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DYNAMICS OF PERMUTABLE TRANSCENDENTAL ENTIRE FUNCTIONS

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ABSTRACT. Let f and g be two permutable transcendental entire functions. Assume that f has the form

$$f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)}.$$

We shall investigate the dynamical properties of f and g and show that they have the same Julia sets and Fatou sets, i.e., $J(f) = J(g)$. This relates to an open question due to Baker.

1. Introduction and main results. Let $f(z)$ be a transcendental entire function, and denote by f^n, $n \in N$, the nth iterate of f. The set of normality, $F(f)$, is defined to be the set of points, $z \in \mathbf{C}$, such that the sequence $\{f^n\}$ is normal in some neighborhood of z, and $J = J(f) = \mathbf{C} - F(f)$. $F(f)$ and $J(f)$ are called the Fatou set and Julia set of f, respectively. Clearly $F(f)$ is open. It is well-known that $J(f)$ is a nonempty perfect set which coincides with $\mathbf{C}$, or is nowhere dense in $\mathbf{C}$. For the basic results in the dynamical system theory of transcendental functions, we refer the reader to books [**12, 17**], the survey paper [**2**] and the papers of Fatou [**9**] and Julia [**13**].

In what follows, we shall use the following standard notations:

$$M(r,f) = \max\{|f(z)| : |z| = r\},$$
$$m(r,f) = \min\{|f(z)| : |z| = r\},$$
$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$
$$\rho = \rho(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

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We call them maximum modulus, minimum modulus, order of f and lower order of f, respectively. We will use $T(r, f)$ to denote the Nevanlina characteristic of f, see for example [**11**] for an introduction to Nevanlina theory.

A point a is called a singular value of f if it is either a critical value or an asymptotic value of f. We denote by $\text{sing}\,(f^{-1})$ the set of all finite singular values of f:

$$\text{sing}\,(f^{-1}) = \{z \in \mathbf{C} : \ z \text{ is a singularity of } f^{-1}\}.$$

If the set $\text{sing}\,(f^{-1})$ is bounded, then we say f is of bounded type. In particular, if the set sing.(f^{-1}) is finite, then f is of finite type, and we denote this by $f \in B$ and $f \in S$, respectively [**2**].

Let f and g denote two meromorphic functions. If

$$f(g) = g(f), \tag{1}$$

then we call f and g *permutable*.

Theorem A [**8, 21**]. *Let R_1 and R_2 be two permutable rational functions. Then*

1. $F(R_1) = F(R_2)$ *and* $J(R_1) = J(R_2)$;

2. *if D is an attractive domain, a parabolic domain or a Siegel disk of period m of R_1, then it is also an attractive domain, a parabolic domain or a Siegel disk of period m of R_2, respectively.*

Question 1 (Baker [**1**]). For two given distinct permutable transcendental entire functions f and g, does it follow that $F(f) = F(g)$?

This is a difficult question to answer. So far, some answers to several special cases or classes of functions of f and g are obtained. Firstly, we recall the following two known results.

Theorem B ([**1, 19**]). *Suppose that f and g are distinct permutable transcendental entire functions, and $g = af + b$ for some constant $a \neq 0$. Then $F(f) = F(g)$.*

Theorem C [**20**]. *Let* $f, g \in S$ *and* $f \circ g = g \circ f$. *Then*

1. $J(f) = J(g)$;

2. *If* D *is a superattractive stable domain, an attractive stable domain, a parabolic stable domain or a Siegel disk of* f, *then* D *is also a superattractive stable domain, an attractive stable domain, a parabolic stable domain or a Siegel disk of* g, *respectively.*

Theorem D [**22**]. *Let* f *and* g *be two distinct permutable transcendental entire functions and* $q(z)$ *be a non-constant polynomial. Suppose that* $q(g) = aq(f) + b$, $a(\neq 0), b \in \mathbf{C}$. *Then* $J(f) = J(g)$.

Theorem E [**16**]. *If* f *and* g *are two permutable transcendental entire functions, and there exists a non-constant polynomial* $\Phi(x, y)$ *in both* x *and* y *such that* $\Phi(f(z), g(z)) \equiv 0$, *then* $J(f) = J(g)$.

Theorem F [**16**]. *Let* f *and* g *be two permutable transcendental entire functions with* $\lambda(g) < \infty$. *If* $f(z) = p(z) + p_1(z)e^{q(z)}$, *where* $p(z)$, $p_1(z)$ *and* $q(z)$ *are polynomials, then* $g(z) = cf(z) + d$ *for some two constants* $c \neq 0$ *and* d.

