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THE STORY OF A TOPOLOGICAL GAME

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ABSTRACT. In the author's dissertation, he introduced a simple topological game. Seemingly minor variations of this game have over the years seen various uses, including the characterization of Corson and Eberlein compacta, and characterizing when certain function spaces with the compact-open topology are Baire. This article is primarily a survey of this game and its applications. Some new results are included, and a number of open problems are stated.

1. Introduction. Let X be a topological space, and $x \in X$. The following four games will be discussed in this paper:

(1) $G_{O,P}(X,x)$: In the nth round, Player O chooses an open neighborhood O_n of x, and Player P chooses a point $p_n \in \cap_{i \le n} O_i$. O *wins* if the sequence $\{p_n\}_{n\in\omega}$ converges to x.

(2) $G_{K,P}(X)$: In the nth round, Player K chooses a compact subset K_n of X, and P chooses a point $p_n \notin \cup_{i\le n} K_i$. K *wins* if the sequence $\{p_n\}_{n\in\omega}$ is a closed discrete subset of X.

(3) $G_{K,L}(X)$: In the nth round, K chooses a compact subset K_n of X, and L chooses a compact subset L_n of X disjoint from $\cup_{i\le n} K_i$. K *wins* if $\{L_n\}_{n\in\omega}$ is a closed discrete collection in X.

(4) $G^o_{K,L}(X)$: Same as (3), except that K *wins* if $\{L_n\}_{n\in\omega}$ has a discrete open expansion.

These four games are variations on the same theme. In fact, note that if X is compact, then $G_{O,P}(X,x)$ is equivalent to the game $G_{K,P}(X\backslash\{x\})$. Of course, $G_{K,L}(X)$ is essentially the game $G_{K,P}(X)$ modified to allow P to choose compact sets instead of single points.

$G_{O,P}(X,x)$ was introduced in [**18**], where it was helpful in solving a problem of Zenor, and studied in detail in [**19**], where it was used to define and study a new convergence property. $G_{K,P}(X)$ was, in effect, introduced in [**20**], where it was used to characterize Corson compact spaces and strong Eberlein compact spaces, as well as the

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metalindelöf property in locally compact spaces. It was explicitly defined in [**21**], where it was used to characterize Eberlein compact spaces and metacompactness in locally compact spaces. $G_{K,L}(X)$ was also introduced there and used to characterize paracompactness in locally compact spaces. Finally, in [**24**], $G^o_{K,L}(X)$ was introduced and used to characterize, for locally compact X, Baireness of the space $C_k(X)$ of continuous real-valued functions on X with the compact-open topology.

In this article, we survey the various results related to the above games, occasionally with proofs or outlines of proofs. A few new results are included, and several open questions are stated. The game $G_{O,P}(X,x)$ will be discussed in Section 2, $G_{K,P}(X)$ in Section 3, and $G_{K,L}(X)$ and $G^o_{K,L}(X)$ in Section 4. In Section 5, we discuss a property related to the game $G_{K,L}(X)$ called the Moving Off Property, which turns out to be relevant to the problem of Baireness in function spaces.

Before proceeding, we should remark that the above definitions of the games are as we defined them in [**21**], but in some other papers we only required P's point or L's set to be in (or not, or disjoint from) his opponent's last choice, *rather than all previous choices*. These games are easily seen to be equivalent to their originals in the sense that a player has a winning strategy in one of the above games if and only if he has a winning strategy in the corresponding game in which the rules say P or L need only avoid his opponent's last move. We stick with the definitions as given above for a more convenient characterization of Eberlein and Corson compacts; the convenience has to do with the fact that the games are not equivalent if one wants to talk about the existence of certain *kinds* of winning strategies. This should become clearer in Section 3.

2. The game $G_{O,P}(X,x)$. This game was studied in detail in [**19**], where we called X a W-*space* if O has a winning strategy in $G_{O,P}(X,x)$ for every $x \in X$. We also defined X to be a w-*space* if for every $x \in X$, P fails to have a winning strategy in $G_{O,P}(X,x)$. Clearly, first-countable spaces are W-spaces. A prototypical W-space which is not first-countable is the one-point compactification of an uncountable discrete space.

2.1 Basic properties. Let us recall that a space X is a *Fréchet space* (respectively, *countably bi-sequential*) if whenever $x \in \overline{A}$ (respectively, $x \in \overline{A_n}$, where $A_0 \supset A_1 \cdots$), then there is a sequence $\{a_n\}_{n\in\omega}$ converging to x with $a_n \in A$ (respectively, $a_n \in A_n$) for all n. Also, X is an α_1-*space*, respectively, α_2-*space* if, whenever $A_0, A_1, \dots$ are sequences converging to $x \in X$, there is a sequence S converging to x with $S \cap A_n$ cofinite in A_n, respectively, $S \cap A_n$ infinite, for all n. (The α_i-spaces, $i = 1, 2, 3, 4$, were introduced by Arhangel'skii [**1**] in his study of Fréchetness in products.)

Recall that $\Sigma = \Sigma_{\alpha\in\kappa} X_\alpha$ is a Σ-*product* of the spaces $\{X_\alpha\}_{\alpha\in\kappa}$ if and only if there is $\vec{x} \in \Pi_{\alpha\in\kappa} X_\alpha$ such that $\Sigma = \{\vec{y} \in \Pi_{\alpha\in\kappa} X_\alpha : |\{\alpha : x_\alpha \neq y_\alpha\}| \le \omega\}$.

The following summarizes the main facts about W-spaces:

Theorem 2.1 (a) *First-countable* $\Rightarrow$ *W-space* $\Rightarrow$ *Fréchet and every countable subset is first-countable* $\Rightarrow$ *Fréchet α_1-space* $\Rightarrow$ *w-space Fréchet α_2-space* $\Rightarrow$ *countably bi-sequential*;

(b) *Separable W-spaces are first-countable*;

(c) *W-spaces are preserved by arbitrary subspaces, open mappings, and Σ-products* (*in particular, countable products*).

(d) *If X is a W-space, and Y has convergence property C, where $C \in \{$countably tight, countably bi-sequential, w-space$\}$, then $X \times Y$ has C*;

(e) *X is a w-space if and only if whenever $x \in \overline{A_n}$ for all $n \in \omega$, there exist $a_n \in A_n$ with $a_n \to x$.*

Statement (e) characterizing w-spaces is due to Sharma [**42**], as is the corollary that w and Fréchet α_2 are equivalent. I had shown that w implies countably bi-sequential, but Sharma's result makes this obvious. The only other nonobvious implication in (a) is the second one, which follows from (b). Why does (b) hold? Because one easily shows that the open neighborhoods of x in the range of a winning strategy for O, restricted to P's points chosen from a countable dense set, must form a base at x.

At the time of [**19**], I didn't know an example of a w-space which was not a W-space. But Hajnal and Juhasz, see [**15**], observed that the one-point compactification $T \cup \{\infty\}$ of an Aronszajn tree T (with the interval topology) is an example: neither player has a winning strategy in $G_{O,P}(T \cup \{\infty\}, \infty)$. Answering a question suggested in [**19**], Nyikos [**35, 37**] discovered several ZFC examples of countable w-spaces which were not first-countable. However, while some of these examples could fail to be α_1 under, say, $MA(\omega_1)$, it was not immediately clear if any were non-α_1 in ZFC. This was settled by Dow [**7**], who showed that in Laver's model for the Borel conjecture, all α_2-spaces are α_1. There is another model constructed by Dow and Steprans [**8**] in which all countable α_1-spaces are first-countable. So any countable non-first-countable w-space constructed in ZFC must be α_1 in the Laver model, but can't be α_1 in the Dow-Steprans model.

We describe one of Nyikos's examples, since it will be relevant also in later sections. It starts with the well-known "Cantor tree space."

Definition 2.2. Let $T = 2^{<\omega}$ be the Cantor tree, and let A be an uncountable subset of the Cantor set 2^ω. The space $T \cup A$, called the "Cantor tree space over A," has points of T isolated, and, for each $a \in A$, the branch $T_a = \{a \restriction n : n \in \omega\}$ is a sequence converging to a.

Now let $T \cup A \cup \{\infty\}$ be the one-point compactification of the Cantor tree space $T \cup A$, and let $X(A)$ be the subspace $T \cup \{\infty\}$. Since $X(A)$ is countable, but not first-countable, at $\{\infty\}$, O has no winning strategy in $G_{O,P}(X(A), \infty)$. Nyikos showed that P also has no winning strategy if A is a λ'-set, i.e., for any countable subset B of the Cantor set, B is G_δ in $B \cup A$. (Miller [**33**] also shows that the converse, i.e., that "P has no winning strategy in $G_{O,P}(X(A), \infty)$ implies A is a λ'-set" is consistent with and independent of ZFC.) Since there are uncountable λ'-sets in ZFC, this provides a ZFC example of a countable w-space which is not a W-space. Nyikos also observed that $X(2^\omega)$ is not w, but is Fréchet and α_3 (which is defined like α_2, but the sequence $S \cap S_n$ needs to be infinite for only infinitely many n, instead of all n).

