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THE FROBENIUS NUMBER AND a-INVARIANT

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ABSTRACT. We will give two different proofs for the fact that the Frobenius number of a numerical semigroup is the a-invariant of the semigroup algebra associated to it. These give rise to two different algorithms for computing the Frobenius number.

1. Introduction. Let $\mathcal{A} = \{w_1, \dots, w_n\}$ be a set of strictly positive integers and Q a subsemigroup of N generated by $\mathcal{A}$, i.e.,

$$Q = \langle \mathcal{A} \rangle = \mathrm{N}w_1 + \cdots + \mathrm{N}w_n.$$

We say that Q is numerical if the greatest common divisor of $\mathcal{A}$, gcd $(\mathcal{A})$, is equal to 1, or equivalently $\mathrm{N} \setminus Q$ is a finite set [**9**, Exercise 10.2.4].

For the numerical semigroup Q the largest integer f^* not in Q is called the Frobenius number of Q, and the problem of finding this number is called the Frobenius problem. In other words, the problem is finding the largest integer f^* which cannot be written as a nonnegative integral combination of the w_i's. Thus the Frobenius number is concerned with a family of linear equations $\sum w_i x_i = f$, as f varies over all positive integers. The Frobenius problem has been examined by many authors ([**5, 6, 7**]).

Let k be a field, $k[\mathbf{x}] := k[x_1, \dots, x_n]$ the polynomial ring over k, $A := [w_1, \dots, w_n]$ an integer $1 \times n$-matrix whose entries generate the numerical semigroup Q, B an integer $n \times (n-1)$-matrix whose columns generate the lattice

$$\mathcal{L}_B := \mathrm{Ker}_{\mathrm{z}} A := \{u \in \mathrm{Z}^n : \ Au = 0\},$$

and $k[Q] \simeq k[t^{w_1}, \dots, t^{w_n}]$ the semigroup algebra associated to Q. For every $u \in \mathrm{Z}^n$ we define the body

$$P_u := \{v \in \mathrm{R}^{n-1} : \ Bv \leq u\}\cdot$$

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Since the matrix B is homogeneous, this body is polytope [**6**, Proposition 2.1]. Two polytopes P_u and $P_{u'}$ are lattice translates of each other if $u - u' \in \mathcal{L}_B$. Disregarding lattice equivalence we write $P_C := P_u$. P_C is called the polytope of fiber $C \in \mathrm{N}^n/\mathcal{L}_B$ [**6**, Construction 2.2]. The polytope P_C is said to be a *maximal lattice point free polytope* if it contains no lattice points in its interior, but every facet of it contains at least one lattice point in its relative interior. We denote by $T(B)$ the set of all maximal lattice point free polytopes P_C associated to the matrix B.

In [**7**] the authors proved that

$$f^* = \max\{Au : \ P_u \text{ is a maximal lattice point free polytope } \} - \sum_{i=1}^{n} w_i,$$

where u varies over all integral vectors.

In this paper we will translate this formula to its algebraic counterpart, Theorem 2.1. In fact, we will prove by two different methods that for a numerical semigroup Q

$$f^*(Q) = a(k[Q]),$$

that is the Frobenius number of Q is the a-invariant of the semigroup algebra $k[Q]$. We will use these methods to give two algorithms for computing f^*.

In Section 2, we will give our first proof and algorithm which are based on the highest minimal syzygies of the semigroup algebra $k[Q]$.

In Section 3, we will give our second proof and algorithm which are based on the Hilbert-Poincaré series of the semigroup algebra $k[Q]$.

2. The highest minimal syzygies. For each monomial $\mathbf{x}^u \in k[\mathbf{x}]$, we define $\deg_Q(\mathbf{x}^u) = Au$. Then the ring $k[\mathbf{x}]$ will have a Q-graded structure. The defining ideal of the semigroup Q is the toric ideal I_A [**8**, Chapter 4] associated to A. The ring $k[\mathbf{x}]$ is *local [**2**, Definition 1.5.13] and we can consider the minimal Q-graded free resolution of $k[Q]$ over $k[\mathbf{x}]$. The highest minimal syzygies of $k[Q]$ over $k[\mathbf{x}]$ are those which are of the highest homological degree. Since $k[Q]$ is one dimensional domain it is Cohen-Macaulay k-algebra. Thus the projective dimension of $k[Q]$

is $codim(I_A)$ and all the syzygies of homological degree codim (I_A) will be the highest minimal syzygies of $k[Q]$ over $k[\mathbf{x}]$.

Theorem 2.1. *With the above notations, suppose $g_1, \dots, g_t$ are the highest minimal syzygies of $k[Q]$ over $k[\mathbf{x}]$ where t is Cohen-Macaulay type of $k[Q]$. Then*

$$f^* = \max\{\deg_Q(g_i) \mid i = 1, \dots, t\} - \sum_{i=1}^{n} w_i.$$

Proof. Let $u \in \mathrm{Z}^n$ be such that P_u is a maximal lattice point free polytope. We chose a lattice point $v_0 \in P_u$ and we consider $P_u - v_0 = P_{u'}$, where $u' = u - Bv_0$ is a non-negative integer vector. Thus $P_{u'}$ is equal to the maximal lattice point free polytope P_C where C is the fiber containing the monomial $\mathbf{x}^{u'}$. Consequently, we have $P_{u'} = P_C \in T(B)$ and by [**6**, Theorem 3.2], $k[Q]$ has the highest minimal syzygy with Q-degree $Au = Au'$. Now the proof follows from [**6**, Theorem 3.8]. $\square$

The relationship between Theorem 2.1 and the a-invariant of $k[Q]$ is given in Remark 3.2. Our first algorithm is based on Theorem 2.1 and goes as follows:

Algorithm 2.2.