From this theorem and Theorem B, we can easily get

Theorem 1. *Let* f *and* g *be two permutable transcendental entire functions with* $\lambda(g) < \infty$, $p(z)$, $q(z)$ *and* $r(z)$ *be three polynomials. Suppose that*

$$f(z) = p(z) + q(z)e^{r(z)}.$$

Then $J(f) = J(g)$.

References [**14, 15, 18, 19, 24**] also studied the dynamics of two transcendental entire functions.

In this paper, we shall prove the following results.

Theorem 2. *Let* f *and* g *be two permutable transcendental entire functions with* $\lambda(g) < \infty$. *Let* $p(z)$ *and* $q_i(z)$, $i = 1, 2$, *be nonconstant* *polynomials and* $p_1(z) \not\equiv 0$ *and* $p_2(z) \not\equiv 0$ *be polynomials. Assume that*

$$f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)}. \tag{2}$$

Then $J(f) = J(g)$.

Example 1. Let $f(z) = z + \gamma \sin z$, $g_1(z) = z + \gamma \sin z + 2k\pi$ and $g_2(z) = -z - \gamma \sin z + 2k\pi$. Then $f \circ g_1 = g_1 \circ f$ and $f \circ g_2 = g_2 \circ f$. Here $\gamma(\neq 0) \in \mathbf{C}$ and $k \in \mathbf{Z}$.

Example 2. Let $f(z) = z + \gamma e^z$, $g(z) = z + \gamma e^z + 2k\pi i$. Then $f \circ g = g \circ f$. Here $\gamma(\neq 0) \in \mathbf{C}$ and $k \in \mathbf{Z}$.

When $p(z)$ is a constant, we have the following result.

Theorem 3. *Let* f *and* g *be two permutable transcendental entire functions with* $\lambda(g) < \infty$ *and*

$$f(z) = p + p_1 e^{q_1(z)} + p_2 e^{q_2(z)}. \tag{3}$$

Let $q_i(z)(i = 1, 2)$ *be nonconstant polynomials such that* q_1'/q_2' *is not constant. Assume that* p, $p_1 \neq 0$ *and* $p_2 \neq 0$ *are three constants. Then* $J(f) = J(g)$.

Proof of Theorem 2.

Lemma 1 [**10**]. *Let* $G_0, G_1, \dots, G_m$ *and* f *be nonconstant entire functions, and let* $h_0, h_0, \dots, h_m$, $m \geq 1$, *be nonzero meromorphic functions. Suppose that* K *is a positive number and* $\{r_i\}$ *is an unbounded monotone increasing sequence of positive numbers such that, for each* $j \geq 1$,

$$T(r_j, h_i) \leq KT(r_j, f), \quad i = 0, \dots, m,$$
$$T(r_j, f') \leq (1 + o(1))T(r_j, f).$$

If

$$h_0 G_0(f) + h_1 G_1(f) + \cdots + h_m G_m(f) \equiv 0,$$

then there exist polynomials $\{p_j\}$, $j = 0, 1, \dots, m$, *not all identically zero such that*

$$p_0(z)G_0(z) + p_1(z)G_1(z) + \cdots + p_m(z)G_m(z) \equiv 0.$$

Lemma 2 [**5**]. *Let* $f_j(z)$, $j = 1, 2, 3, \dots n$, *and* $g_j(z)$, $j = 1, 2, 3, \dots n$, $n \geq 2$, *be two systems of entire functions satisfying the following conditions*:

1. $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
2. *for* $1 \leq j, k \leq n$, $j \neq k$, $g_j(z) - g_k(z)$ *is nonconstant*;
3. *for* $1 \leq h, k \leq n$, $h \neq k$, $1 \leq j \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$.

Then $f_j(z) \equiv 0 (j = 1, 2, 3, \dots, n)$.

Lemma 3 [**23**]. *Let* f *and* g *be two permutable entire functions satisfying*

1. $\lambda(f) < \infty$ *and* $\lambda(g) < \infty$;
2. $\rho(f) > 0$.