Statement (d), and part of (c), of Theorem 2.1 shows that W-spaces behave well with respect to products. Similar statements cannot be made for w-spaces. Todorčević [**46**] constructed two monolithic Fréchet function spaces $C_p(X)$ and $C_p(Y)$ whose product is not countably tight. *Monolithic* means that closures of countable subsets are second countable, so by Theorem 2.1(a) these spaces are w-spaces.

An earlier example of a $C_p(X)$ which is w but not W came out of some interesting work of Gerlits and Nagy [**15**]. They say a space X has *property* (γ) if, given any collection $\mathcal{U}$ of open sets such that any finite subset of X is contained in some member of $\mathcal{U}$, there are $U_0, U_1, \dots$ in $\mathcal{U}$ such that $X \subset \cup_{n\in\omega}\cap_{i\geq n} U_i$. Also, recall that the "point-open game" of Telgársky goes as follows: at the nth play, P chooses a point p_n and O chooses an open set O_n containing p_n; P *wins* if $\cup_{n\in\omega} O_n = X$. Now we can state their key result:

Theorem 2.3. (a) $C_p(X)$ *is Fréchet if and only if* $C_p(X)$ *is a* w*-space if and only if* X *has property* (γ);

(b) $C_p(X)$ *is a* W*-space if and only if the point picker has a winning strategy in the point-open game on* X.

A subset of the real line having property (γ) is called a γ*-set.* γ-sets do not exist in ZFC, but spaces having property (γ) do; e.g., Galvin showed that any Lindelöf P-space, i.e., G_δ-sets are open, has (γ). There is a Lindelöf P space X in which Telǵarsky's game is undetermined, see [**44**]. Thus for this X, $C_p(X)$ is a w-space but not a W-space.

Gerlits [**14**] later showed that "$C_p(X)$ is a k-space" can be added to the list of properties in Theorem 2.3 (a).

An example due to Isbell, appearing in [**39**], produces two countable w-spaces whose product is not Fréchet. It is constructed from a Hausdorff gap. The assumption $2^{\aleph_0} < 2^{\aleph_1}$ is used in [**39**] in describing Isbell's example, and it is only claimed that they are countably bi-sequential (= Fréchet α_4). Nyikos [**35**] noticed that the examples are w-spaces, and that no special set-theoretic assumptions are needed. Since this seems not to be widely known, see, e.g., [**5**] and [**6**] where weaker examples have recently been published, we describe the example here (a bit differently than it was described in [**39**]). Recall that a collection $\{(A^0_\alpha, A^1_\alpha)\}_{\alpha<\omega_1}$ of pairs of infinite subsets of ω is a *Hausdorff gap* if

(a) $A^0_\alpha \subset^* A^0_\beta \subset^* A^1_\beta \subset^* A^1_\alpha$ for all $\alpha < \beta < \omega_1$;

(b) There is no C such that $A^0_\alpha \subset^* C \subset^* A^1_\alpha$ for all $\alpha < \omega_1$.

(Recall $A \subset^* B$ means $|A \backslash B| < \omega$.)

Example 2.4. Let $\{(A^0_\alpha, A^1_\alpha)\}_{\alpha<\omega_1}$ be a Hausdorff gap. Let

$$\mathcal{I}^0 = \{A \subset \omega : |A \cap A^0_\alpha| < \omega \text{ for all } \alpha < \omega_1\},$$
$$\mathcal{I}^1 = \{A \subset \omega : |A \cap (\omega \backslash A^1_\alpha)| < \omega \text{ for all } \alpha < \omega_1\}.$$

Let X^e be the space $\omega \cup \{\infty\}$ with ω isolated and complements of the ideal $\mathcal{I}^e$ forming the neighborhood base at ∞. Then X^e is a w-space, i.e., Fréchet α_2, for each $e < 2$, but $X^0 \times X^1$ is not Fréchet.

The verification of the above example is straightforward. The proof of non-Fréchetness of the product follows standard form by showing that point (∞, ∞) is a limit point of $A = \{(n,n)\}_{n\in\omega}$ but is not the limit of any convergent sequence from A. Note that $X^0 \times X^1$ is α_2, since Nogura [**34**] showed that the α_i-spaces for $i \in \{1,2,3\}$ are countably productive.

On the other hand, Nogura [**34**] proved that the class of w-spaces does have the following nice product property:

Theorem 2.5. *If X^n is a w-space for each $n \in \omega$, then X^ω is a w-space.*

In fact, Nogura proved the analogous result true for the classes of Fréchet α_i-spaces, $i = 1,2,3$. The analogue is not true, at least consistently, for all Fréchet spaces: I gave a counterexample under MA in [**22**]. But I do not know of any ZFC example. Except for the case $n = 1$ answered by the Isbell example, I also do not know the answer in ZFC to:

Question 2.6. Is there, for each positive integer n, a w-space X such that X^n is a w-space but X^{n+1} is not a w-space (equivalently, not Fréchet)?

Tamano [**43**] used the method of my example above to obtain, under MA, a (compact) Fréchet space X such that X^n is Fréchet but X^{n+1} is not. But these are not w-spaces, since there cannot be compact examples for the above question (Arhangel'skii [**1**] proved that the product of a Fréchet α_3-space and a compact Fréchet space is Fréchet). However, Szeptycki, see [**25**], recently constructed (noncompact) examples under CH.

2.2 Two variations. One natural variation of the game $G_{O,P}(X,x))$ is to allow P to choose a finite set of points instead of just one point. Let us denote this variation by $G^{fin}_{O,P}(X,x))$. We noted in [**19**] that this game is equivalent for player O, since, given a winning strategy for O for the usual game, O can consider what his move would be for each of P's finitely many points, and intersect these moves for his response.

However, this variation is not equivalent for Player P. Reznichenko and Sipacheva [**40**] call a space *Fréchet-Urysohn for finite sets*, or FU_f for short, at the point x if, whenever $\mathcal{F}$ is a collection of finite sets such that every neighborhood of x contains a member of $\mathcal{F}$, there are $F_n \in \mathcal{F}$, $n \in \omega$, such that $\cup_{n\in\omega} F_n$ is a sequence converging to x. If X is not FU_f at x, witnessed by $\mathcal{F}$, then P is assured of winning in $G^{fin}_{O,P}(X,x))$ by always choosing sets in $\mathcal{F}$. That shows one direction of the following result of Szeptycki and I [**25**]:

Theorem 2.7. *A space X is FU_f at x if and only if P has no winning strategy in $G^{fin}_{O,P}(X,x)$.*

Combining this with Proposition 2.1 (a), we see that the games $G^{fin}_{O,P}(X,x)$ and $G_{O,P}(X,x)$ are inequivalent for P if and only if there is a Frechét α_2-space which is not FU_f. There are many consistent examples of this, see [**25**], e.g., under CH, or under the assumption that there is a λ'-set of reals which is not a γ-set. The latter example has the form $X(A)$ of Nyikos above. It turns out that P has a winning strategy in $G^{fin}_{O,P}(X(A),\infty)$ if and only if A is a γ-set, see [**25**] or [**33**]. But this won't yield a ZFC example, since Miller [**33**] showed that every λ'-set is γ in the standard model of $MA_{\sigma-\text{centered}}(\omega_1)$.

However, a simple modification of the Hausdorff gap space of Example 2.4 does yield a ZFC example:

Example 2.8 [**26**]. There is a countable space $X = \omega \cup \{\infty\}$ such that P has no winning strategy in $G_{O,P}(X,\infty)$, but P has a winning strategy in $G^{fin}_{O,P}(X,\infty)$, even if restricted to choosing only two points at each turn. (In other words, X is Fréchet α_2, but, in the terminology of [**25**], not FU_2.)

Proof. Let X^0 and X^1 be the w-spaces of Example 2.4 built from a Hausdorff gap. Let Y^1 be the space X^1 using a disjoint copy ω' of ω, and let X be the space obtained by identifying the points ∞ of X^0 and Y^1. Note that X is also a "gap space" by the Hausdorff gap $\{(A^0_\alpha \cup (\omega \setminus A^1_\alpha)', A^1_\alpha \cup (\omega \setminus A^0_\alpha)')\}_{\alpha<\omega_1}$ in $\omega \cup \omega'$. (Here C' denotes the copy in ω' of a subset C of ω.) So X is Fréchet α_2.

