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DERIVATIVES OF THE HYPERBOLIC DENSITY NEAR AN ISOLATED BOUNDARY POINT

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ABSTRACT. Suppose that c is an isolated boundary point of a hyperbolic domain Ω in the complex plane, and let λ_Ω denote the density of the hyperbolic metric on Ω. We show that for each pair of nonnegative integers n and m

$$\lim_{w\to c}(w-c)^n\overline{(w-c)}^m\,|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{m+n}\lambda_\Omega(w)}{\partial\overline{w}^m\partial w^n} = \frac{1}{2}\,c_n c_m,$$

where $c_0 = 1$ and $c_n = ((-1)^n/2^n)\,1\cdot 3\cdot 5\cdots\ (2n-1)$ for $n = 1,2,3,\dots$. Also we find the asymptotic limit of $\partial^{m+n}\lambda_\Omega(w)/\partial\overline{w}^m\partial w^n$ as $w\to\infty$ when Ω is a hyperbolic domain containing a neighborhood of ∞.

1. Introduction. Let Ω be a hyperbolic domain in the complex plane $\mathcal{C}$, and let λ_Ω denote the density of the hyperbolic metric on Ω normalized so that the curvature is -4 . Suppose that c is an isolated boundary point of Ω. In [**4**] Yamada proved that

$$\lim_{w\to c}|w-c|\log\frac{1}{|w-c|}\,\lambda_\Omega(w) = \frac{1}{2}. \tag{1}$$

This also was shown by Yamashita in [**5**] and by Minda in [**3**] using different arguments. Yamashita found the order of the growth of $(\partial\lambda_\Omega(w)/\partial w)$, $(\partial^2\lambda_\Omega(w)/\partial w^2)$ and $(\partial^2\lambda_\Omega(w)/\partial\overline{w}\partial w)$ as $w\to c$. This was improved by Minda who determined the asymptotic limits of these three derivatives as $w\to c$. For example, Minda proved that

$$\lim_{w\to c}(w-c)|w-c|\log\frac{1}{|w-c|}\,\frac{\partial\lambda_\Omega(w)}{\partial w} = -\frac{1}{4}. \tag{2}$$

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These results are extended in this paper to partial derivatives of all orders. The main theorem asserts that

$$\lim_{w\to c}(w-c)^n\overline{(w-c)}^m|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{m+n}\lambda_\Omega(w)}{\partial\overline{w}^m\partial w^n}=c_{n,m}$$

for each pair of nonnegative integers n and m, where $c_{n,m}$ are explicit constants. Similar asymptotic limits are obtained for derivatives of $\lambda_\Omega(w)$ as $w \to \infty$ when Ω is a hyperbolic domain containing a neighborhood of ∞.

Our approach is the one used by Minda. It depends on a result of Marden, Richards and Rodin in [**2**]. Namely, there exists an analytic covering projection f from $\Delta_0 = \{z \in \mathcal{C} : 0 < |z| < 1\}$ onto Ω which extends to an analytic function from $\Delta = \{z \in \mathcal{C} : |z| < 1\}$ onto $\Omega \cup \{c\}$ with $f(0) = c$ and $f'(0) \neq 0$. Further, the condition $f'(0) > 0$ determines a unique covering. The conformal invariance of the hyperbolic metric implies that

$$\lambda_\Omega(w)=\frac{\lambda_{\Delta_0}(z)}{|f'(z)|}, \tag{3}$$

where $w = f(z)$ and $z \in \Delta_0$.

Equation (3) and

$$\lambda_{\Delta_0}(z)=\frac{1}{2|z|\log(1/|z|)} \tag{4}$$

for $z \in \Delta_0$ form the starting point for our arguments. First we obtain the asymptotic limits of derivatives of $\log\lambda_\Omega(w)$ as w approaches an isolated boundary point of Ω. Those limits are then used to derive asymptotic limits of derivatives of λ_Ω. Finally, the asymptotic limits at ∞ are deduced from the facts about limits at an isolated boundary point.