Input: A strictly positive integer $1 \times n$-matrix $A = [w_1, \dots, w_n]$.

Output: f^*, the Frobenius number of $Q = \mathrm{N}w_1 + \cdots + \mathrm{N}w_n$.

Steps of the Algorithm:

1. Compute the toric ideal I_A [**4**, Theorem 12.24], [**8**, Chapter 12].

2. Find Q-degree of each of the highest minimal syzygies of $k[Q]$ over $k[\mathbf{x}]$ by computing the minimal Q-graded free resolution of $k[Q] \simeq k[\mathbf{x}]/I_A$ over $k[\mathbf{x}]$ [**3**].

3. Use Theorem 2.1 to compute f^*.

Example 2.3 [**5**]. Suppose $A = [271, 277, 281, 283]$ and $R = k[t, x, y, z]$. Using a computer algebra system one can see that

$$I_A = \langle x^2 - tz, y^3 - xz^2, t^{48} - xy^2z^{43}, t^{47}x - y^2z^{44}, t^{47}y - z^{46}\rangle$$

and the last term in the minimal Q-graded free resolution of $k[Q]$ over R is $R(-13566) \oplus R(-14134)$. Thus, using Theorem 2.1, we have

$$f^* = \max\{13566, 14134\} - 271 - 277 - 281 - 283 = 13022.$$

3. Hilbert-Poincaré series. Again we consider Q-graded k-algebra $k[Q] \simeq k[\mathbf{x}]/I_A$. Its Hilbert function and Hilbert-Poincaré series are defined by

$$H(k[Q], i) = \dim_k k[Q]_i$$

and

$$F(k[Q], t) = \sum_{i=0}^{\infty} H(k[Q], i) t^i,$$

respectively, where $k[Q]_i$ is the k-vector space generated by all monomials of Q-degree i. By the Hilbert-Serre theorem, we know that

$$F(k[Q], t) = \frac{h(t)}{\prod_{i=1}^{n}(1 - t^{w_i})}$$

where $h(t) \in \mathrm{Z}[t]$. The degree of $F(k[Q], t)$ as a rational function is denoted by $a(k[Q])$ and is called the a-invariant of $k[Q]$ [**9**, Definition 4.1.5].

Theorem 3.1. *With the above notations, we have*

$$f^*(Q) = a(k[Q]).$$

Proof. Suppose $\theta(t) = \sum_{i \in \mathrm{N} \setminus Q} t^i$. Since Q is a numerical semigroup, $\theta(t)$ is a polynomial and by definition of the Frobenius number, $\deg \theta(t) = f^*$. Clearly, we have

$$\begin{aligned} F(k[Q], t) = \sum_{j \in Q} t^j &= \frac{1}{1-t} - \sum_{i \in \mathrm{N} \setminus Q} t^i \\ &= \frac{1}{1-t} - \theta(t). \end{aligned}$$

By the Hilbert-Serre theorem we also have

$$F(k[Q],t) = \frac{h(t)}{\prod_{i=1}^{n}(1-t^{w_i})}.$$

Thus we conclude that

$$(1-t)h(t) = \prod_{i=1}^{n}(1-t^{w_i}) - (1-t)\prod_{i=1}^{n}(1-t^{w_i})\theta(t).$$

Since the degrees of the left- and right-hand side of the above equality are the same, we have

$$1 + \deg h(t) = 1 + \sum_{i=1}^{n} w_i + \deg \theta(t).$$

This implies the result. $\square$

Remark 3.2. Theorem 3.1 together with [**9**, Proposition 4.2.3] will give us another proof for the Theorem 2.1.

The second algorithm is based on Theorem 3.1 and goes as follows:

Algorithm 3.3.

Input: A strictly positive integer $1 \times n$-matrix $A = [w_1, \dots, w_n]$.

Output: f^*, the Frobenius number of $Q = \mathrm{N}w_1 + \cdots + \mathrm{N}w_n$.

Steps of the Algorithm:

1. Compute the toric ideal I_A [**4**, Theorem 12.24], [**8**, Chapter 12].

2. Compute the Hilbert-Poincaré series of $k[Q]$ [**1**], [**4**, Theorem 12.24].

3. Use Theorem 3.1 to compute f^*.

Example 3.4 (continued from Example 2.3). We can see that the numerator of Hilbert-Poincaré series is

$$1 - t^{554} - t^{843} + t^{1397} - t^{13008} - t^{13014} - t^{13018} + t^{13285} + t^{13289} + t^{13291} + t^{13295} - t^{13566} + t^{13580} - t^{14134}.$$

Thus, using Theorem 3.1, we have

$$f^* = 14134 - 271 - 277 - 281 - 283 = 13022.$$

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