Then the same argument that shows that $X^0 \times X^1$ is not Fréchet shows that P wins in $G^{fin}_{O,P}(X,\infty)$ by choosing in the nth round any legal doubleton of the form $\{n, n'\}$. For example, if not, then $C \cup C'$ would be convergent for some infinite $C \subset \omega$. C' convergent implies $C \cap (\omega \setminus A^1_\alpha)$ is infinite for some α. But then C is not convergent, a contradiction. □

We also considered in [**19**] the modification $G^c_{O,P}(X,x)$ in which O wins if P's chosen points merely cluster at the point x. It was not difficult to show that, for Player O, this game is equivalent to the original. For Player P, the situation is different. For one thing, the statement "P has no winning strategy in $G^c_{O,P}(X,x)$" does not necessarily imply that X is Fréchet at x. However, Hrušák [**28**] showed that it is consistent with ZFC that even in Fréchet spaces the two games are not equivalent for Player P. If $\mathcal{I}$ is a subbase for a proper ideal on ω, let $X(\mathcal{I})$ be the space $\omega \cup \{\infty\}$, where ω is the set of isolated points and neighborhoods of ∞ are complements of finite unions from $\mathcal{I}$.

Theorem 2.9. *If $\mathfrak{a} < \mathfrak{ra}$ or $\mathfrak{ra} = \mathfrak{c}$, then there is an almost-disjoint family $\mathcal{A}$ of subsets of ω such that $X(\mathcal{A})$ is a Fréchet space in which P has a winning strategy in the convergence game $G_{O,P}(X,\infty)$, but not in the clustering game $G^c_{O,P}(X,\infty)$.*

Recall that $\mathfrak{a}$ is the least cardinal of an infinite, maximal almost-disjoint family of subsets of ω, and $\mathfrak{r}$ is the least cardinal of a family $\mathcal{R}$ of infinite subsets of ω such that, for any subset X of ω, either X

or $\omega\backslash X$ almost contains a member of $\mathcal{R}$. Hrušák's proof of the above theorem uses a result of LaFlamme [**30**], who showed that for a proper ideal $\mathcal{I}$ on ω, P has a winning strategy in the game $G^{c}_{O,P}(X(\mathcal{I}),\infty)$ if and only if I is not "+-Ramsey", where I is *+-Ramsey* if and only if every tree $T\subset\omega^{<\omega}$ where for each node σ, $\{n\in\omega:\sigma^\frown\langle n\rangle\}$ is not in $\mathcal{I}$, has a branch whose range is not in $\mathcal{I}$. Hrušák shows that, under the assumptions stated in the theorem above, there is a MAD family on ω such that the ideal generated by $\mathcal{A}$ is +-Ramsey, and that if there is such a MAD family, then there is a corresponding almost-disjoint family $\mathcal{A}'$ so that the space $X(\mathcal{A}')$ is a Fréchet space in which P has a winning strategy in the game $G_{O,P}(X,\infty)$.

It is apparently not known if there is a ZFC example like this:

Question 2.10 [**28**]. Is there in ZFC a MAD family $\mathcal{A}$ of subsets of ω such that the corresponding ideal is $+$-Ramsey?

If so, there would be a ZFC example as in Theorem 2.9. We don't know the answer to this MAD family question, but we do have a positive answer to Hrušák's more general question whether there is a space in ZFC of the form $Y=\omega\cup\{\infty\}$ in which the two games are inequivalent.

Example 2.11. There is, in ZFC, an almost-disjoint family $\mathcal{A}$ on ω such that $X(\mathcal{A})$ is a Fréchet space in which P has a winning strategy in the game $G_{O,P}(Y,\infty)$, but P does not have a winning strategy in the game $G^{c}_{O,P}(Y,\infty)$.

Proof. Instead of ω, we define the almost-disjoint family on the rationals in $[0,1]$, which we denote by $\mathbf{Q}$. Let $\{D_\alpha\}_{\alpha<\mathfrak{c}}$ index all dense subsets of $\mathbf{Q}$. Let $B=\{x_\alpha:\alpha<\mathfrak{c}\}$ be any subset of $[0,1]\backslash\mathbf{Q}$ which has cardinality $\mathfrak{c}$ but does not contain any perfect set, e.g., one can take B to be a Bernstein set. At step α, choose a sequence S_α in D_α which converges to x_α. For convenience, if $x\in[0,1]$, we also use S_x to denote S_α, if $x=x_\alpha$, and we let $S_x=\varnothing$ if $x\notin B$. Let $\mathcal{A}=\{S_\alpha\}_{\alpha<\mathfrak{c}}=\{S_x:x\in B\}$.

Claim 1. *$X(\mathcal{A})$ is Fréchet.*

Suppose ∞ is a limit point of $C \subset \mathbf{Q}$. Then C cannot be covered by finitely many members of $\mathcal{A}$. Let L be the set of limit points of C. If L is finite, then $S = C \setminus \cup_{y \in L} S_y$ has finite intersection with every member of $\mathcal{A}$, hence converges to ∞. If L is infinite, let y be a limit point of L. Then one can construct a sequence $S \subset C$ converging to y in the real line but disjoint from S_y. Again, S converges to ∞.

Claim 2. *P has a winning strategy in $G_{O,P}(X(\mathcal{A}), \infty)$.*

Clearly P can make sure his chosen points are dense, which wins for him by the construction of $\mathcal{A}$.

It remains to prove the following claim.

Claim 3. *P has no winning strategy in $G^c_{O,P}(X(\mathcal{A}), \infty)$.*

By way of contradiction, suppose P has a winning strategy Ψ. Then for each complete play of the game with P using Ψ, the set of limit points of P's chosen points must be a finite subset of B. Let $\mathcal{S}$ be all finite initial segments of plays of the game with P using Ψ. If s' extends s, we denote this by $s' \supset s$. For each $s \in \mathcal{S}$, let P_s denote the set of limit points of P's points for all complete plays of the game starting with s.

Suppose there exists $s_0 \in \mathcal{S}$ such that $\cap_{s \supset s_0} \overline{P_s} \neq \varnothing$. In this case, O can defeat Ψ as follows. Let $x \in \cap_{s \supset s_0} \overline{P_s}$. O begins by playing the moves of s_0. Then he plays the complement of S_x. At the nth play after that, there will be a finite extension of the play so far such that one of P's points will lie within $1/2^n$ of x. At the end of the game, P's points will include a sequence converging to x but missing S_x, so P has lost, a contradiction.

It follows that, given any $s \in \mathcal{S}$, the set of closures of the P_t's for $t \supset s$ does not have the finite intersection property. So one can construct a finitely branching tree $\mathcal{T} \subset \mathcal{S}$ such that

(a) For each $s \in \mathcal{T}$, if F_s is the set of immediate successors of s in $\mathcal{T}$, then $\cap_{t \in F_s} \overline{P_t} = \varnothing$;

(b) If s is at the nth level of $\mathcal{T}$, then at some stage in each play $t \in F_s$, P has chosen a point q_t within $1/2^n$ of the set P_s.

Each branch of $\mathcal{T}$ is a play of the game with P using Ψ, and hence the set of rationals representing P's choices along a branch has only finitely many limit points, all in B. The set L of all limit points for all the branches of $\mathcal{T}$ is easily seen to be analytic. (We thank Howard Becker for pointing this out to the author.)

So we have our final contradiction once we prove that L is uncountable (hence must contain a Cantor set, hence a point not in B). Suppose $L = \{x_n : n < \omega\}$. There is $t_0 \in F_\varnothing$ with $x_0 \notin \overline{P_{t_0}}$. Then find $t_1 \in F_{t_0}$ with $x_1 \notin \overline{P_{t_1}}$. And so on. This defines a branch b of $\mathcal{T}$. By condition (b), some limit point of P's choices in b is in $\cap_{n<\omega}\overline{P_{t_n}}$. But any limit point is x_n for some n, which is not in $\overline{P_{t_n}}$, a contradiction. □

A variation of the clustering game $G^{c}_{O,P}(X,x)$ was considered by Bouziad [**3**]; the difference in Bouziad's game is that O wins if P's points merely cluster *somewhere*, in X. A space was defined to be a $\mathcal{G}$-space if O had a winning strategy at every point. Recall that X is a *q-space* if, for each $x \in X$, there are neighborhoods U_n, $n \in \omega$, of x such that $\{x_n : n \in \omega\}$ has a cluster point whenever $x_n \in U_n$ for all n. Bouziad's $\mathcal{G}$-spaces generalize q-spaces in the same way that W-spaces generalize first-countability. Later, Garcia-Ferreira, González-Silva and Tomita [**12, 13**] studied the version of Bouziad's game with "cluster point" replaced by "p-limit point" for some $p \in \omega^*$.

2.3 Continuously perfectly normal spaces. Let me finish this section by mentioning our original motivation for considering W-spaces. It came from a problem of Zenor [**48**], who defined a space X to be *continuously perfectly normal* (CPN) if there is a continuous function $\phi : X \times 2^X \to [0,1]$, where 2^X is the space of closed subsets of X with the Vietoris topology, such that, for any $x \in X$ and closed set H, $\phi(x,H) = 0$ if and only if $x \in H$. He asked if there was a nonmetrizable CPN-space. He had proven that CPN-spaces must be Fréchet. I discovered they must be W-spaces, which led me to the following example. Let $X = (\omega \times \omega_1) \cup \{\infty\}$, where the points of $\omega \times \omega_1$ are isolated, and a basic neighborhood of ∞ has the form

$$B(n,F) = \{\infty\} \cup \{(m,\alpha) \in X : m > n \text{ and } \alpha \notin F\},$$

where $n \in \omega$ and F is a finite subset of ω_1. It is not hard to show X is a non-first-countable W-space. It turns out X is also CPN [**18**].

