

DIFFERENTIAL OPERATORS AND WEIGHTED ISOBARIC POLYNOMIALS

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ABSTRACT. We characterize those sequences of weighted isobaric polynomials which belong to the kernel of the linear operator $D_{11} - \sum_{j=1}^k a_j t_j D_{2j} - mD_2$, $m \geq 1$, $k \geq 2$, and we characterize those linear operators of this form which have a nonzero kernel in terms of the coefficients a_j .

1. Introduction. In [5] the following linear operator:

$$\mathfrak{T}_m = D_{11} - \sum_{j=1}^k t_j D_{2j} - mD_2,$$

where $m \in \mathbf{Z}$ and $D_{ij} = \partial/\partial t_i \partial t_j$, acting on a ring of polynomials $\mathbf{Z}[t_1, t_2, \dots, t_k]$, was introduced. In [6] the ring of *isobaric polynomials* was discussed, and a certain proper subset, the *weighted isobaric polynomials* (WIP's) was defined. In this paper we shall consider the effect of the operator \mathfrak{T}_m on sequences of WIP's.

An *isobaric polynomial* of degree n , see [6, Definition 1], is a polynomial of the form $P_{k,n} = \sum_{\alpha} A_{\alpha} t^{\alpha}$, where $\alpha = (\alpha_1, \dots, \alpha_k)$, $t^{\alpha} = t_1^{\alpha_1} \dots t_k^{\alpha_k}$ and $(1^{\alpha_1}, \dots, k^{\alpha_k})$ is a partition of n ; thus, $\sum_{j=1}^k j\alpha_j = n$. (The monomials of $P_{k,n}$ can be thought of as indexed by Young diagrams.) The isobaric polynomials form a graded ring, with the grading induced by $\deg t_i = i$. In fact, this ring is isomorphic to the ring of symmetric polynomials with the isomorphism given by $t_i \leftrightarrow (-1)^{i-1} e_i$, where e_i is the i th elementary symmetric polynomial. We will refer to this isomorphism as the *isobaric reflection*. The *weighted isobaric polynomials* $P_{k,n,\omega}$ are distinguished by the fact that their coefficients depend on a fixed vector ω . It is the effect of \mathfrak{T}_m on sequences, $\{P_{k,n,\omega}\}_n$,

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of the weighted isobaric polynomials that we are interested in here. These sequences of WIP's form a \mathbf{Z} -module in the natural way which we call the WIP-module [6].

The purpose of our paper is to discuss the portion of the kernel of \mathfrak{T}_m lying in the WIP-module. In other words, we are concerned with those "weighted" sequences of polynomials such that every term of the sequence lies in the kernel of the above operator.

The interest in this question came with the realization in [3] that some well-known sequences of polynomials (which belong to the WIP-module) arising in number theory satisfy this equation. They are the generalized Fibonacci polynomials (GFP), $\{F_n\}_n$, for $m = 2$, and the sequence of generalized Lucas polynomials (GLP), $\{G_n\}_n$, for $m = 1$. In terms of symmetric polynomials (via the reflection isomorphism defined above) they are, respectively, the isobaric reflects of the complete symmetric polynomials (CSP) and the power symmetric polynomials (PSP), see also [2-4]. And since these two families of polynomials are particular cases of weighted isobaric polynomials it is natural to ask whether any other family is annihilated by this differential operator. In this paper we shall answer the following questions.

- (1) Are there any other WIP sequences which lie in the kernel of the operator for these two values of m ?
- (2) Are there any other WIP sequences which lie in the kernel of the operator for other integer values of m ?
- (3) Are there any reasonable generalizations of these operators with nontrivial kernels in the WIP-module?

The answer to these questions is rather surprising: only families that have weights close enough to the GFP and GLP satisfy $\mathfrak{T}_m = 0$ for $m = 1$ or $m = 2$ and none for other m 's (or slightly more general operators).

The paper is organized as follows. In the next section we make precise the definition of the weighted isobaric polynomials and recall a few results that were proved in [6]. In Section 3 we answer the first question, i.e., by considering the operators \mathfrak{T}_1 and \mathfrak{T}_2 which admit nontrivial WIP-kernels. Section 4 is concerned with the last two questions for which, as we shall see, the answers are somewhat intertwined.

The results contained in this paper are a far-reaching generalization of a result for Fibonacci and Lucas Polynomials in [1] and are of special interest as a result of the weighted isobaric structures involving Schur polynomials introduced in [6].

2. Weighted isobaric polynomials. Most of the definitions and results of this section are taken from [6]. We say that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ with $\alpha_i \in \mathbf{Z}_{\geq 0}$ a *partition* of n if

$$\sum_{i=1}^k i \alpha_i = n,$$

and we denote this by $\alpha \vdash_k n$, i.e., α is a partition with at most k parts. (This is an abbreviation of the usual notation, $(1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k})$, for a partition of n).

Definition 1. A *weight* ω is a vector $\omega = (\omega_1, \omega_2, \dots, \omega_k) \in \mathbf{Z}^k$. In [6] we also worked with countably many variables t_1, t_2, \dots , and infinite vectors as weights. In this paper, however, we shall work with finitely many variables. For a fixed weight ω we introduce a weight function $(wt)_\omega$ on monomials $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \dots t_k^{\alpha_k}$ defined inductively by

$$\begin{aligned} (wt)_\omega(t_i) &:= \omega_i, \quad i = 1, \dots, k, \\ (wt)_\omega(t_1^{\alpha_1} t_2^{\alpha_2} \dots t_k^{\alpha_k}) &:= \sum_{\substack{i=1 \\ \alpha_i > 0}}^k (wt)_\omega(t_1^{\alpha_1} t_2^{\alpha_2} \dots t_i^{\alpha_i - 1} \dots t_k^{\alpha_k}). \end{aligned}$$

A *weighted isobaric polynomial of weight* ω is an isobaric polynomial defined by

$$P_{k,n,\omega} := \sum_{\alpha \vdash_k n} (wt)_\omega(t^\alpha) t^\alpha.$$

Note. We shall omit the subscript k in the remainder of the paper.