Then there exists a sequence $\{r_j\}$ *tending to* ∞ *and a positive constant* K *so that*

$$T(r_j, g') \leq KT(r_j, f) \quad \text{and} \quad T(r_j, g'') \leq KT(r_j, f).$$

Proof of Theorem 2. If $q_1(z) - q_2(z)$ is identically constant, then Theorem 2 reduces to Theorem 1. Next we assume that with

$$q_1(z) - q_2(z) \not\equiv \text{constant}.$$

Note that $\rho(f) = \lambda(f) = \max\{\deg(q_1),\ \deg(q_2)\}$. From (1) we have

$$f'(g) = \frac{f'}{g'}\, g'(f) \tag{4}$$

and, hence,

$$f''(g) = \frac{f''g' - f'g''}{g'^3}\, g'(f) + \left(\frac{f'}{g'}\right)^2 g''(f). \tag{5}$$

From

$$f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)} \tag{6}$$

we get

$$f'(z) = p'(z)+[p_1'(z)+p_1(z)q_1'(z)]e^{q_1(z)}+[p_2'(z)+p_2(z)q_2'(z)]e^{q_2(z)} \tag{7}$$

and

$$\begin{aligned} f''(z) = p''(z) &+ [p_1''(z) + 2p_1'(z)q_1'(z) + p_1(z)q_1''(z) + p_1(z)q_1'(z)^2]e^{q_1(z)} \\ &+ [p_2''(z) + 2p_2'(z)q_2'(z) + p_2(z)q_2''(z) + p_2(z)q_2'(z)^2]e^{q_2(z)}. \end{aligned} \tag{8}$$

By eliminating the factors $e^{q_1(z)}$ and $e^{q_2(z)}$ from the three equations (6), (7) and (8), we derive

$$P_2(z)f''(z) + P_1(z)f'(z) + P_0(z)f(z) + P(z) = 0, \tag{9}$$

where

$$P_2 = p_1p_2' - p_1'p_2 - p_1p_2(q_1' - q_2'), \tag{10}$$

$$\begin{aligned} P_1 &= -p_1p_2'' + p_1''p_2 - 2p_1p_2'q_2' + 2p_1'p_2q_1' + p_1p_2(q_1'' - q_2'') \\ &\quad + p_1p_2(q_1'^2 - q_2'^2) \\ &= -P_2' - P_2(q_1' + q_2'), \end{aligned} \tag{11}$$

$$\begin{aligned} P_0 = &-(p_1'' + 2p_1'q_1' + p_1q_1'' + p_1q_1'^2)(p_2' + p_2q_2') \\ &+ (p_2'' + 2p_2'q_2' + p_2q_2'' + p_2q_2'^2)(p_1' + p_1q_1'), \end{aligned} \tag{12}$$

$$\begin{aligned} P = &\ p''P_2 + (p_1'' + 2p_1'q_1' + p_1'q_1'' + p_1q_1'^2)[(p_2' + p_2q_2')p - p_2p'] \\ &+ (p_2'' + 2p_2'q_2' + p_2'q_2'' + p_2q_2'^2)[-(p_1' + p_1q_1')p + p_1p']. \end{aligned} \tag{13}$$

Claim 1. $P_2 \not\equiv 0$.

Proof of Claim 1. In fact, if, on the contrary, $P_2 \equiv 0$, then

$$p_1p_2' - p_1'p_2 = p_1p_2(q_1' - q_2'),$$

this contradicts the fact that $q_1(z) - q_2(z) \not\equiv$ constant. Claim 1 follows. $\square$

Replacing z by $g(z)$ in equation (9) yields

$$P_2(g)f''(g) + P_1(g)f'(g) + P_0(g)f(g) + P(g) = 0.$$

Combining this with (1), (4) and (5), we get

$$(14)\quad P_2(g)\left(\frac{f'}{g'}\right)^2 g''(f) + \left[P_2(g)\frac{f''g' - f'g''}{g'^3} + P_1(g)\frac{f'}{g'}\right]g'(f) + P_0(g)g(f) + P(g) = 0.$$

By Lemmas 1 and 3, there exist four polynomials $Q(z)$, $Q_0(z)$, $Q_1(z)$ and $Q_2(z)$, not all identically zero, such that

$$(15)\qquad Q_2(z)g''(z) + Q_1(z)g'(z) + Q_0(z)g(z) + Q(z) = 0.$$

Substituting z by $f(z)$ in this equation, we get

$$(16)\qquad Q_2(f)g''(f) + Q_1(f)g'(f) + Q_0(f)g(f) + Q(f) = 0.$$

Eliminating the term $g''(f)$ from this and (14), we have

$$(17)\qquad H_1 g'(f) + H_0 g(f) + H = 0,$$

where

$$(18)\quad H_1 = Q_1(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)\left[P_2(g)\,\frac{f''g' - f'g''}{g'^3} + P_1(g)\frac{f'}{g'}\right],$$