Except for a modification in which the space is made first-countable by blowing up the nonisolated point to $[0,1]$, this X remains the only known example of a non-metrizable CPN-space.

3. The game $G_{K,P}(X)$. Recall that in this game, at the nth play K chooses a compact $K_n \subset X$, and P chooses a point $p_n \in X \setminus \cup_{i \leq n} K_i$; K wins if the p_n's are closed discrete in X. If X is locally compact, note that this game is essentially equivalent to $G_{O,P}(X \cup \{\infty\}, \infty)$, where $X \cup \{\infty\}$ is the one-point compactification of X.

3.1 Covering properties. This game played on a locally compact space X turns out to be related to covering properties of X. Recall that X is *metacompact* (respectively, *σ-metacompact, metalindelöf*) if every open cover of X has a point-finite (respectively, σ-point-finite, point-countable) open refinement.

In [**20**], we proved that for locally compact spaces of countable tightness, K has a winning strategy in $G_{K,P}(X)$ if and only if X is metalindelöf. The countable tightness assumption is not necessary for the "if" direction, but we don't know if it is necessary for the "only if" direction.

Question 3.1. Let X be locally compact. If K has a winning strategy in $G_{K,P}(X)$, must X be metalindelöf?

The original proof of the metalindelöf result is, in retrospect, tailor-made for elementary submodels. We outline an elementary submodel style proof below, and also include a new special case where the tightness assumption is replaced by a "locally small" assumption.

Theorem 3.2. *Let X be locally compact and either countably tight or locally of cardinality not greater than $\aleph_1$. Then K has a winning strategy in $G_{K,P}(X)$ if and only if X is metalindelöf.*

Proof. If X is metalindelöf, then there is a point-countable cover $\mathcal{U}$ of X by open sets with compact closures. K wins by looking at the countably many members of $\mathcal{U}$ containing P's chosen point at each round and choosing an increasing sequence of compact sets that eventually cover every one of these members of $\mathcal{U}$. It is easy to check that this wins for K.

Now suppose K has a winning strategy σ, and let $\mathcal{U}$ be a cover of X by open sets with compact closures (of cardinality $\leq \omega_1$ if X is locally of cardinality $\leq \omega_1$). Let M be an elementary submodel (of some sufficiently large $H(\theta)$) with $X, \mathcal{U}, \sigma \in M$.

Claim 1. $\overline{M \cap X} \subset \cup(M \cap \mathcal{U})$.

Proof of Claim 1. Suppose $p \in \overline{M \cap X} \setminus \cup (M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$. Suppose $F = \{x_0, x_1, \ldots, x_n\} \subset U_p \cap (M \cap X)$. Then $\sigma(F)$ is compact and in M so there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ in M covering $\sigma(F)$. Since M also contains a finite subset of $\mathcal{U}$ covering $\overline{\cup \mathcal{U}_0}$, we have $p \notin \overline{\cup \mathcal{U}_0}$. So there exists $x_{n+1} \in U_p \cap (M \cap X) \setminus \overline{\cup \mathcal{U}_0}$. It follows that if K uses the strategy σ, P can always choose a point in $U_p \cap (M \cap X)$. But then K loses the game, a contradiction which completes the proof of Claim 1.

Since there is an M with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.

Claim 2. *There is a point-countable open refinement $\mathcal{V}_M$ of $M \cap \mathcal{U}$ covering $\cup(M \cap \mathcal{U})$.*

Proof of Claim 2. This is obvious if $|M| = \omega$. Suppose $|M| = \kappa$ and Claim 2 is true whenever $|M| < \kappa$. Let $\{M_\alpha : \alpha < \kappa\}$ be a continuous increasing sequence of elementary submodels of cardinality less than κ whose union is M. Let $X_\alpha = M_\alpha \cap X$ and $\mathcal{U}_\alpha = M_\alpha \cap \mathcal{U}$. Note that for limit α, $X_\alpha = \cup_{\beta<\alpha} X_\beta$ and $\mathcal{U}_\alpha = \cup_{\beta<\alpha} \mathcal{U}_\beta$. If the closure of each U in $\mathcal{U}$ has cardinality $\leq \omega_1$, and $U \in \mathcal{M}_\alpha$, we may assume $\overline{U} \subset \mathcal{M}_{\alpha+\omega_1}$.

For each $U \in \mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha$, $\overline{U} \cap \overline{X_\alpha}$ is compact and in $M_{\alpha+1}$, and by Claim 1 a subset of $\cup \mathcal{U}_\alpha$, so there is an open set $S(U) \in M_{\alpha+1}$ with $\overline{U} \cap \overline{X_\alpha} \subset S(U) \subset \overline{S(U)} \subset \cup \mathcal{U}_\alpha$. Let $U' = U \setminus \overline{S(U)}$, and let $U^*(\alpha) = \cup\{U' : U \in \mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha\}$. Note that $U^*(\alpha)$ covers $\cup \mathcal{U}_{\alpha+1} \setminus \cup \mathcal{U}_\alpha$ and that $\{U^*(\alpha) : \alpha < \kappa\}$ covers $\cup(M \cap \mathcal{U})$.

Subclaim. $\{U^*(\alpha) : \alpha < \kappa\}$ is point-countable. Suppose not; then we can find a point p, and $U_\delta \in \mathcal{U}_{\alpha_\delta}$ for $\delta < \omega_1$, such that $p \in \cap_{\delta<\omega_1} U'_\delta$. Without loss of generality, we may assume that $\delta < \delta' \Rightarrow \alpha_\delta < \alpha_{\delta'}$. Let $\gamma = sup\{\alpha_\delta : \delta < \omega_1\}$. Since $U'_\delta \in M_{\alpha_\delta}$, the set $\{U_\delta \cap X_\gamma : \delta < \omega_1\}$ has the finite intersection property, so there exists $y \in \cap\{\overline{U}_\delta : \delta < \omega_1\} \cap \overline{X}_\gamma$. If X has countable tightness, then $y \in \overline{X}_\gamma = \cup_{\delta<\omega_1} \overline{X}_{\alpha_\delta}$. If X is locally of cardinality $\leq \omega_1$, then $|\overline{U_0}| \leq \omega_1$, and hence $y \in \overline{U_0} \subset M_\gamma \cap X = X_\gamma = \cup_{\delta<\omega_1} X_{\alpha_\delta}$. In either case, $y \in \overline{X}_{\alpha_\delta}$ for some $\delta < \omega_1$. But $\delta' > \delta \Rightarrow \overline{U}'_\delta \cap \overline{X}_{\alpha_\delta} = \varnothing$, a contradiction.

Now to finish the proof of Claim 2 and the proposition, for $\alpha < \kappa$, let $\mathcal{V}_\alpha$ be the assumed point-countable open refinement of $\mathcal{U}_\alpha$, let

$$\mathcal{V}^p_\alpha = \{V \cap U^*(\alpha) : V \in \mathcal{V}_\alpha\},$$

and let $\mathcal{V}_M = \cup_{\alpha<\kappa} \mathcal{V}'_\alpha$. It is easy to check that $\mathcal{V}_M$ is a point-countable open refinement of $M \cap \mathcal{U}$ covering $\cup(M \cap \mathcal{U})$. □

Corollary 3.3. *Assume CH. Let X be locally compact and locally of cardinality not greater than $\mathfrak{c}$. Then K has a winning strategy in $G_{K,P}(X)$ if and only if X is metalindelöf.*

A similar kind of argument is used to show the following; but note that a countable tightness assumption is not needed here:

Theorem 3.4. *Suppose X is locally compact. Then X is metacompact if and only if K has a winning strategy in $G_{K,P}(X)$ which depends only on P's last move, i.e., a stationary winning strategy, and X is σ-metacompact if and only if K has a winning strategy in $G_{K,P}(X)$ depending only on P's last move and what round it is, i.e., a Markov winning strategy.*

Theorem 3.5. *Suppose X is a locally compact scattered space. Then X is metacompact if and only if K has a winning strategy in $G_{K,P}(X)$.*

Regarding the question of omitting the countable tightness assumption in Theorem 3.2, it may be useful to note a connection with a property studied in [**2**]. A cover $\mathcal{U}$ of X is said to be *finite-in-countable* if, for any infinite $A \subset X$, there is some finite $F \subset A$ such that $\{U \in \mathcal{U} : F \subset U\}$ is countable.