The definition of the weight of a monomial and the isobaric polynomial can be easily understood by considering the lattice of monomials, also called the *differential lattice* denoted by $\mathcal{L}(t)$. The covering relations are given by $\alpha < \beta$ if there exists an i such that

$\beta_i = \alpha_i + 1$ and $\beta_j = \alpha_j$ for $j \neq i$. The rank of the lattice is given by $\text{rank}(t^\alpha) = \sum_{i=1}^k \alpha_i =: |\alpha|$. The name *differential* is appropriate because, as is obvious, the lattice is formed by partial differentiation (forgetting the differentiation constant).

Remark 1. This lattice is isomorphic to the lattice of natural numbers where the partial order is given by divisibility. The isomorphism is given by identifying t_j with p_j , the j th rational prime. The join and the meet of two numbers are therefore the *lcm* and *gcd*, respectively.

We will denote by $\mathcal{L}(t^\alpha)$ the portion of $\mathcal{L}(t)$ with t^α as the top element.

Example 1. The lattice $\mathcal{L}(t_1 t_2^2 t_3^2)$. (We omit the variables and write the corresponding exponent).

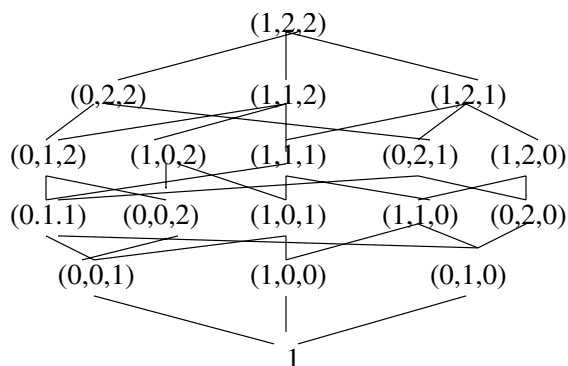


FIGURE 1.

Let us compute the weight of $t_1 t_2^2 t_3^2$, written $(1,2,2)$ as mentioned in Example 1, assuming $k = 3$ and $\omega = (\omega_1, \omega_2, \omega_3)$. After assigning the weights ω_j to the variable t_j in the previous example, the monomial (whose exponent vector is) $(1, 0, 1)$ gets the weight $\omega_1 + \omega_3$, while the monomial $(0, 2, 0)$ gets the weight ω_2 , and, after a rather tedious calculation using the assignment rule, the monomial $(1, 2, 2)$ gets the

weight $6(\omega_1 + 2\omega_2 + 2\omega_3)$. Fortunately, we can avoid this calculation using the following theorem [6, Theorem 1].

Theorem 2.1. *Given a weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_k)$, the weight assigned to the monomial whose exponent vector is $(\alpha_1, \dots, \alpha_k)$ is*

$$(2.1) \quad (wt)_{\omega}(t^{\alpha}) = \binom{|\alpha|}{\alpha_1 \cdots \alpha_k} \frac{\sum_{j=1}^k \alpha_j \omega_j}{\sum_{j=1}^k \alpha_j}.$$

Each weight vector determines a unique sequence of WIP's. Two such sequences are the generalized Fibonacci polynomials $\{F_n\}_n$ and the generalized Lucas polynomials $\{G_n\}_n$; they are the weighted isobaric sequences for the weight vector $\omega_F = (1, 1, \dots, 1)$ (the unit vector) for the F -sequence, and $\omega_G = (1, 2, \dots, k)$ (the natural vector) for the G -sequence. The coefficients for the F -sequence are given by

$$(wt)_{\omega_F}(t^{\alpha}) = \binom{|\alpha|}{\alpha_1 \cdots \alpha_k}$$

and for the G -sequence

$$(wt)_{\omega_G}(t^{\alpha}) = \frac{(|\alpha| - 1)!}{\prod_{j=1}^k \alpha_j!} n.$$

In [6, Theorem 2.3], it is shown that the sequences of weighted isobaric polynomials form a free \mathbf{Z} -module where addition is defined as addition of weight vectors, that is, the sum of two sequences of weights ω and ω' is the sequence of weight $\omega'' = \omega + \omega'$. It is also shown there that isobaric reflects of hook Schur polynomials, i.e., the Schur polynomials determined by hook Young diagrams, are in the WIP-module. (The weight of the hook reflect determined by the hook partition $(n - r, 1^r)$ is $(-1)^r(0, \dots, 0, 1, 1, \dots)$ with r 0's.) The hook reflects in fact form a basis for the WIP-module, [6, Theorem 3.5]. As an application of the WIP-module structure we have the following isobaric version of a well-known theorem of symmetric polynomials.

Theorem 2.2. $G_n = \sum_r (-1)^r H_r$, where H_r is the reflect of the hook Schur polynomial $S_{(n-r, 1^r)}$.

The symmetric polynomial version of this is the statement that a complete symmetric polynomial is an alternating sum of hook Schur polynomials.