$$(19)\quad H_0 = Q_0(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P_0(g),$$

$$(20)\quad H = Q(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P(g).$$

From (1), (4) and (17) we deduce that

$$(21)\qquad H_1\frac{g'}{f'}\,f'(g) + H_0 f(g) + H = 0.$$

Replacing z by $g(z)$ in the equations (6) and (7) first and then substituting them into (21), we obtain that

$$H_1 \frac{g'}{f'} p'(g) + H_0 p(g) + H + \left[H_1 \frac{g'}{f'} (p_1'(g) + p_1(g) q_1'(g)) + H_0 p_1(g) \right] \times e^{q_1(g)} + \left[H_1 \frac{g'}{f'} (p_2'(g) + p_2(g) q_2'(g)) + H_0 p_2(g) \right] e^{q_2(g)} = 0.$$

It follows from Lemmas 2 and 3 that

$$H_1 \frac{g'}{f'} (p_1'(g) + p_1(g) q_1'(g)) + H_0 p_1(g) = 0 \tag{22}$$

and

$$H_1 \frac{g'}{f'} (p_2'(g) + p_2(g) q_2'(g)) + H_0 p_2(g) = 0. \tag{23}$$

Claim 2. $H_1 \equiv 0$.

Proof of Claim 2. If $H_1 \not\equiv 0$, then from (22) and (23) we get

$$\frac{p_1'(g) + p_1(g) q_1'(g)}{p_1(g)} = \frac{p_2'(g) + p_2(g) q_2'(g)}{p_2(g)} \quad \text{if} \quad H_0 \neq 0$$

or

$$(p_1(z) e^{q_1(z)})' = 0 \quad \text{and} \quad (p_2(z) e^{q_2(z)})' = 0 \quad \text{if} \quad H_0 = 0.$$

Thus

$$\frac{p_1'(z)}{p_1(z)} + q_1'(z) = \frac{p_2'(z)}{p_2(z)} + q_2'(z) \tag{24}$$

or $p_1(z)e^{q_1(z)} = c_1$ and $p_2(z)e^{q_2(z)} = c_2$ for some constants c_1 and c_2, which is a contradiction. But, from (24), we have $p_1(z)e^{q_1(z)} = cp_2(z)e^{q_2(z)}$ for some constant c. This obviously contradicts to the assumptions of the theorem. Claim 2 follows. □

By Claim 2, (17) becomes $H_0 f(g) + H = 0$. It follows from Lemmas 2 and 3 again that $H_0 \equiv H \equiv 0$. Hence,

(25)
$$Q_1(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)\left[P_2(g)\frac{f''g' - f'g''}{g'^3} + P_1(g)\frac{f'}{g'}\right] = 0$$

(26)
$$Q_0(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P_0(g) = 0$$

and

(27)
$$Q(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P(g) = 0.$$

Claim 3. $P_0 \not\equiv 0$.

Proof of Claim 3. If $P_0 \equiv 0$, then from (12) we deduce that

$$\frac{(p_1' + p_1 q_1')'}{p_1' + p_1 q_1'} - \frac{(p_2' + p_2 q_2')'}{p_2' + p_2 q_2'} = q_1' - q_2',$$

which yields

$$\frac{p_1' + p_1 q_1'}{p_2' + p_2 q_2'} = ce^{q_1 - q_2}$$

for some nonzero constant c; this implies that $q_1 - q_2$ is a constant, a contradiction. Claim 3 follows. □

Claim 4. $Q_2 \not\equiv 0$.

Proof of Claim 4. Suppose on the contrary that $Q_2 \equiv 0$. From Claim 1 we know that $P_2 \not\equiv 0$, then from (26) and (27) we get that $Q_0 \equiv Q \equiv 0$, and therefore $Q_1 \equiv 0$ from (15), a contradiction. Claim 4 follows. □

Claim 5. $Q_0 \not\equiv 0$.

Proof of Claim 5. This follows from (26), Claim 3 and Claim 4. □

Note that the term with the highest degree in (12) is $-p_1p_2q_1'q_2'(q_1'-q_2')$, and the term with the highest degree in (13) is $pp_1p_2q_1'q_2'(q_1'-q_2')$. Since $p(z)\not\equiv a$ constant, it follows from (12) and (13) that $P(z)\not\equiv 0$ and $P_0(z)/P(z)$ is not constant, and so, by (27), $Q(z)\not\equiv 0$. From (26) and (27), we have

$$\frac{Q_0(f)}{Q(f)}=\frac{P_0(g)}{P(g)}. \tag{28}$$

We rewrite this as

$$\frac{Q_0(f)P(g)-Q(f)P_0(g)}{Q(f)P(g)}=0$$

and consider two subcases.

