{H}(k,s,x)$ are

$$\mathcal{H}_m(k,s,x) = \frac{1}{x^s} + \frac{1}{s-1} H(k,s,x)$$

$$+ \sum_{n=1}^{m} \left[\frac{1}{(n+x)^s} + \frac{1}{s-1} \left(H(k,s,n+x) - H(k,s,n+x-1) \right) \right]$$

$$= \sum_{n=0}^{m} \frac{1}{(n+x)^s} + \frac{1}{s-1} H(k,s,m+x).$$

By the definition of the function H, namely (3.3), we have that $\lim_{m \to \infty} H(k, s, m + x) = 0$ if $\Re s > 1$, and consequently,

$$\lim_{m \to \infty} \mathcal{H}_m(k, s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} = \zeta(s, x).$$

The formula (4.3) is actually rewritten (4.1) by using Proposition 3.1 and formula (3.6). This proves part (c) and completes the proof of the theorem.

4.3 Proof of Theorem 4.2. We fix a natural number r such that $|s| \le r$.

We choose a sequence $\{g_n\}$ of natural numbers in such a way that the following conditions are satisfied: There is n_0 such that for $n > n_0$ we have

$$(4.8) \qquad g_0 = 1, \quad \lim_{n \to \infty} g_n = \infty, \quad 0 \le g_n - g_{n-1} \le 1, \quad g_n{}^{g_n} \le n.$$

As an example of a sequence that satisfies (4.8) we can take

$$(4.9) \qquad g_n = [\![\sqrt{\ln(n + 3)}\,]\!] \quad \text{for} \quad n = 0, 1, \ldots .$$

The sequence (4.9) obviously satisfies the first three conditions of (4.8). To see the last one, we have the following inequalities:

$$g_n{}^{g_n} = \left([\![\sqrt{\ln(n + 3)}\,]\!]\right)^{[\![\sqrt{\ln(n+3)}\,]\!]} \le \left(\sqrt{\ln(n + 3)}\right)^{\sqrt{\ln(n+3)}},$$

and taking logarithm of the above expressions, we get

$$g_n \ln g_n \le \frac{1}{2}\sqrt{\ln(n+3)}\,\ln(\ln(n+3)) \le \frac{1}{2}\ln(n+3) \le \ln n \quad \text{for} \quad n \ge 3,$$

which implies the last condition of (4.8).

The general term of the series in (4.2) multiplied by $(n+x)^s$ can be written as:

(4.10)
$$1 + \frac{(n+x)^s}{s-1} \left[H(g_n, s, n+x) - H(g_{n-1}, s, n+x-1)\right]$$
$$= \underbrace{\frac{(n+x)^s}{s-1} \left[H(g_n, s, n+x) - H(g_{n-1}, s, n+x)\right]}_{I_n}$$

(4.11)
$$+ \underbrace{1 + \frac{(n+x)^s}{s-1} \left[H(g_{n-1}, s, n+x) - H(g_{n-1}, s, n+x-1)\right]}_{II_n}$$

We notice that I_n and II_n depend on n, x, and s.

In estimates of these two expressions, we need the following simple inequality: for a natural number l, $l \le g_n$, we have

(4.12)
$$(lg_n)! \le (lg_n)^{lg_n} \le [g_n^2]^{lg_n} \le [g_n^{g_n}]^{2l} \le n^{2l}.$$

Estimate of I_n. First consider the expression I_n. From the properties of the sequence $\{g_n\}$, this expression is either zero or contains exactly one term (see the definition of H). If it contains one term, take n large enough such that $g_n \ge r$. Then, using (4.12), we have

$$|I_n| = \left| \frac{1}{s-1} B_{2g_n} \binom{s+2g_n-2}{2g_n} \frac{1}{(n+x)^{2g_n-1}} \right|$$
$$\le \left| \frac{B_{2g_n}}{(2g_n)!} \right| |s(s+1)\dots(s+2g_n-2)| \frac{1}{(n+x)^{2g_n-1}}$$
$$\le \frac{4}{(2\pi)^{2g_n}} r(r+1)\dots(r+2g_n-2) \frac{1}{(n+x)^{2g_n-1}}$$
$$\le \frac{4}{(2\pi)^{2g_n}} (3g_n)! \frac{1}{(n+x)^{2g_n-1}}$$
$$\le \frac{1}{(2\pi)^{2g_n}} n^6 \frac{1}{(n+x)^{2g_n-1}}$$
$$\le \frac{1}{(2\pi)^{2g_n}} \frac{1}{(n+x)^{2g_n-7}}.$$