3. The kernel of \mathfrak{T}_m . We now turn our attention to the linear operator \mathfrak{T}_m and find that for certain choices of the parameter m , the F -sequence and the G -sequence belong to the kernel of \mathfrak{T}_m .

Theorem 3.1 [5, Theorem 4]. $\mathfrak{T}_m(F_n) = (D_{11} - \sum_{j=1}^k t_j D_{2j} - m D_2)(F_n) = 0$ for $m = 2$, and $\mathfrak{T}_m(G_n) = (D_{11} - \sum_j t_j D_{2j} - m D_2)(G_n) = 0$ for $m = 1$.

This theorem will follow from Theorem 3.2 below. Theorem 3.1 tells us that the F - and G -sequences are solutions to the operator equation when the parameter is $m = 1$ in the case of the G -polynomials and $m = 2$ in the case of the F -polynomials, but it turns out that these solutions are determined by other more basic solutions which, while dependent on the weights of the F - and G -sequences, are not themselves WIP's. We might refer to these isobaric polynomials as *satellites*. They will contain certain monomials that we call *strings*.

Definition 2. We first define the concept of string. For a given n we first choose exponent vectors of the following two kinds:

(1) Vectors of type $(0, \alpha_2, \alpha_3, \dots, \alpha_k)$, where $\alpha_3, \dots, \alpha_k$ is a fixed $(k-2)$ -tuple and α_2 is the largest second element with respect to the condition that $\alpha \vdash_k n$, i.e., $\sum_i i\alpha_i = n$.

(2) Vectors of type $(1, \alpha_2, \alpha_3, \dots, \alpha_k)$ where $\alpha_3, \dots, \alpha_k$ is a fixed $(k-2)$ -tuple and α_2 is the largest second element with respect to the same condition as in (1).

Then we select sequences of these vectors of the following form

$$\begin{array}{ll}
 (0, \alpha_2, \alpha_3, \dots, \alpha_k) & (1, \alpha_2, \alpha_3, \dots, \alpha_k) \\
 (2, \alpha_2 - 1, \alpha_3, \dots, \alpha_k) & (3, \alpha_2 - 1, \alpha_3, \dots, \alpha_k) \\
 \dots & \dots \\
 (2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k) & (2j + 1, \alpha_2 - j, \alpha_3, \dots, \alpha_k) \\
 \dots & \dots \\
 (2\alpha_2, 0, \alpha_3, \dots, \alpha_k) & (2\alpha_2 + 1, 0, \alpha_3, \dots, \alpha_k)
 \end{array}$$

$$j = 0, 1, \dots, \alpha_2.$$

Such a sequence is called a *string (of n)*. The first element in the sequence is the *string generator*. If the string generator starts with 0, we call it an *even string*, and if it starts with 1, we call it an *odd string*. The left-hand column above is an *even string*, while the right-hand column is an *odd string*. We say that two elements in a string are *adjacent* if they are adjacent in the ordering of the string.

A string is generated by starting with the string generator and increasing the first entry by 2 at each stage and decreasing the second entry by 1 until the second entry becomes 0. For example, if (1, 3, 1) is a string generator, then the ordered set of the vectors

$$\{(1, 3, 1), (3, 2, 1), (5, 1, 1), (7, 0, 1)\}$$

is an odd string. It is not difficult to see that for a given *n* all of the exponent vectors that arise from partitions of *n* occur in some even or odd string. Thus, every isobaric polynomial is just the sum of its strings with “remembered” coefficients. In particular, for a sequence of weighted isobaric polynomials, each polynomial is just the weighted sum of its strings. Theorem 3.1 will follow from this fact. We shall say that a string *belongs* to a weighted isobaric polynomial if it is a weighted string in that polynomial.

Example 2. For example, the (three) strings that *belong* to F_4 , where $F_4 = t_1^4 + 3t_1^2t_2 + t_2^2 + 2t_1t_3 + t_4$, are (0,2,0,0), (2,1,0,0), (4,0,0,0); and (1,0,1,0); and (0,0,0,1).

Definition 3. Assume \mathcal{S} is a given string, i.e., a sequence of vectors as above. Given a weight vector ω we consider the following isobaric polynomial $P_{\mathcal{S},\omega} := \sum_{\alpha \in \mathcal{S}} (wt) \omega(t^\alpha) t^\alpha$. We call such a polynomial a *satellite*.

From these definitions and observations we have that a weighted isobaric polynomial is the sum of its satellites:

$$P_{n,\omega} = \sum_{\mathcal{S}} P_{\mathcal{S},\omega}.$$

Example 3. In the previous example the satellites of F_4 are $t_1^4 + 3t_1t_2$; $2t_1t_3$; and t_4 .

We will denote by $P_{\mathcal{S},F}$ a satellite belonging to a polynomial in the sequence F and by $P_{\mathcal{S},G}$ a satellite belonging to a polynomial in the sequence G . The remark made after Theorem 3.1 says

Theorem 3.2.

1. If $P_{\mathcal{S},F}$ is a satellite belonging to F , then $\mathfrak{T}_2(P_{\mathcal{S},F}) = 0$.
2. If $P_{\mathcal{S},G}$ is a satellite belonging to G , then $\mathfrak{T}_1(P_{\mathcal{S},G}) = 0$.