If $Q_0(x)P(y)-Q(x)P_0(y)$ is identically constant, then the constant will be zero by the above equation. Thus,

$$Q_0(x)P(y)=Q(x)P_0(y)$$

for any x and y. In particular,

$$\frac{Q_0(z)}{Q(z)}=\frac{P_0(z)}{P(z)}:=R(z)$$

for a rational function $R(z)$. It follows from (28) that

$$R(f)=R(g).$$

Therefore, $f=\pm g+c$ for a constant c. By Theorem D, we get the conclusion $J(f)=J(g)$.

If $Q_0(x)P(y)-Q(x)P_0(y)\not\equiv$ constant, then the conclusion follows from this, (1) and Theorem F. ◻

3. Proof of Theorem 3. Now we consider the case where p, $p_1\neq 0$ and $p_2\neq 0$ are three constants. From (12) and (13), we have

$$P(z)\equiv -pP_0(z).$$

By (28),

$$Q(z) \equiv -pQ_0(z).$$

From (26), we get

$$\left(\frac{f'}{g'}\right)^2 = \frac{Q_2(f)P_0(g)}{Q_0(f)P_2(g)}. \tag{29}$$

By differentiating this equality, we derive that

$$\begin{aligned}\frac{f''g' - f'g''}{g'^3} &= \frac{[Q_2'(f)Q_0(f) - Q_2(f)Q_0'(f)]P_0(g)}{2[Q_0(f)]^2P_2(g)} \\ &\quad + \frac{[P_0'(g)P_2(g) - P_0(g)P_2'(g)]Q_2(f)}{2Q_0(f)[P_2(g)]^2} \cdot \frac{g'}{f'} \\ &= R_1(f,g) + R_2(f,g) \cdot \frac{g'}{f'}\end{aligned} \tag{30}$$

where

$$R_1(f,g) = \frac{[Q_2'(f)Q_0(f) - Q_2(f)Q_0'(f)]P_0(g)}{2[Q_0(f)]^2P_2(g)} \tag{31}$$

and

$$R_2(f,g) = \frac{[P_0'(g)P_2(g) - P_0(g)P_2'(g)]Q_2(f)}{2Q_0(f)[P_2(g)]^2} \tag{32}$$

are two rational functions of f and g. Substituting (29) and (30) into (25), we obtain that

$$\begin{aligned}&\frac{Q_1(f)Q_2(f)P_0(g)}{Q_0(f)} - Q_2(f)P_2(g)R_1(f,g) \\ &\qquad = Q_2(f)P_2(g)R_2(f,g) \cdot \frac{g'}{f'} + P_1(g)Q_2(f) \cdot \frac{f'}{g'}.\end{aligned} \tag{33}$$

Now squaring both sides of (33) and then substituting (29) into it, we derive that

$$\begin{aligned}&\left[\frac{Q_1(f)Q_2(f)P_0(g)}{Q_0(f)} - Q_2(f)P_2(g)R_1(f,g)\right]^2 \\ &\qquad = \frac{Q_0(f)Q_2(f)[P_2(g)]^3[R_2(f,g)]^2}{P_0(g)} \\ &\qquad\quad + 2P_1(g)P_2(g)[Q_2(f)]^2R_2(f,g) \\ &\qquad\quad + \frac{[P_1(g)]^2[Q_2(f)]^3P_0(g)}{Q_0(f)P_2(g)}.\end{aligned} \tag{34}$$

Substituting (31) and (32) into (34), then simplifying and rearranging terms, we obtain that

$$\begin{aligned}(35)\quad &\left\{2Q_0(f)Q_1(f) - [Q_2'(f)Q_0(f) - Q_2(f)Q_0'(f)]\right\}^2 P_0(g)^3 P_2(g)\\ &= \left\{2P_0(g)P_1(g) - [P_2'(g)P_0(g) - P_2(g)P_0'(g)]\right\}^2 Q_0(f)^3 Q_2(f).\end{aligned}$$

Let

(36)

$$\begin{aligned}R(x,y) = &\left\{2Q_0(x)Q_1(x) - [Q_2'(x)Q_0(x) - Q_2(x)Q_0'(x)]\right\}^2 P_0(y)^3 P_2(y)\\ &- \left\{2P_0(y)P_1(y) - [P_2'(y)P_0(y) - P_2(y)P_0'(y)]\right\}^2 Q_0(x)^3 Q_2(x).\end{aligned}$$

Then

$$(37)\qquad R(f,g) = 0.$$

If $R(x,y) \not\equiv$ constant, then the conclusion follows from Theorem F. So what we need to do is to show that $R(x,y) \not\equiv$ constant.