Thus we have obtained

$$(4.13) \qquad |I_n| \le \frac{1}{(2\pi)^{2g_n}} \frac{1}{(n+x)^{2g_n-7}}.$$

Estimate of II_n. Using (3.9), (4.12) and taking n large enough such that $n > 4g_{n-1} + 2r + 3$ and $g_n \ge 2r + 2$, we obtain

$$|II_n| = \left| 1 + \frac{(n+x)^s}{s-1} \left[H(g_{n-1}, s, n+x) - H(g_{n-1}, s, n+x-1) \right] \right|$$

$$\le \frac{2\pi}{[2\pi(n+x)]^{2g_{n-1}+2}} (4g_{n-1} + 2r + 2)! \left(\frac{n+x}{n+x-1} \right)^{4g_{n-1}+2r+3}$$

$$\le \frac{e}{(2\pi)^{2g_n-1}} \frac{1}{(n+x)^{2g_n}} (4g_n + 2r + 2)!$$

$$\le \frac{e}{(2\pi)^{2g_n-1}} \frac{1}{(n+x)^{2g_n}} (5g_n)!$$

$$\le \frac{e}{(2\pi)^{2g_n-1}} \frac{1}{(n+x)^{2g_n}} n^{10}$$

$$\le \frac{e}{(2\pi)^{2g_n-1}} \frac{1}{n^{2g_n-10}}.$$

Thus we have obtained

$$(4.14) \qquad |II_n| \le \frac{e}{(2\pi)^{2g_n-1}} \frac{1}{n^{2g_n-10}}$$

Combining (4.13) and (4.14) we obtain:

$$\left| \frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(g_n, s, n+x) - H(g_{n-1}, s, n+x-1) \right] \right|$$

$$\le \frac{\text{const}}{|n^{s+2g_n-10}|}$$

for n sufficiently large. This estimate proves part (a) of Theorem 4.2.

Part (b) of the theorem follows from the fact that each term of the series (4.2) is smooth with respect to x and s, holomorphic with respect to s and by using Remark 3.1 and estimates of the same kind as in (4.7).

Proof of (c). Now take $s \in \mathbf{C}$, $\Re s > 1$, and $x > 0$. Then the partial sums $\mathcal{H}_m^g(s, x)$ of $\mathcal{H}^g(s, x)$ are

(4.15)

$$
\mathcal{H}_m^g(s, x) = \frac{1}{x^s} + \frac{1}{s-1} H(1, s, x) + \sum_{n=1}^{m} \left[\frac{1}{(n+x)^s} + \frac{1}{s-1} \left[H(g_n, s, n+x) \right. \right.
$$
$$
\left. \left. - H(g_{n-1}, s, n+x-1) \right] \right]
$$
$$
= \sum_{n=0}^{m} \frac{1}{(n+x)^s} + \frac{1}{s-1} H(g_m, s, m+x).
$$

The limit of the first term when $m \to \infty$ on the right-hand side of (4.15) is $\zeta(s, x)$. Now we prove that the limit of the second term is zero as $m \to \infty$.

By definition of $H(g_m, s, m+x)$, using the assumptions $\Re s > 1$ and $|s| \leq r$, we have

$$
\frac{1}{s-1} |H(g_m, s, m+x)| \leq \sum_{j=0}^{2g_m} \frac{|B_j|}{j!} |s \dots (s+j-2)| \frac{1}{|(m+x)^{s+j-1}|}
$$
$$
\leq \frac{1}{|m^{s-1}|} \sum_{j=0}^{2g_m} \frac{4}{(2\pi)^j} \frac{(r+j-2)!}{m^j}.
$$

To prove that the very right-hand side converges to zero, it is enough to show that $(j+r-2)! \leq m^{j-1}$ for large j and $j \leq 2g_m$. This follows from the following sequence of estimates: For large m we have

$$
(j+r-2)! \leq (j+r-2)^{j+r-2} \leq (j+r-2)^{j+r} \leq (3g_m)^{j+r} \leq m^j.
$$

To see the last inequality, observe that the equivalent statement

$$
(j+r)\ln(3g_m) \leq j \ln m \quad \Longleftrightarrow \quad \frac{j+r}{j} \leq \frac{\ln m}{\ln(3g_m)}
$$

is obvious.

Consequently, we get

$$
|H(g_m, x, m+x)| \leq \frac{\mathrm{const}}{|m^{s-1}|},
$$

where the constant depends on r only. Since $\Re s > 1$, we see $\lim_{m \to \infty} H(g_m, x, m + x) = 0$.