This theorem will follow from

Lemma 3.3.

a) $(2j + 2, \alpha_2 - j - 1, \alpha_3, \dots, \alpha_k)$ and $(2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k)$ are adjacent elements in the even string generated by $(0, \alpha_2, \alpha_3, \dots, \alpha_k)$. The coefficient of $D_{11}(2j+2, \alpha_2-j-1, \alpha_3, \dots, \alpha_k)$ equals the coefficient of $-(\mathfrak{T}_m - D_{11})(2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k)$ whenever the weight vector is $(1, 1, \dots, 1)$ and $m = 2$, or the weight vector is $(1, 2, \dots, k)$ and $m = 1$. D_{11} applied to the string generator is 0 and $(\mathfrak{T}_m - D_{11})$ applied to the last element in the string is also 0.

b) $(2j + 3, \alpha_2 - j - 1, \alpha_3, \dots, \alpha_k)$ and $(2j + 1, \alpha_2 - j, \alpha_3, \dots, \alpha_k)$ are adjacent elements in the odd string generated by $(1, \alpha_2, \alpha_3, \dots, \alpha_k)$. The coefficient of $D_{11}(2j+3, \alpha_2-j-1, \alpha_3, \dots, \alpha_k)$ equals the coefficient

of $(\mathfrak{F}_m - D_{11})(2j+1, \alpha_2 - j, \alpha_3, \dots, \alpha_k)$ whenever the weight vector is $(1, 1, \dots, 1)$ and $m = 2$, or the weight vector is $(1, 2, \dots, k)$ and $m = 1$. D_{11} applied to the string generator is 0 and $(\mathfrak{F}_m - D_{11})$ applied to the last element in the string is also 0.

Proof of Lemma.

a) That the elements mentioned in the lemma belong to the string and are adjacent is obvious. The fact that the first and last elements of the string are mapped to 0 by the operators D_{11} and $(\mathfrak{F}_m - D_{11})$ as claimed is also obvious. We shall prove then that the coefficients of the elements $D_{11}(2j+2, \alpha_2 - j - 1, \alpha_3, \dots, \alpha_k)$ and $(\mathfrak{F}_m - D_{11})(2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k)$ are negatives of one another.

By Theorem 2.1 we have that

$$(3.1) \quad (wt)\omega(t^{s_{2j+2}}) = \frac{(\sum_{i=2}^k \alpha_i + j)!}{(2j+2)!(\alpha_2 - j - 1)! \prod_{i=3}^k \alpha_i!} \times \left[(2j+2)\omega_1 + (\alpha_2 - j - 1)\omega_2 + \sum_{i=3}^k \alpha_i \omega_i \right]$$

$$(3.2) \quad (wt)\omega(t^{s_{2j}}) = \frac{(\sum_{i=2}^k \alpha_i + j - 1)!}{(2j)!(\alpha_2 - j)! \prod_{i=3}^k \alpha_i!} \times \left[(2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^k \alpha_i \omega_i \right],$$

where $s_{2j+2} = (2j+2, \alpha_2 - j - 1, \dots, \alpha_k)$ and $s_{2j} = (2j, \alpha_2 - j, \dots, \alpha_k)$. The coefficient due to D_{11} applied to $t^{s_{2j+2}}$ is

$$(3.3) \quad (2j+2)(2j+1),$$

and the coefficient due to $\mathfrak{F}_m - D_{11}$ applied to $t^{s_{2j}}$ is

$$(3.4) \quad (\alpha_2 - j) \left(\sum_{i=2}^k \alpha_i + j + m - 1 \right).$$

Multiplying equation (3.1) by (3.3) and equation (3.2) by (3.4) and using the values given by the hypothesis of the lemma for m and for

the weight vector and comparing gives the result. It is useful to record the last steps in the computation, beginning just before the hypotheses on m and the weight vectors are applied. We have this expression

$$(3.5) \quad \left(\sum_{i=2}^k \alpha_i + j \right) \left((2j+2)\omega_1 + (\alpha_2 - j - 1)\omega_2 + \sum_{i=3}^k \alpha_i \omega_i \right) \\ - \left(\sum_{i=1}^k \alpha_i + j + m - 1 \right) \left((2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^k \alpha_i \omega_i \right).$$

Letting $m = 1$ gives $2\omega_1 - \omega_2 = 0$, after applying the hypothesis on the weights, which gives the result required no matter what the weights ω_j are for $j > 2$. Thus we have proved more in this case, that is, we have infinitely many WIP sequences as solutions. Letting $m = 2$ gives the expression

$$(3.6) \quad \left(\sum_{i=1}^k \alpha_i + j \right) (2\omega_1 - \omega_2) - \left((2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^k \alpha_i \omega_i \right),$$

but now we need our weight hypothesis on all of the weights to achieve the cancelation, thus this expression is 0 if we assume that $\omega_j = \omega_1$, for all j .

b) The proof of part b) is similar to that of part a) and will be omitted. \square

But then Theorem 3.2 now follows from Lemma 3.3. Theorem 3.1 follows from Theorem 3.2 by linearity.

It is an interesting consequence of the proof that the lattices of the string elements intersect for the first time exactly at the nodes determined by D_{11} operating on the string. We give an example.

Example 4. Consider the string generated by $(0, 2, 1)$, $n = 7$, $k = 3$. The lattices are given by

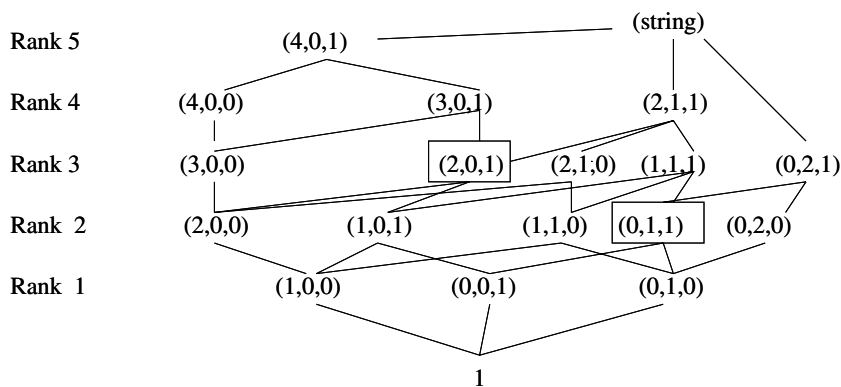


FIGURE 2.