Claim 6. $R(x,y) \not\equiv$ constant.

Proof of Claim 6. If on the contrary $R(x,y) \equiv$ constant, then by (37), $R(x,y) \equiv 0$, and therefore

$$\begin{aligned}(38)\quad &\frac{\{2Q_0(x)Q_1(x) - [Q_2'(x)Q_0(x) - Q_2(x)Q_0'(x)]\}^2}{Q_0(x)^3 Q_2(x)}\\ &\qquad \equiv \frac{\{2P_0(y)P_1(y) - [P_2'(y)P_0(y) - P_2(y)P_0'(y)]\}^2}{P_0(y)^3 P_2(y)}.\end{aligned}$$

If the left-hand side is a nonconstant rational function of x, then there exist two different values a and b, and two different roots x_1 and x_2 such that

$$\frac{\{2Q_0(x_1)Q_1(x_1) - [Q_2'(x_1)Q_0(x_1) - Q_2(x_1)Q_0'(x_1)]\}^2}{Q_0(x_1)^3 Q_2(x_1)} \equiv a$$

and

$$\frac{\{2Q_0(x_2)Q_1(x_2) - [Q_2'(x_2)Q_0(x_2) - Q_2(x_2)Q_0'(x_2)]\}^2}{Q_0(x_2)^3 Q_2(x_2)} \equiv b.$$

It follows from (38) that

$$a \equiv \frac{\{2P_0(y)P_1(y) - [P_2'(y)P_0(y) - P_2(y)P_0'(y)]\}^2}{P_0(y)^3 P_2(y)}$$

and

$$b \equiv \frac{\{2P_0(y)P_1(y) - [P_2'(y)P_0(y) - P_2(y)P_0'(y)]\}^2}{P_0(y)^3 P_2(y)};$$

this is a contradiction. Therefore, the left-hand side of (38) is a constant, say c, and we have, by (38),

$$\frac{\{2P_0(y)P_1(y) - [P_2'(y)P_0(y) - P_2(y)P_0'(y)]\}^2}{P_0(y)^3 P_2(y)} \equiv c.$$

Eliminating $P_1(y)$ by substituting (11) into the above equation, we get

$$\left[\frac{P_0'(y)}{P_0(y)} - 3\frac{P_2'(y)}{P_2(y)} - 2(q_1'(y) + q_2'(y))\right]^2 = c\,\frac{P_0(y)}{P_2(y)}. \tag{39}$$

Note that p, $p_1 \neq 0$ and $p_2 \neq 0$ are three constants. We deduce from (10) and (12) that

$$P_0 = -\,p_1 p_2 q_1' q_2'(q_1' - q_2'), \quad P_2 = -\,p_1 p_2 q_1' q_2'.$$

Substituting these into (39), we have

$$\left[\frac{q_1''(y)}{q_1'(y)} + \frac{q_2''(y)}{q_2'(y)} - 2\frac{q_1''(y) - q_2''(y)}{q_1'(y) - q_2'(y)} - 2(q_1'(y) + q_2'(y))\right]^2 = c\,q_1'(y)q_2'(y). \tag{40}$$

Note that

$$\frac{q_1''(y)}{q_1'(y)} + \frac{q_2''(y)}{q_2'(y)} - 2\frac{q_1''(y) - q_2''(y)}{q_1'(y) - q_2'(y)}$$

is a rational function and is of the form

$$\frac{a_1}{y - y_1} + \cdots + \frac{a_k}{y - y_k},$$

note also that $(q_1(y)+q_2(y))'$ and $q_1'(y)q_2'(y)$ are polynomials, it follows from (40) that

$$\frac{q_1''(y)}{q_1'(y)} + \frac{q_2''(y)}{q_2'(y)} - 2\frac{q_1''(y) - q_2''(y)}{q_1'(y) - q_2'(y)} \equiv 0.$$

Substituting this into (40) implies that

$$[-2(q_1'(y) + q_2'(y))]^2 = c\, q_1'(y) q_2'(y).$$

This implies that q_1'/q_2' is a constant, which contradicts the assumption of the theorem. □

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