Part (d) of the theorem follows from general considerations: As we know, the function $\mathcal{H}^g(s, x)$ is smooth with respect to both variables and holomorphic with respect to $s \in \mathbf{C} \setminus \{1\}$. For fixed $x > 1$, the function $s \to \mathcal{H}^g(s, x) - x^{-s+1}/(s - 1)$ is holomorphic in $s \in \mathbf{C}$ and is $\zeta(s, x)$ for $\Re s > 1$. Because the holomorphic extension is unique, therefore we get independence of $g = \{g_n\}$.

5. Applications: series representations of the Dirichlet L-functions. In this section we apply the theorems from Section 3 to the Dirichlet L-functions. That such simple evaluations are possible was observed earlier by Stark in [**48**]. Actually the Hurwitz zeta function can be considered as a generalization of both the Riemann zeta function and the Dirichlet L-functions, see [**3**, p. 249]. Here we remind the reader in brief of the definition of the Dirichlet L-functions.

Let χ be a Dirichlet character. Then the Dirichlet L-function is defined by

$$(5.1) \qquad L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

if $\Re s > 1$.

If χ is a character mod m, then we can rearrange the terms in the series for $L(s, \chi)$ according to the residue classes mod m, and we get

$$n = qm + r, \quad \text{where} \quad 1 \le r \le m \quad \text{and} \quad q = 0, 1, 2, \dots,$$

$$
\begin{aligned}
L(s, \chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{r=1}^{m} \sum_{q=0}^{\infty} \frac{\chi(qm + r)}{(qm + r)^s} \\
&= m^{-s} \sum_{r=1}^{m} \chi(r) \zeta\left(s, \frac{r}{m}\right),
\end{aligned}
$$

(5.2)

where $\zeta(s, x)$ is the Hurwitz zeta function. We already dealt with the case when χ was the principal character, namely this case corresponds

to the standard Riemann zeta function. Therefore now we assume that χ is nonprincipal, and as a consequence we get, see [**3**, p. 140], that

$$(5.3) \qquad \sum_{j=1}^{m} \chi(j) = 0.$$

Actually, following [**48**], we shall consider an even more general situation, namely when χ is a periodic function on $\mathbf{Z}$ with period m and moreover χ satisfies (5.3).

In all statements below we assume that χ is a nonprincipal character. In this case $L(s, \chi)$ is an entire function, see [**3**, Theorem 12.5]. Nevertheless, because in the formulas below we divide by $s - 1$, we exclude the value $s = 1$.

Theorem 5.1. *For any natural number k, the Dirichlet L-function corresponding to character χ mod m can be represented as the series*

$$(5.4) \quad L(s, \chi) = \sum_{r=1}^{m} \chi(r) r^{-s} + \frac{m^{-s}}{s-1} \sum_{r=1}^{m} \chi(r) H\left(k, s, \frac{r}{m}\right) + m^{-s}$$
$$\times \sum_{r=1}^{m} \chi(r) \sum_{n=1}^{\infty} \left[\frac{1}{(n+(r/m))^s} + \frac{1}{s-1} \left[H\left(k, s, n + \frac{r}{m}\right) \right.\right.$$
$$\left.\left. - H\left(k, s, n + \frac{r}{m} - 1\right) \right] \right],$$

where the series converges for $\Re s > -2k - 1$, $s \neq 1$.

Theorem 5.2. *There is a nondecreasing integer-valued sequence $\{g_n\}_{n=0}^{\infty}$, $g_0 = 1$, such that the Dirichlet L-function corresponding to character χ mod m can be represented as the series*

$$L(s, \chi) = \sum_{r=1}^{m} \chi(r) r^{-s} + \frac{m^{-s}}{s-1} \sum_{r=1}^{m} \chi(r) H\left(1, s, \frac{r}{m}\right) + m^{-s}$$
$$\times \sum_{r=1}^{m} \chi(r) \sum_{n=1}^{\infty} \left[\frac{1}{(n+(r/m))^s} + \frac{1}{s-1} \left[H\left(g_n, s, n + \frac{r}{m}\right) \right.\right.$$
$$\left.\left. - H\left(g_{n-1}, s, n + \frac{r}{m} - 1\right) \right] \right],$$

where the series (in the second line) converges in $\mathbf{C} \setminus \{1\}$.

The other consequences for the Dirichlet L-function come from Theorems 4.1 and 4.2 when we write more explicitly the differences of the H functions.