In this case the intersection nodes are $(2, 0, 1)$ and $(0, 1, 1)$. The string consists of the three nodes $(4, 0, 1)$, $(2, 1, 1)$ and the string generator $(0, 2, 1)$. Note that this is consistent with the remark made previously that the intersection of sublattices is determined by the gcd associated with relevant nodes of the divisor lattice of natural numbers. It is also the case that the intersection nodes again form a string, this time for polynomials of degree $n - 2$.

4. Other solutions. We stress here that what we mean by a solution to our problem is the entire sequence of WIP's determined by a particular weight vector ω ; calling such a sequence P_ω , $P_\omega = \{P_{n,\omega}\}_n$, we have as solutions the polynomials generated by the satellites of WIP-solutions. We claim that the WIP sequences of solutions of the PDE $\mathfrak{T}_=0$, that is, for $m = 1$, are exactly those solutions generated by linearity from the satellites in which $2\omega_1 = \omega_2$ with the ω_j arbitrary for $j > 2$, but fixed throughout the string: G -polynomials, for example.

When $m = 2$, the solutions of $\mathfrak{T}_2 = 0$ consist just of the scalar multiples of the F -polynomials. The kernel of the operator operating on the WIP-module is 0 when $m \neq 1, 2$.

We prove these assertions now.

Proposition 4.1. *Let ω be a weight vector and suppose $\{P_{n,\omega}\}$ lies in the kernel of \mathfrak{T}_m , where $P_{n,\omega}$ is a WIP-sequence, then either*

- (1) $m = 1$ and $2\omega_1 = \omega_2$ or
 (2) $m = 2$ and $\omega_1 = \omega_2$.

Proof. It is only necessary to look at the second and third terms of the sequence, namely, at

$$\begin{aligned} P_{2,\omega} &= \omega_1 t_1^2 + \omega_2 t_2, \\ P_{3,\omega} &= \omega_1 t_1^3 + (\omega_1 + \omega_2)t_1 t_2 + \omega_3 t_3. \end{aligned}$$

Requiring that $P_{2,\omega}$ satisfies the operator equation implies that $m\omega_2 = 2\omega_1$; requiring that $P_{3,\omega}$ satisfies the operator equation yields $m(\omega_1 + \omega_2) = (5\omega_1 - \omega_2)$.

Setting the two values equal and solving the resultant quadratic in \mathbf{Z} gives the two possibilities $2\omega_1 = \omega_2$, $\omega_1 = \omega_2$ or $\omega = 0$. Solving for m in each case gives $m = 1$ and $m = 2$, or the trivial case for any m , respectively. And we know that the first two cases are realized with the G -polynomial sequence and the F -polynomial sequence respectively. \square

We have the following important result which says that looking at WIP solutions is in fact equivalent to looking at their satellites.

Theorem 4.2. $P_{n,\omega}$ is a solution of $\mathfrak{T}_m = 0$ if and only if the satellites belonging to $P_{n,\omega}$ are solutions.

Before we proceed we introduce some notation and prove a lemma.

Notation. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a vector we denote by $\mathfrak{T}_m(\alpha)$ the set of exponents of t appearing in $\mathfrak{T}_m(t^\alpha)$.

More precisely, $\mathfrak{T}_m(\alpha) = \{(\alpha_1 - 2, \alpha_2, \dots, \alpha_k), (\alpha_1, \alpha_2 - 1, \alpha_3, \dots, \alpha_k)\}$, if the vectors listed are nonnegative. If \mathcal{S} is a string we denote by

$$\mathfrak{T}_m(\mathcal{S}) := \cup_{s \in \mathcal{S}} \mathfrak{T}_m(s).$$

Lemma 4.3.

1. $\mathfrak{T}_m(\mathcal{S})$ is a string of degree $n - 2$.
2. If \mathcal{S} and \mathcal{T} are two different strings then $\mathfrak{T}_m(\mathcal{S}) \cap \mathfrak{T}_m(\mathcal{T}) = \emptyset$.

Proof.

1. Assume that $(\mathcal{S}) = \{(2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k)\}_{j=1, \dots, \alpha_2}$ is an even string. An easy computation yields that $\mathfrak{T}_m((\mathcal{S})) = \{(2j, \alpha_2 - j - 1, \alpha_3, \dots, \alpha_k)\}_{j=1, \dots, \alpha_2 - 1}$. (Here we assume that $\alpha_2 \neq 0$).