Lemma 3.6. *Suppose X has a finite-in-countable cover $\mathcal{U}$ by open sets with compact closures. Then K has a winning strategy in $G_{K,P}(X)$.*

Proof. Clearly K can devise a strategy such that, for any finite subset F of P's chosen points, if $\mathcal{U}_F = \{U \in \mathcal{U} : F \subset U\}$ is countable, then for any $U \in \mathcal{U}_F$, $K_n \supset \overline{U}$ for all sufficiently large n. Suppose the set C of P's chosen points do not form a discrete set. Let x be a limit point of C, with $x \in U \in \mathcal{U}$. Then $U \cap C$ is infinite, so contains a finite set F with $\mathcal{U}_F$ countable. But then P cannot choose a point in $\overline{U}$ after some finite stage, a contradiction. $\square$

Hence it follows that if a space satisfies the hypotheses of Lemma 3.6, and has countable tightness or is locally of cardinality $\leq \omega_1$, then it is metalindelöf. But again we don't know if the countably tight or locally small assumption is necessary. A natural place to look for a possible counterexample is ω^*:

Question 3.7. Let O be an open non-metalindelöf subset of ω^*. Could K have a winning strategy in $G_{K,P}(O)$? Could O have a finite-in-countable cover by compact open sets?

3.2 Corson and Eberlein compacts. A space X is *Eberlein compact* (*EC*) if it is homeomorphic to a weakly compact subset of a Banach space. However, the following well-known embedding characterizations are handier for our discussion of EC's and related classes.

Definition 3.8. A compact X is

(a) *Corson compact* if and only if for some κ, X embeds in

$$\{\vec{x} \in \mathbf{R}^\kappa : x_\alpha = 0 \text{ for all but countably many } \alpha \in \kappa\};$$

(b) *Eberlein compact* if and only if for some κ, X embeds in

$$\{\vec{x} \in \mathbf{R}^\kappa : \forall\, \varepsilon > 0,\ |x_\alpha| \geq \varepsilon \text{ for only finitely many } \alpha \in \kappa\};$$

(c) *strong Eberlein compact* if and only if for some κ, X embeds in

$$\{\vec{x} \in \mathbf{R}^\kappa : x_\alpha = 0 \text{ for all but finitely many } \alpha \in \kappa\}.$$

Note that, by Theorem 2.1 (c), all of these classes of spaces are W-spaces. It is also the case that strong EC's are exactly the scattered EC's. A prototypical example of a nonmetrizable (strong) EC is the one-point compactification of an uncountable discrete space. See [**47**] for a nice survey of EC's and some related classes from a topological point of view.

The following characterizations of Corson and (strong) Eberlein compact spaces appear in [**20, 21**].

Theorem 3.9. *Let X be a compact space and Δ the diagonal in X^2. Then*:

(a) *X is Corson compact if and only if K has a winning strategy in $G_{K,P}(X^2\backslash\Delta)$ if and only if $X^2\backslash\Delta$ is metalindelöf if and only if X^2 is hereditarily metalindelöf*;

(b) *X is Eberlein compact if and only if K has a winning strategy in $G_{K,P}(X^2\backslash\Delta)$ depending only on P's last move and the number of the move if and only if $X^2\backslash\Delta$ is σ-metacompact if and only if X^2 is hereditarily σ-metacompact.*

(c) *If X is compact scattered, then X is strong Eberlein compact if and only if K has a winning strategy in $G_{O,P}(X,p)$ for every $p \in X$, i.e., X is a W-space as defined in Section 2, if and only if X is hereditarily metacompact.*

Note that, for X compact, $X^2\backslash\Delta$ is locally compact. So the covering property equivalences vis-a-vis the game follow from the results of the previous subsection. Regarding part (c), of course there is also an off-diagonal characterization of strong Eberlein compact, but the W-space characterization is simpler and hence more useful.

Thus we have in the above theorem both covering property and game characterizations of various classes of compacta. These classes have been extensively studied in the literature and there are many other characterizations as well. But there do seem to be occasions when the game characterizations are perhaps the most convenient; e.g., see

Example 3.9 in [**20**] and Theorem 4.6 in [**27**]. As an illustration, we outline a short proof of the following fact, due to Efimov and Certanov [**9**] and, independently, Rudin.

Theorem 3.10. *A Corson compact linearly ordered space X must be metrizable.*

Outline of proof. By Theorem 3.8 (b), K has a winning strategy in $G_{K,P}(X^2 \backslash \Delta)$. K also has a winning strategy in the game in which P is allowed to choose a finite set of points in each round. Let l be the least point and m the maximum point of X. Now suppose K has played $K_0, K_1, \dots, K_n$, and let $U_n = X^2 \backslash \cup_{i \leq n} K_i$. Then U_n is an open superset of the diagonal, and it is easy to use compactness of X to see that P can find a finite sequence of points $l = x_{n,0} < x_{n,1} < \cdots < x_{n,k_n} = m$ such that for each $i < k_n$, either the point $(x_{n,i}, x_{n,i+1})$ is in U_n or the corresponding open interval from $x_{n,i}$ to $x_{n,i+1}$ is empty. P chooses in round n all points $(x_{n,i}, x_{n,i+1})$ that are in U_n. We suppose K has used a winning strategy. Then it is not difficult to use the fact that all limit points in X^2 of P's chosen points are on the diagonal to show that the set of all open intervals with endpoints in the set $\{x_{n,i} : n \in \omega,\ i \leq k_n\}$ is a countable base for X; see Theorem 4.5 in [**20**] for the details. □

There are several other classes of compacta related to the ones we have been discussing. For example, X is *uniform Eberlein compact* (*UEC*) if and only if it is homeomorphic to a weakly compact subset of a Hilbert space. UEC's also have an embedding characterization as in 3.8: they are exactly those compact spaces such that, for some function $N : \mathbf{R} \to \mathbf{N}$ and for some κ, X embeds in $\{\vec{x} \in \mathbf{R}^\kappa : \forall\, \varepsilon > 0,\ |x_\alpha| \geq \varepsilon$ for at most $N(\varepsilon)-$many $\alpha \in \kappa\}$. Another class is the class of *Gul'ko compact* spaces, which are those compacta X such that $C_p(X)$ is a Lindelöf Σ-space, i.e., a continuous image of a perfect pre-image of a separable metric space. I do not know of game characterizations of these classes.

Question 3.11. Are there useful game characterizations of other classes of compacta, e.g., uniform Eberlein compacta or Gul'ko compacta?

Possibly relevant is an off-diagonal covering property of Gul'ko compacta given in [**23**]; it was recently proven to be an equivalence by Garcia, Oncina and Orihuela [**11**].

Before moving on to the next game, we should mention that Nyikos's article [**36**] contains, among other things, a nice survey of many of the results of this section and the preceding one that were known at the time, and includes some related ideas and results not mentioned here.

4. The games $G_{K,L}(X)$ and $G^o_{K,L}(X)$. $G_{K,L}(X)$ was introduced by the author in [**21**] and independently by McCoy and Ntantu in [**32**] where it was denoted $\Gamma_1(X)$. It is defined just like $G_{K,P}(X)$, except that P, who is renamed L, chooses compact sets instead of points, i.e., L's nth play is a compact set L_n missing all of K's previous moves $K_0, K_1, \dots, K_n$. K wins if and only if $\{L_i\}_{i\in\omega}$ is a discrete collection. $G^o_{K,L}(X)$ is the same as $G_{K,L}(X)$, except that K wins if and only if $\{L_i\}_{i\in\omega}$ has a discrete open expansion.

Since this makes it harder for K to win vis-a-vis $G_{K,P}(X)$, the property "K has a winning strategy" is stronger for this game. It is easy to see that K has a winning strategy in any locally compact σ-compact space: K simply chooses at the nth play the nth set in an increasing sequence of compact sets whose interiors cover the space. It is nearly as easy to see that K wins if X is a topological sum of locally compact σ-compact spaces, i.e., whenever X is locally compact and paracompact. The next theorem shows we have an equivalence:

Theorem 4.1. *Let X be a locally compact space. Then the following are equivalent*:

(a) *K has a winning strategy in $G_{K,L}(X)$*;

(b) *K has a winning strategy in $G^o_{K,L}(X)$*;

(c) *X is paracompact.*

Proof. That (c) implies (b) is easy, as indicated above. That (b) implies (a) follows because it is easier for K to win in $G_{K,L}(X)$ than in $G^o_{K,L}(X)$.

It remains to prove (a) implies (c). This proof is in the same spirit as the proof of Theorem 3.2, but is a bit simpler. We give an elementary submodel version of our original proof in [**21**].