Corollary 5.1. *For any natural number k, the Dirichlet L-function corresponding to character χ mod m can be represented as the series*

(5.5)

$$
L(s,\chi) = \sum_{r=1}^{m} \chi(r) r^{-s} + \frac{m^{-s}}{s-1} \sum_{r=1}^{m} \chi(r) H\left(k, s, \frac{r}{m}\right) - m^{-s}
$$

$$
\times \sum_{r=1}^{m} \left[\chi(r) \sum_{n=1}^{\infty} \sum_{\beta=2k+2}^{\infty} \left[s\ldots(s+\beta-1) \sum_{j=0}^{2k} B_j \frac{1}{j!\,(\beta-j+1)!} \right] \right.
$$

$$
\left. \times \frac{1}{(n+(r/m))^{\beta+s}} \right],
$$

where the series converges for $\Re s > -2k-1$, $s \neq 1$.

Corollary 5.2. *There is a nondecreasing integer-valued sequence $\{g_n\}_{n=0}^{\infty}$, $g(0)=1$, such that the Dirichlet L-function corresponding to character χ mod m can be represented as the series*

$$
L(s,\chi) = \sum_{r=1}^{m} \chi(r) r^{-s} + \frac{m^{-s}}{s-1} \sum_{r=1}^{m} \chi(r) H\left(1, s, \frac{r}{m}\right)
$$

$$
+ m^{-s} \sum_{r=1}^{m} \left\{ \chi(r) \sum_{n=1}^{\infty} \left[\sum_{\beta=2g_{n-1}+2}^{2g_n} B_\beta \frac{s\ldots(s+\beta-2)}{\beta!} \right. \right.
$$

$$
\left. \left. \times \frac{1}{(n+(r/m))^{s+\beta-1}} \right] \right\}
$$

$$
- m^{-s} \sum_{r=1}^{m} \left[\chi(r) \sum_{n=1}^{\infty} \sum_{\beta=2g_{n-1}+2}^{\infty} \left[s\ldots(s+\beta-1) \right. \right.
$$

$$
\left. \left. \times \sum_{j=0}^{2g_{n-1}} B_j \frac{1}{j!\,(\beta-j+1)!} \right] \frac{1}{(n+(r/m))^{\beta+s}} \right],
$$

where the series converges for all $s \in \mathbf{C} \setminus \{1\}$.

As some applications of the above formulas for the Dirichlet L-function, we compute the values of $L(s, \chi)$ for negative integers s. Take any nonnegative integer a and for this integer fix a natural number k such that $a < 2k + 1$. When we plug $s = -a$ into (5.5), we see that the infinite sum on the right-hand sides of this formula vanishes. Thus, the formula for L becomes

$$(5.6) \qquad L(-a, \chi) = \sum_{r=1}^{m} \chi(r) r^a - \frac{m^a}{a+1} \sum_{r=1}^{m} \chi(r) H\left(k, -a, \frac{r}{m}\right).$$

Using (3.5), the formula (5.6) can be written as

$$(5.7) \qquad L(-a, \chi) = \sum_{r=1}^{m} \chi(r) r^a - \frac{m^a}{a+1} \sum_{r=1}^{m} \chi(r) B_{a+1}\left(\frac{r}{m} + 1\right).$$

In particular, if $a = 0$, we get

$$(5.8) \qquad L(0, \chi) = -\sum_{r=1}^{m} \chi(r)\left(\frac{r}{m} + \frac{1}{2}\right) = -\frac{1}{m} \sum_{r=1}^{m} r\,\chi(r)$$

since the character is nonprincipal.

If $a = 1$, we get

$$\begin{aligned}
L(-1, \chi) &= \sum_{r=1}^{m} r\,\chi(r) - \frac{m}{2} \sum_{r=1}^{m} \chi(r) B_2\left(\frac{r}{m} + 1\right) \\
&= \sum_{r=1}^{m} r\,\chi(r) - \frac{m}{2} \sum_{r=1}^{m} \chi(r)\left(\frac{r^2}{m^2} + \frac{r}{m} + \frac{1}{6}\right) \\
&= \sum_{r=1}^{m} r\,\chi(r) - \frac{1}{2m} \sum_{r=1}^{m} r^2 \chi(r) - \frac{1}{2} \sum_{r=1}^{m} r\,\chi(r) \\
&= -\frac{1}{2m} \sum_{r=1}^{m} r^2 \chi(r) + \frac{1}{2} \sum_{r=1}^{m} r\,\chi(r).
\end{aligned}$$

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