2. Clearly the new string of $n - 2$ preserves the initial parity. So we can assume that \mathcal{S} and \mathcal{T} have the same parity (say even) generated by $(0, \alpha_2, \alpha_3, \dots, \alpha_k)$ and $(0, \alpha'_2, \alpha'_3, \dots, \alpha'_k)$. If $\mathfrak{T}_m(\mathcal{S}) \cap \mathfrak{T}_m(\mathcal{T})$ is nonempty, then there exists i, j such that $(2i, \alpha_2 - i, \alpha_3, \dots, \alpha_k) = (2j, \alpha'_2 - j, \alpha'_3, \dots, \alpha'_k)$. In particular $\alpha_3 = \alpha'_3, \dots, \alpha_k = \alpha'_k$ and since the elements are partitions of n we have $\alpha_2 = \alpha'_2$, and thus the two strings are the same. \square

Proof of Theorem 4.2. Clearly since $P_{n, \omega}$ is the sum of its satellites we need only prove necessity. Since the monomials appearing in each $\mathfrak{T}_m((\mathcal{S}))$ for every satellite are disjoint (previous lemma), we must have that cancelation of the operator holds at the level of each satellite. \square

Moreover, we note that the ranks of elements in a string \mathcal{S} form a strictly monotonic increasing sequence and that the monomials $\mathfrak{T}_m(P_{\mathcal{S}, \omega})$ form also a string. Therefore, if $\mathfrak{T}_m(P_{\mathcal{S}, \omega}) = 0$ then cancelation must occur for the coefficient of every monomial, which means that the conditions in Lemma 3.3 are satisfied. We call these conditions *domino* conditions. We therefore have

Lemma 4.4. *A satellite is a solution of $\mathfrak{T}_m = 0$ if and only if the “domino” conditions of Lemma 3.3 hold.*

We can now use the previous results (Proposition 4.1, Theorem 3.2 and Lemma 4.4) to determine completely the general WIP solutions for our operator in case $m = 1, 2$. We have

Theorem 4.5.

1. *The WIP sequence $\{P_{n, \omega}\}_n$ is a solution for $\mathfrak{T}_1 = 0$ if and only if $2\omega_1 = \omega_2$.*
2. *The WIP sequence $\{P_{n, \omega}\}_n$ is a solution for $\mathfrak{T}_2 = 0$ if and only*

if $\omega_i = a$, for all $i = 1, \dots, k$. That is, all such WIP-solutions are of the form aF_n , $n \in \mathbf{N}$, and all solutions are exactly those generated in the WIP-module by F -satellites.

Proof. In both cases we look at the necessary condition only, as sufficiency was proved in Theorem 3.1. In both cases looking at the WIP solution is equivalent to looking at the satellite solution (Theorem 4.2) and requiring that *domino* cancelations hold.

1. The proof consists of looking at the proof of Lemma 3.3 more carefully and noting that in light of Proposition 4.1 (1), the cancelation occurs independently of the choice of ω_j for $j > 2$.

2. In this case the proof in Lemma 3.3 arrives at the equation $\sum_j a\alpha_j = \sum_j \alpha_j\omega_j$ with $\omega_1 = \omega_2 = a$, which must hold for all exponent vectors α and for a fixed weight vector ω ; thus, $\omega_j = a$ for all j . \square

So now we come to the three questions posed in the Introduction. It turns out that we shall be able to answer these questions completely once we have answered the third one. So our aim is to prove

Theorem 4.6. *The partial differential equation $D_{11} - \sum_j a_j t_j D_{i,j} - mD_2 = 0$, where $a_j \in \mathbf{Z}$, has WIP-sequence solutions only when $a_j = 1$ and $m = 1$ or $m = 2$, where the a_j and m are assumed to be arbitrary real numbers, not all zero. (Though, they could be taken from any field of characteristic 0 as far as the proof is concerned).*

The statement of this theorem makes clear what we have chosen to mean by a generalized operator. When one tries to find the other second order, linear partial differential equations that have sufficient resemblance to the one at hand, the lack of left-right symmetry among the partitions of n as n increases becomes more apparent. This is due to the fact that 1's will appear in the decompositions of n many times, but n itself can appear only once; small numbers have the advantage over big ones. This is reflected in the futility of trying to find new PDE's by varying the suffixes of the operators, $D_{i,j}$. However, a tack that appears promising is to provide \mathfrak{T}_m with arbitrary (real) coefficients. Thus, we want to ask what is the kernel of any operator of

the sort $D_{11} - \sum_j a_j t_j D_{2,j} - mD_2$, a_j arbitrary (real) scalars? (Here we assume that the coefficient of D_{11} is not 0, so we can, without loss of generality, assume that it is 1.) By the way, the resemblance of the operator equation $\mathfrak{T}_m = 0$ to the “Newton identity” satisfied by the WIP-polynomials, see [5, Theorem 4.1], is striking, and probably significant, though the anomalous role of the D_2 -term is puzzling.

Before we prove the theorem we note that the proof of Proposition 4.1 contains the following fact which, together with its proof, also holds in the generalized operator case.

Lemma 4.7. *If $P_{2,\omega}$ satisfies the generalized operator equation, then $m\omega_2 = 2\omega_1$.*

Proof of Theorem 4.6. We have seen that Theorem 4.2 and Lemma 4.4 imply that $P_{n,\omega}$ satisfies the operator equation if and only if the domino conditions of Lemma 3.3 hold. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k)$ be an arbitrary element in a string \mathcal{S} . If α is the only element of the string, then clearly the corresponding satellite satisfies the operator equation. So suppose that α is not an element of least rank, that is, it is not the generating element of the string. In this case, there is an element of rank one less than the rank of α . Let us suppose that $D_{11}((wt)\omega(t^\alpha)t^\alpha) = (\mathfrak{T}_m - D_{11})((wt)\omega(t^\beta)t^\beta)$, that is, suppose that the “domino” proof applies. (Note that if α is an element of greatest rank, then $D_{11}(t^\alpha) = 0$.) The picture looks like this:

$$\begin{array}{ccc}
 (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k) = \alpha & & \\
 | & & \\
 (\alpha_1 - 1, \alpha_2, \alpha_3, \dots, \alpha_k) & \quad (\alpha_1 - 2, \alpha_2 + 1, \alpha_3, \dots, \alpha_k) = \beta & \\
 | & \quad / & \\
 (\alpha_1 - 2, \alpha_2, \alpha_3, \dots, \alpha_k) & &
 \end{array}$$