Let $\mathcal{U}$ be a cover of X by open sets with compact closures. Let σ be a winning strategy for K. We may assume the domain of σ is the set of all finite sequences of compact subsets of X. Let M be an elementary submodel containing $\sigma, X, \mathcal{U}, \dots ,$. It suffices to prove the following:

Claim. *$M \cap \mathcal{U}$ has a locally finite refinement covering $\cup(M \cap \mathcal{U})$.*

The proof of the claim is by induction on $|M|$. If M is countable, then $\cup(M \cap \mathcal{U})$ is σ-compact (noting that $U \in M \cap \mathcal{U}$ implies there is a finite subset of $M \cap \mathcal{U}$ which covers $\overline{U}$), so the claim holds in this case. Now suppose $|M| = \kappa$ and the result is claim whenever $|M| < \kappa$. Let $\{M_\alpha : \alpha < \kappa\}$ be a continuous increasing sequence of elementary submodels of cardinality less than κ whose union is M. Let $\mathcal{U}_\alpha = M_\alpha \cap \mathcal{U}$. If each $\cup U_\alpha$ is clopen, then it is easy to construct a locally finite refinement of $M \cap \mathcal{U}$ from locally finite refinements of the $\mathcal{U}_\alpha$'s. So there exists $\delta < \kappa$ and a point $p \in \overline{\cup \mathcal{U}_\alpha} \backslash (\cup \mathcal{U}_\alpha)$. Let N be a compact neighborhood of P. Suppose $L_0, L_1, \dots , L_n$ is a sequence of legitimate moves of L with K using the winning strategy σ, with each $L_i \in M_\delta$. We obtain a contradiction to σ being a winning strategy by showing that L has a legitimate response $L_{n+1} \in M_\delta$ such that $L_{n+1} \cap N \neq \varnothing$.

To this end, for $i \leq n + 1$, let $K_i = \sigma(L_0, L_1, \dots , L_{i-1})$. Then $K_i \in M_\delta$ for each i, and so some finite subcollection $\mathcal{V}$ of $M_\delta \cap \mathcal{U}$ covers $\cup_{i \leq n+1} K_i$. The point p cannot be in $\overline{\cup \mathcal{V}}$, so there is some $U \in \mathcal{U}_\delta$ with $N \cap [U \backslash \overline{\cup \mathcal{V}}] \neq \varnothing$. Then letting $L_{n+1} = \overline{U \backslash \cup \mathcal{V}}$ does the trick. □

In locally compact spaces, the games $G_{K,L}(X)$ and $G^o_{K,L}(X)$ are equivalent for Player L as well as K, a fact we apparently did not notice when writing [**24**]. For, if L has a winning strategy in $G_{K,L}(X)$, he can construct one in $G^o_{K,L}(X)$ essentially by selecting compact neighborhoods of the sets L_n given to him by a winning strategy in $G_{K,L}(X)$. Why introduce $G^o_{K,L}(X)$ at all? Because it seems to be the more natural one for attacking the following problem:

Question 4.2. For what (completely regular) spaces X is $C_k(X)$ a Baire space?

Here, $C_k(X)$ is the space of continuous real-valued functions on X with the compact-open topology. The following lemma, which except for a minor reformulation is due to McCoy and Ntantu [**32**], shows the connection. Recall that Baire spaces have the following game characterization: Players Empty and Nonempty successively choose nonempty open subsets sets of their opponent's previous move. Empty wins if the intersection of the chosen open sets is Empty, otherwise Nonempty wins. Then a space is Baire if and only if Empty has no winning strategy. If Nonempty has a winning strategy, the space is said to be *Choquet* or *weakly α-favorable*. The latter term is the more classical one; however, we will follow Kechris [**29**] and use the former. We also follow [**29**] by calling the game the *Choquet game*.

Lemma 4.3. (a) *If $C_k(X)$ is Choquet, then K has a winning strategy in $G^o_{K,L}(X)$;*

(b) *If $C_k(X)$ is Baire, then L has no winning strategy in $G^o_{K,L}(X)$.*

Proof. We prove (b), (a) being similar. Suppose L has a winning strategy in $G^o_{K,L}(X)$. We show that $C_k(X)$ fails to be Baire by showing that Empty has a winning strategy in the Choquet game on $C_k(X)$.

Without loss of generality, in the nth round, Nonempty chooses a basic open set of the form $B(K_n, f_n, \varepsilon_n) = \{g \in C_k(X) : \forall x \in K_n(|g(x) - f_n(x)| < \varepsilon_n\}$, where K_n is compact. Let L_n be L's response to K_n in $G^o_{K,L}(X)$ using a winning strategy. Then Empty plays $B(K_n, f_n, \varepsilon_n\} \cap B(L_n, c_n, 1/2)$. If $\phi \in C_k(X)$ is in all chosen sets, then $\phi(L_n) \subset (n - 1/2, n + 1/2)$ for all n. It easily follows that $\{L_n : n \in \omega\}$ has a discrete open expansion, a contradiction. Hence this strategy is winning for Empty. $\square$

We do not know if the converse of either (a) or (b) of the above lemma holds:

Question 4.4. Is it true that for any completely regular space X, $C_k(X)$ is Baire if and only if L has no winning strategy in $G^o_{K,L}(X)$?

That $C_k(X)$ is Choquet if and only if K has a winning strategy in $G^o_{K,L}(X)$?

The answer to Question 4.4 is positive for locally compact spaces. For Baire, this was done by Ma and myself in [**24**]. For normal X, the Choquet case was done earlier by McCoy and Ntantu. However, they didn't have Theorem 4.1 in hand, which yields the following stronger result, essentially due to Ma [**31**].

Theorem 4.5. *Let X be locally compact. Then the following are equivalent*:

(a) $C_k(X)$ *is Choquet*;

(b) $C_k(X)$ *is a product of completely metrizable spaces*;

(c) K *has a winning strategy in* $G_{K,L}(X)$;

(d) X *is paracompact.*

Proof. That (b) implies (a) is well known [**29**]. That (a) implies (c) is part of Lemma 4.3 and (c) implies (d) is Theorem 4.1. But (d) plus locally compact implies X is the topological sum of locally compact, σ-compact spaces, which in turn implies that $C_k(X)$ is the product of spaces of the form $C_k(Y)$, where Y is locally compact and σ-compact. But such $C_k(Y)$ are completely metrizable [**32**]. $\square$

We also give in [**24**] an internal, nongame-theory characterization of L having no winning strategy in $G^o_{K,L}(X)$ by defining a property we call the *Moving Off Property* (MOP for short), and showing that "X has the MOP" is equivalent to "L has no winning strategy in $G^o_{K,L}(X)$."

Definition. A collection $\mathcal{L}$ of nonempty compact subsets of X is said to *move off* the compact sets if, for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \varnothing$. The space X is said to have the MOP if and only if every collection $\mathcal{L}$ which moves off the compact sets contains an infinite subcollection which has a discrete open expansion. A useful equivalence, proved in [**24**], is: X has the MOP if and only if, for every sequence $\mathcal{L}_0, \mathcal{L}_1, \dots$, of moving off collections in X, one can choose $L_i \in \mathcal{L}_i$ such that $\{L_i\}_{i\in\omega}$ has a discrete open expansion.

Here, then, is the main theorem of [**24**], except that the equivalence of (c) was not mentioned there:

Theorem 4.6. *Let X be a locally compact space. Then the following are equivalent*:

(a) $C_k(X)$ *is a Baire space*;

(b) L *has no winning strategy in* $G^o_{K,L}(X)$;

(c) L *has no winning strategy in* $G_{K,L}(X)$;

(d) X *has the moving off property.*

Remarks on the proof. We already have seen (a)⇒(b) for any space, and (b) $\iff$ (c) for locally compact spaces is easy and was already noted. That (c)⇒(d) is also easy: if there is a moving off collection $\mathcal{L}$ of compact sets which witnesses failure of the moving off property, then all L has to do to win in choose sets in $\mathcal{L}$. So (d)⇒(a) is the meat of the theorem. The proof, which we will not repeat here, goes back and forth between the game $G_{K,L}(X)$ and the Choquet game on $C_k(X)$. $\square$

The above result was stated in [**24**] for q-spaces, a common generalization of local compactness and first-countability. However, the theorem for q-spaces is not fundamentally more general, as McCoy and Ntantu [**32**] had shown that for a q-space, $C_k(X)$ Baire implies X is locally compact (see also Lemma 5.1). Recently, Nyikos [**38**] showed that the statement "the subspace of $C_k(X)$ consisting of the functions which vanish at infinity is Fréchet" can be added to the list of equivalences in Theorem 4.6.

The conditions (b) and (d) of Theorem 4.6 are equivalent for all spaces [**24**]. Thus, the question of whether or not the MOP characterizes Baireness of $C_k(X)$ is equivalent to Question 4.4.

Question 4.7. Is "exactly those (completely regular) spaces X which have the MOP" an answer to Question 4.2, i.e., is $C_k(X)$ Baire if and only if X has the MOP?

For this question, requiring that the sequence $\{L_n\}_{n\in\omega}$ of compact sets (in the game or in the MOP) have discrete open expansions, instead of just being discrete, is important, since there are spaces in which these differ.

Example 4.8. There is a completely regular space X such that L has a winning strategy in $G^o_{K,L}(X)$ but not in $G_{K,L}(X)$ (in fact, K has a winning strategy in $G_{K,L}(X)$).

Proof. Let Z be the space $\omega_1 + 1$ with all $\alpha < \omega_1$ isolated, and ω_1 retaining its usual order-theoretic neighborhoods. Let

$$X = \{(\alpha, \beta) \in (\omega + 1) \times Z : \alpha \neq \omega \text{ or } \beta \neq \omega_1\}.$$

For $\alpha < \omega_1$, let $H_\alpha = (\omega + 1) \times \{\alpha\}$. Observe that any sequence $\{L_n\}_{n\in\omega}$ of disjoint compact sets such that each H_α meets only finitely many of them is discrete. It is easy for K to use this to devise a winning strategy in $G_{K,L}(X)$. However, L can always choose a set of the form (n, ω_1), $n < \omega$. No infinite collection of such points has a discrete open expansion. Thus L has a winning strategy in $G^o_{K,L}(X)$. $\square$

Of course, by Lemma 4.3, if X is as above, then $C_k(X)$ is not Baire.

5. What spaces have the MOP**?** Suppose one really wants to know if $C_k(X)$ is Baire for a certain locally compact space X. By Theorem 4.6, it reduces to determining if L has a winning strategy in $G_{K,L}(X)$, or equivalently, if X has the MOP. This certainly seems less daunting than dealing with $C_k(X)$ directly. Still, it is often not clear what spaces, even locally compact ones, have the MOP. The following result from [**24**] (this was essentially done in [**32**] as well) gives some useful necessary, but far from sufficient, conditions. (See the paragraph immediately preceding subsection 2.3 for the definition of q-space.)

Lemma 5.1. *Suppose a regular space X has the MOP. Then*:

(a) *Every closed pseudocompact subspace of X is compact*;

(b) *If X is a q-space, then X must be locally compact.*

So, for example, (a) implies the space of countable ordinals does not have the MOP, while (b) together with Theorems 4.5 and 4.6 show that the metrizable spaces having the MOP are exactly the locally compact ones.

That it can be difficult to tell if a locally compact space has the MOP is shown in Ma's study [**31**] of these properties for Cantor tree spaces over a subset A of 2^ω (see Definition 2.2).

Theorem 5.2. *The following are equivalent*:

(a) $C_k(T \cup A)$ *is a Baire space*;

(b) A *is a* γ*-set.*

Todorčević, see [**10**], showed that it is consistent for there to be two γ-sets A_0 and A_1 whose topological sum is not a γ-set. Since $C_k(X_0 \oplus X_1) \cong C_k(X_0) \times C_k(X_1)$, Ma obtained the following corollary.

Corollary 5.3. *There are, consistently, two locally compact spaces having the* MOP *whose topological sum does not, and two function spaces with the compact-open topology which are Baire but whose product is not.*

But we don't know about ZFC examples.

Question 5.4. Are there examples in ZFC of two Baire function spaces whose product is not Baire? Of two (locally compact) spaces having the MOP whose topological sum does not?

The Cantor tree spaces are special cases of spaces $\psi(\mathcal{A}) = \omega \cup \mathcal{A}$ built from an almost-disjoint family $\mathcal{A}$ of subsets of ω, where ω is the set of isolated points, $\mathcal{A}$ the set of nonisolated points, with each set $A \in \mathcal{A}$ essentially being a sequence of points of ω which limit to the point A. So it may be interesting to answer the following:

Question 5.5. For what almost-disjoint families $\mathcal{A}$ of subsets of ω does $\psi(\mathcal{A})$ have the MOP? What if ω is replaced by a larger cardinal?

The Cantor tree spaces are also special cases of trees with the interval topology, i.e., a basic neighborhood of a node t of the tree T is, for any $s < t$, the interval $\{u \in T : s < u \leq t\}$.

Question 5.6. What trees with the interval topology have the *MOP*?

The following result, which is new, answers this question for Aronszajn trees.

Theorem 5.7. *Every Aronszajn tree with the interval topology has the MOP.*

Proof. Suppose T is Aronszajn and $\mathcal{L}$ is a moving off collection of nonempty compact sets which does not contain an infinite discrete subcollection.

Fact 1. *If $\mathcal{F}$ is an uncountable collection of finite antichains of T, then there are $F_n \in \mathcal{F}$ such that $\cup_{n\in\omega} F_n$ is an antichain.* The poset of finite antichains is well known to be *ccc*, e.g., see Lemma 9.2 of [**45**]. By the Erdös-Rado theorem $\omega_1 \to (\omega, \omega_1)^2$, any uncountable subcollection of a *ccc* poset has an infinite pairwise-compatible subset. Applying this to $\mathcal{F}$ proves Fact 1.

In the sequel, if L is a compact set, we let $m(L)$ denote the (finite) set of T-minimal elements of L. And we let T_α denote the set of all nodes of T of level $< \alpha$.

Fact 2. *There is a $\delta < \omega_1$ such that, for every $L \in \mathcal{L}$, $L \cap T_\delta \neq \varnothing$.* If not, we can easily find an uncountable disjoint subcollection $\mathcal{L}'$ of $\mathcal{L}$. By Fact 1, there are $L_n \in \mathcal{L}$ such that $\cup\{m(L_n) : n \in \omega\}$ is an antichain. Then $\{L_n : n \in \omega\}$ is discrete, a contradiction.

Fact 3. *If $\mathcal{F}$ is a moving off collection of finite sets, there are arbitrarily large $\delta < \omega_1$ such that, to every $F \in \mathcal{F}$, one can assign $\{F^n : n \in \omega\} \subset \mathcal{F}$ satisfying*:

(a) *For each* $n \in \omega$, $F^n \cap T_\delta = F \cap T_\delta$;

(b) $F^n \cap F^m \backslash T_\delta = \varnothing$ *if* $n \neq m$;

(c) $\cup_{n\in\omega}(m(F^n)\backslash T_\delta)$ *is an antichain.*

To see this, let M be a countable elementary submodel containing T and $\mathcal{F}$. We claim that $\delta = M \cap \omega_1$ works. Take $F \in \mathcal{F}$. If $F \subset T_\delta$, we can simply let $F^n = F$ for every n. If on the other hand, $F\backslash T_\delta \neq \varnothing$, then a standard elementary submodel argument gives that, for each $\alpha > \delta$, there is $F_\alpha \in \mathcal{F}$ such that $F_\alpha \cap T_\delta = F \cap T_\delta$ and $(F_\alpha\backslash T_\delta)\cap T_\beta = \varnothing$. Then one obtains Fact 3 by passing to an uncountable subcollection such that the $F_\alpha\backslash T_\delta$'s are disjoint and applying Fact 1.

Now let's complete the proof of the theorem. Let δ satisfy the conditions of Facts 2 and 3. Let $\{t_n^\alpha\}_{n\in\omega}$ index the αth level of T. Let U_n^α be the compact open set $\{t \in T : \exists i \leq n(t \leq t_i^\alpha)\}$. Now use the fact that $\mathcal{L}$ moves off to find disjoint $L_n \in \mathcal{L}$ such that $L_n \cap U_n^\delta = \varnothing$. Then apply Fact 3 to $\{m(L) : L \in \mathcal{L}\}$ to find $L_n^i \in \mathcal{L}$, $i \in \omega$, satisfying:

(a) For each $n \in \omega$, $m(L_n^i \cap T_\delta = L_n \cap T_\delta$;

(b) $L_n^i \cap L_n^j \backslash T_\delta = \varnothing$ if $i \neq j$;

(c) $\cup_{i\in\omega}(m(L_n^i)\backslash T_\delta)$ is an antichain.

Find $\nu > \delta$ such that $\cup_{n,i<\omega} L_n^i \subset T_\nu$. It is not difficult to use conditions (b) and (c) to construct $n_0 < n_1 < \cdots$ and i_k, $k \in \omega$, such that $L_{n_k}^{i_k} \cap (U_k^\nu \cup \cup_{i<k} L_{n_k}^{i_k}) = \varnothing$. Then $\{L_{n_k}^{i_k} : k \in \omega\}$ is discrete. $\square$

Suppose in the statement of the MOP we had merely required the L_n's to be discrete instead of having a discrete open expansion. This property was called the *weak moving off property* ($WMOP$) by Bouziad [**4**]. He points out that any pseudocompact noncompact space in which all countable subsets are discrete is a space satisfying $WMOP$ but not MOP. Example 4.8 is another example of this. It is easy to see, however, that $WMOP$ is equivalent to MOP for either normal or locally compact spaces and could have been added to the list of equivalences in Theorem 4.6. Bouziad uses MOP and $WMOP$ to study Prohorov spaces and coincidence of certain hyperspace topologies.