Recalling the proof of Theorem 3.2 at this point, we needed to equate the product of the coefficient $(wt)\omega(t^\alpha)$ of t^α and the coefficient of $D_{11}(t^\alpha)$ with the product of the coefficient $(wt)\omega(t^\beta)$ of t^β and

$(\mathfrak{T}_m - D_{11})(t^\beta)$. The new ingredient here is the coefficient of $(\mathfrak{T}_m - D_{11})(t^\beta)$ which is $(\alpha_2 + 1)(\sum_{j=1}^k a_j \alpha_j + m - 2a_1)t^{\beta'}$, where $\beta' = (\alpha_1 - 2, \alpha_2, \alpha_3, \dots, \alpha_k)$. After making the calculation indicated above and allowing the dust to settle, this gives

$$(4.1) \quad \left(\sum_{j=1}^k \alpha_j - 1 \right) \left(\sum_{j=1}^k \alpha_j \omega_j \right) = \left(\sum_{j=1}^k \alpha_j \omega_j + \omega_2 - 2\omega_1 \right) \left(\sum_{j=1}^k a_j \alpha_j + m - 2a_1 \right)$$

as a necessary condition for the generalized operator to have a solution. We shall assume throughout that P_ω is not trivial, that is, that $\omega \neq 0$. Equation (4.1) can be rewritten as

$$(4.2) \quad \left(\sum_{j=1}^k \alpha_j - 1 \right) \left(\sum_{j=1}^k \alpha_j \omega_j \right) - \left(\sum_{j=1}^k \alpha_j \omega_j \right) \left(\sum_{j=1}^k a_j \alpha_j + m - 2a_1 \right) - (\omega_2 - 2\omega_1) \left(\sum_{j=1}^k a_j \alpha_j \right) = (\omega_2 - 2\omega_1)(m - 2a_1).$$

The left-hand side of (4.2) depends on α , which is a variable, while the right-hand side depends only on the choice of ω and the constants a_1 and m . Hence, the left-hand side and the right-hand side of (4.2) are independently 0. And hence, either $\omega_2 - 2\omega_1 = 0$ or $m - 2a_1 = 0$.

In the first case, we have then that $\omega_2 = 2\omega_1$ and, by Lemma 4.4, $m = 1$. The left-hand side of (4.2) becomes

$$(4.3) \quad \left(\sum_{j=1}^k \alpha_j \omega_j \right) \left(\sum_{j=1}^k \alpha_j - \sum_{j=1}^k a_j \alpha_j + 2 - 2a_1 \right) = 0.$$

$(\sum_{j=1}^k \alpha_j \omega_j) = 0$ implies that $\omega = 0$, that is, the solution is trivial. Thus, $(\sum_{j=1}^k \alpha_j - \sum_{j=1}^k a_j \alpha_j + 2 - 2a_1) = 0$, and so we have as above that $\sum_{j=1}^k \alpha_j - \sum_{j=1}^k a_j \alpha_j = 0$ and $2 - 2a_1 = 0$. And so we have that

$$(4.4) \quad a_1 = 1 \quad \text{and} \quad \sum_{j=1}^k \alpha_j = \sum_{j=1}^k a_j \alpha_j.$$

From these equations we have that $a_j = 1$ for $j = 1, \dots, k$. This is just the case of the original operator for which the G -polynomials were solutions.

In the second case, from $m - 2a_1 = 0$, we have $m = 2a_1$, and from (4.1) we have

$$(4.5) \quad \sum_{j=1}^k \alpha_j \omega_j \left(\sum_{j=1}^k \alpha_j - \sum_{j=1}^k \alpha_j a_j - 1 \right) = (\omega_2 - 2\omega_1) \left(\sum_{j=1}^k \alpha_j a_j \right).$$

We first suppose that $a_j \neq 0$ for all $1 \leq j \leq k$ and for all k . Consider the monomial $\omega_n t_n$. It follows from the definition of a string that $\omega_n t_n$ is a string, or, in the case that $n = 2$, is the generator of a two element string, so we apply Theorem 4.2. Here $\alpha_n = 1$ and $\alpha_j = 0$ otherwise. From (4.5) we arrive at

$$(4.6) \quad \omega_n a_n = (2\omega_1 - \omega_2)(a_n).$$

In particular, when $n = 2$, we have that $\omega_2 = \omega_1$, and thus $\omega_n = \omega_1$ for all n . From (4.1) it then follows that

$$\omega_1 \left(\sum_{j=1}^k \alpha_j - 1 \right) \left(\sum_{j=1}^k \alpha_j \right) = \left(\sum_{j=1}^k \alpha_j - 1 \right) \left(\sum_{j=1}^k a_j \alpha_j \right) \omega_1.$$

If $\omega \neq 0$, that is, if the solution is not the trivial solution, we have that $\sum_{j=1}^k \alpha_j = \sum_{j=1}^k a_j \alpha_j$ from which it follows that $a_j = 1$ for $j = 1, \dots, k$. Moreover, $m = 2$; and this is just the case of solutions generated by the strings of F -polynomials and the original operator.

Finally, we show the assumption that no $a_j = 0$ is justified if we are to have a nontrivial kernel for the generalized operator. First note that for our case 2 assumption, $a_1 = 0$ implies that $m = 0$, and so (4.1) becomes

$$\sum_{j=1}^k (\alpha_j - 1) \left(\sum_{j=1}^k \alpha_j \omega_j \right) = \left(\sum_{j=1}^k \alpha_j \omega_j + \omega_2 - 2\omega_1 \right) \left(\sum_{j=1}^k \alpha_j a_j \right).$$

Thus, if both $a_1 = 0$ and $a_2 = 0$, then $\sum_{j=1}^k (\alpha_j - 1) \left(\sum_{j=1}^k \alpha_j \omega_j \right) = 0$ for suitable choices of values for the α_j (say, $\alpha_1 = 2, \alpha_2 = 1, \alpha_j = 0$

otherwise) and hence $\omega_j = 0$ for $j = 1, 2$, and for any $k > 1$. Thus, we have arrived at the equation

$$\sum_{j=1}^k (\alpha_j - 1) \left(\sum_{j=1}^k \alpha_j \omega_j \right) = \left(\sum_{j=1}^k \alpha_j \omega_j \right) \left(\sum_{j=1}^k \alpha_j a_j \right).$$

But then, either $\omega_j = 0$ for all j (the trivial solution), or $\sum_1^k \alpha_j - 1 = \sum_1^k \alpha_j a_j$, which implies that $a_j = 0$ for all j . And there are no solutions of the type we have required for such a partial differential equation. Thus, either $a_1 \neq 0$ or $a_2 \neq 0$, but then $\omega_2 = \omega_1$. If $a_1 = 0$, we can again choose α_j 's to show that $\omega_1 = 0$. Thus, $\omega_j = 0$ or $a_j = 0$, and again we arrive at the conclusion that either all of the a_j 's = 0, or the solution is the trivial solution. Thus, both a_1 and a_2 are different from zero. So suppose that k is the earliest subscript for which $a_k = 0$; then the last result tells us that for sequences of isobaric degree k , the only solutions for the partial differential equation $\mathfrak{T}_{m,k}$, are those which are already solutions of $\mathfrak{T}_{m,k-1}$. So we are back to the case $m = 2$, $a_j = 1$, $1 \leq j \leq k$. \square

Remark 2. It is rather interesting that if P_ω is a solution of the operator equation and if $\omega_2 = 0$, then either $\omega_j = 0$ for all j , or $m = 1$. This follows easily by applying Theorem 4.2, Lemma 4.4 and the assumptions to the strings generated by first $(0, 1, \dots, 1, 0, 0)$ and then the string generated by $(1, 1, \dots, 1, 0, 0, 0)$, with the last 1 being the exponent of t_n in each case.

We have then that the answer to question (3) is that the only WIP solutions for the generalized operator equation occur when $a_j = 1$ for all $j \in \mathbf{N}$, and $m = 1$ or 2. Thus, the generalized operator has a zero kernel except in the case we started with, thus generalizing the operator does not produce new solutions. Clearly, we have also answered question (2); allowing m to vary beyond 1 and 2, in fact, over any field of characteristic 0, produces no new solutions. The answer to question (1), we learn here, is yes and no. If $m = 2$, then the answer is unique up to a scalar multiple, that is, all WIP-solutions are scalar multiples of the F -sequence; but, if $m = 1$, then not only are scalar multiples of the G -sequence solutions, but also so is the sequence P_ω anytime that

$2\omega_1 = \omega_2$, the remaining weights being arbitrary. However, we also have the *satellite* solutions that get their life from the WIP's, but are not themselves WIP's.

It is tempting to think that a weight vector for an initial string of $P_{n,\omega}$, i.e., the "degree" string (the string whose terminal element is $(n, 0, \alpha_3, \dots, \alpha_k)$), might be weighted as $(\omega_1, \omega_2, 0, \dots, 0, \dots)$, while the $\omega = (\omega_j)$, where ω_j is different from 0. The following example shows what goes wrong here.

Example 5. Consider $P_{n,\omega} = \omega_1 t_1^4 + (2\omega_1 + \omega_2)t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3)t_1 t_3 + \omega_4 t_4$. The strings are:

$$\begin{array}{c} \text{Initial String} \\ \left\{ \begin{array}{l} (0, 2, 0, 0) \\ (2, 1, 0, 0) \\ (4, 0, 0, 0) \end{array} \right\} \quad \left\{ (1, 0, 1, 0) \right\} \quad \left\{ (0, 0, 0, 1) \right\} \end{array}$$

Try weight vector $(\omega_1, \omega_2, 0, 0)$. But now by Theorem 3.1, if the satellite of the initial string is a WIP, the monomial $t_1 t_3$, has coefficient $\omega_1 + \omega_3$, while $t_1 t_3$ has weight ω_1 in the new weighting; recall, we have to assign a weight to each of the monomials induced by a partition of n , thus $t_1 t_3$ would appear in the initial string if $\omega_1 \neq 0$. This contradiction would appear more generally. We omit the proof.

It is also interesting to note the rather special companionable role that the F -sequences and G -sequences play among the isobaric polynomials, especially among the WIP's. In addition to the properties shown in this paper, we have, for example, that the G 's are related to the F 's by partial differentiation as follows: $\partial/\partial t_j(G_n) = nF_{n-j}$. In general, $\partial/\partial t_j(P_\omega)$ is not a WIP, and, in fact, there is good reason to believe that this is the only case. We pursue this observation in a later paper.

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