We are nowhere near figuring out which nonlocally compact spaces have the MOP or $WMOP$. Bouziad showed that any Fréchet fan, i.e., the quotient space obtained by identifying the limits of the topological sum of convergent sequences, has the MOP. Granado [**16**] showed more generally that the $WMOP$ property is always preserved by closed images, and hence any closed normal image of a space with the MOP also has the MOP. Granado and I [**17**] also recently characterized the MOP, equivalently, $WMOP$, in ordered spaces as follows:

Proposition 5.8. *Let X be an ordered space, and let $LC(X)$ be the open subspace consisting of the points of locally compactness. Let $\mathcal{I}$ be the partition of $LC(X)$ into maximal convex sets. Then X has the MOP if and only if the following holds*:

(a) *If $x \in X$ and $(x, \to)$ has countable coinitiality, then $[x, y]$ is compact for some $y > x$; similarly, if $(\leftarrow, x)$ has countable cofinality, then $[y, x]$ is compact for some $y < x$*;

(b) *For each $I \in \mathcal{I}$ and $x \in I$, if the interval $(x, sup\ I)$ is not paracompact, then $[y, sup\ I]$ is compact for some $y \in I$; the analogous statement holds for* $(\inf\ I, x)$.

We were able to use this characterization to characterize the Baire property of $C_k(X)$ for ordered spaces X:

Corollary 5.9. (a) *If X is an ordered space, then $C_k(X)$ is Baire if and only if $C_k(X)$ is Choquet if and only if X has the MOP*;

(b) *If X is a locally compact ordered space, then $C_k(X)$ is Baire if and only if $C_k(X)$ is Choquet if and only if X is paracompact.*

REFERENCES

1. A.V. Arhangel'skiĭ, *Classes of topological groups*, Russian Math. Surv. **36** (1981), 151–174 (in English); Uspekhy Mat. Nauk **36** (1981), 127–146 (in Russian).

2. Z. Balogh, S.W. Davis, W. Just, S. Shelah and P. Szeptyczki, *Strongly almost disjoint sets and weakly uniform bases*, Trans. Amer. Math. Soc. **352** (2000), 4971–4987.

3. A. Bouziad, *The Ellis theorem and continuity in groups*, Topology Appl. **50** (1993), 73–80.

4. ———, *Coincidence of the upper Kuratowski topology with the co-compact topology on compact sets, and the Prohorov property*, Topology Appl. **120** (2002), 283–299.

5. C. Costantini, *On a problem of Nogura about the product of Fréchet-Urysohn* $\langle\alpha_4\rangle$*-spaces*, Comment. Math. Univ. Carolin. **40** (1999), 537–549.

6. C. Costantini and P. Simon, *An* α_4*, not Fréchet product of* α_4 *Fréchet spaces*, Topology Appl. **108** (2000), 43–52.

7. A. Dow, *Two classes of Fréchet-Urysohn spaces*, Proc. Amer. Math. Soc. **108** (1990), 241–247.

8. A. Dow and J. Steprans, *Countable Fréchet* α_1 *spaces may be first-countable*, Arch. Math. Logic **32** (1992), 33–50.

9. V.A. Efimov and G.I. Čertanov, *On subspaces of* Σ*-products of intervals*, Comment. Math. Univ. Carolin. **19** (1978), 569–592 (in Russian).

10. F. Galvin and A.W. Miller, γ*-sets and other singular sets of real numbers*, Topology Appl. **17** (1984), 145–155.

11. F. Garca, L. Oncina and J. Orihuela, *Network characterization of Gul'ko compact spaces and their relatives*, J. Math. Anal. Appl. **297** (2004), 791–811.

12. S. García-Ferreira and R.A. González-Silva, *Topological games defined by ultrafilters*, IV Iberoamer. Conf. on Topology and Its Appl., Topology Appl. **137** (2004), 159–166.

13. S. García-Ferreira, R.A. González-Silva and A.H. Tomita, *Topological games and product spaces*, Comment. Math. Univ. Carolin. **43** (2002), 675–685.

14. J. Gerlits, *Some properties of* $C(X)$ II, Topology Appl. **15** (1983), 255–262.

15. J. Gerlits and Zs. Nagy, *Some properties of* $C(X)$ I, Topology Appl. **14** (1982), 151–161.

16. M. Granado, *On the Moving Off Property and weak additivity of local connectedness and metrizability*, Ph.D. Dissertation, Auburn University, 2004.

17. M. Granado and G. Gruenhage, *Baireness of* $C_k(X)$ *for ordered* X, Comment. Math. Univ. Carol. **47** (2006), 103–111.

18. G. Gruenhage, *Continuously perfectly normal spaces and some generalizations*, Trans. Amer. Math. Soc. **224** (1976), 323–338.

19. ———, *Infinite games and generalizations of first-countable spaces*, Gen. Topology Appl. **6** (1976), 339–352.

20. ———, *Covering properties on* $X^2\backslash\Delta$*,* W*-sets, and compact subsets of* Σ*-products*, Topology Appl. **17** (1984), 287–304.

21. ———, *Games, covering properties and Eberlein compacts*, Topology Appl. **23** (1986), 291–297.

22. ———, *A note on the product of Fréchet spaces*, Topology Proc. **3** (1978), 109–115.

23. ———, *A note on Gul'ko compact spaces*, Proc. Amer. Math. Soc. **100** (1987), 371–376.

24. G. Gruenhage and D.K. Ma, *Baireness of* $C_k(X)$ *for locally compact* X, Topology Appl. **80** (1997), 131–139.

25. G. Gruenhage and P.J. Szeptycki, *Fréchet-Urysohn for finite sets*, Topology Appl. **151** (2005), 238–259.

26. ———, *Fréchet-Urysohn for finite sets*, II, Topology Appl., to appear.

27. J.E. Hart and K. Kunen, *On the regularity of the topology of separate continuity*, Proc. Janos Bolyai Math. Soc. 8th Inter. Topology Conf. (Gyula, 1998), Topology Appl. **123** (2002), 103–123.

28. M. Hrušák, *Selectivity of almost disjoint families*, Acta Univ. Carolin.-Math. Phys. **41** (2000), 13–21.

29. A.S. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995.

30. C. LaFlamme, *Filter games and combinatorial properties of strategies*, in *Collection*: *Set theory* (Boise, ID, 1992–1994), Contemp. Math., vol. 192, Amer. Math. Soc., Providence, RI, 1996, pp. 51–67.

31. D.K. Ma, *The Cantor tree, the γ-property, and Baire function spaces*, Proc. Amer. Math. Soc. **119** (1993), 903–913.

32. R.A. McCoy and I. Ntantu, *Completeness properties of function spaces*, Topology Appl. **22** (1986), 191–206.

33. A. Miller, *On λ'-sets*, Topology Proc. **28** (2004), 179–187.

34. T. Nogura, *The product of $\langle\alpha_i\rangle$-spaces*, Topology Appl. **21** (1985), 251–259.

35. P. Nyikos, *The Cantor tree and the Fréchet-Urysohn property*, Ann. New York Acad. Sci. **552** (1989), 109–123.

36. ———, *Classes of compact sequential spaces*, in *Set theory and its applications* (Toronto, ON, 1987), Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 135–159.

37. ———, *Subsets of ${}^{\omega}\omega$ and the Fréchet-Urysohn and α_i-properties*, Topology Appl. **48** (1992), 91–116.

38. ———, *Moving off collections and spaces of continuous functions*, in preparation.

39. R.C. Olson, *Bi-quotient maps, countably bi-sequential spaces, and related topics*, Gen. Topology Appl. **4** (1974), 1–28.

40. E. Reznichenko and O. Sipacheva, *Fréchet-Urysohn type properties in topological spaces, groups and locally convex vector spaces*, Moscow Univ. Math. Bull. **54** (1999), 33–38.

41. D. Shakmatov, *α_i-properties in Fréchet-Urysohn topological groups*, Topology Proc. **15** (1990), 143–183.

42. P.L. Sharma, *Some characterizations of W-spaces and w-spaces*, Proc. Amer. Math. Soc. **83** (1981), 793–801.

43. K. Tamano, *Products of compact Frchet spaces*, Proc. Japan Acad. Math. Sci. **62** (1986), 304–307.

44. R. Telgársky, *Spaces defined by topological games*, Fund. Math. **88** (1975), 193–223.

45. S. Todorčević, *Trees and linearly ordered sets*, in *Handbook of set-theoretic topology* (K. Kunen and J. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 235–293.

46. ———, *Some applications of S and L combinatorics*, in *The work of Mary Ellen Rudin* (Madison, WI, 1991), Ann. New York Acad. Sci. **705** (1993), 130–167.

47. M.L. Wage, *Weakly compact subsets of Banach spaces*, in *Surveys in general topology* (G.M. Reed, ed.), Academic Press, New York, 1980, pp. 479–494.

48. P.L. Zenor, *Some continuous separation axioms*, Fund. Math. **90** (1975/76), 143–158.

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