2. Asymptotic limits at an isolated boundary point.

Lemma 2.1. *Suppose that Ω is a hyperbolic domain and c is an isolated boundary point of Ω. Let $\lambda = \lambda_\Omega$. Then for all positive integers* *n and m*

$$\lim_{w\to c}(w-c)^n \frac{\partial^n \log\lambda(w)}{\partial w^n} = \frac{(-1)^n(n-1)!}{2}, \tag{5}$$

$$\lim_{w\to c}\left(\overline{w-c}\right)^n \frac{\partial^n \log\lambda(w)}{\partial \overline{w}^n} = \frac{(-1)^n(n-1)!}{2}, \tag{6}$$

and

$$\lim_{w\to c}(w-c)^n \left(\overline{w-c}\right)^m \frac{\partial^{n+m} \log\lambda(w)}{\partial \overline{w}^m \partial w^n} = 0. \tag{7}$$

Proof. Let f be the unique analytic covering projection from Δ_0 onto Ω which extends analytically to Δ and satisfies $f(0) = c$ and $f'(0) > 0$. Then (3) and (4) imply

$$\log\lambda(w) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log\overline{f'(z)} = -\log 2 - \log|z| - \log\left(\log\frac{1}{|z|}\right) \tag{8}$$

where $w = f(z)$ and $0 < |z| < 1$. If we differentiate both sides of (8) with respect to z and use $\partial w/\partial z = f'(z)$, we obtain

$$\frac{\partial\log\lambda(w)}{\partial w} = \frac{1}{zf'(z)}\left\{-\frac{1}{2} - \frac{1}{2}\frac{zf''(z)}{f'(z)} + \frac{1}{2\log(1/|z|)}\right\}. \tag{9}$$

We claim that for each positive integer n,

$$\frac{\partial^n\log\lambda(w)}{\partial w^n} = \frac{1}{[zf'(z)]^n}\sum_{j=0}^{n} g_{j,n}(z)\frac{1}{[\log(1/|z|)]^j}, \tag{10}$$

where each function $g_{j,n}$ is analytic in Δ and

$$g_{0,n}(0) = \frac{(-1)^n(n-1)!}{2}. \tag{11}$$

When $n = 1$ this claim follows from (9) and the fact that $f'(z) \neq 0$ for $|z| < 1$. Suppose that (10) and (11) hold for some positive integer n and each function $g_{j,n}$ is analytic in Δ. Differentiating (10) with respect to z yields

$$\frac{\partial^{n+1}\log\lambda(w)}{\partial w^{n+1}} = \frac{1}{[zf'(z)]^{n+1}}\sum_{j=0}^{n+1} g_{j,n+1}(z)\frac{1}{[\log(1/|z|)]^j}, \tag{12}$$

where

$$g_{0,n+1}(z) = -n\left[1+\frac{zf''(z)}{f'(z)}\right]g_{0,n}(z) + zg'_{0,n}(z), \tag{13}$$

$$g_{j,n+1}(z) = -n\left[1+\frac{zf''(z)}{f'(z)}\right]g_{j,n}(z) + zg'_{j,n}(z) + \frac{j-1}{2}\,g_{j-1,n}(z), \tag{14}$$

for $j = 1, 2, \dots, n$, and

$$g_{n+1,n+1}(z) = \frac{n}{2}\,g_{n,n}(z). \tag{15}$$

Since each function $g_{j,n}$ is analytic in Δ and $f'(z) \neq 0$ for $|z| < 1$, equations (13), (14) and (15) show that each function $g_{j,n+1}$ is well-defined and analytic in Δ. Equations (13) and (11) yield $g_{0,n+1}(0) = ((-1)^{n+1}n!)/2$. This completes an inductive proof of our claim.

Equations (10) and (11) imply that

$$\begin{aligned}\lim_{w\to c}(w-c)^n\frac{\partial^n \log\lambda(w)}{\partial w^n} &= \lim_{z\to 0}\left\{\left[\frac{f(z)-f(0)}{zf'(z)}\right]^n\sum_{j=0}^{n} g_{j,n}(z)\,\frac{1}{[\log(1/|z|)]^j}\right\}\\ &= g_{0,n}(0) = \frac{(-1)^n(n-1)!}{2}.\end{aligned}$$

This proves (5).

Since $\log\lambda$ is real-valued and infinitely differentiable,

$$\frac{\partial^n \log\lambda(w)}{\partial\overline{w}^n} = \overline{\left[\frac{\partial^n \log\lambda(w)}{\partial w^n}\right]}$$

for $n = 1, 2, \dots$. Hence (5) implies (6).

We claim that for each pair of positive integers m and n,

$$\begin{aligned}&\frac{\partial^{n+m}\log\lambda(w)}{\partial\overline{w}^m\partial w^n}\\ &\quad= \frac{1}{[zf'(z)]^n}\,\frac{1}{\overline{[zf'(z)]}^m}\sum_{j=0}^{n}\sum_{k=1}^{m}\overline{h_{j,k,m}(z)}\,g_{j,n}(z)\,\frac{1}{[\log(1/|z|)]^{j+k}},\end{aligned} \tag{16}$$

where each function $h_{j,k,m}$ is analytic in Δ. To prove this we give an inductive argument on m with n a fixed positive integer. Differentiation of (10) with respect to $\overline{z}$ shows that

$$\frac{\partial^{n+1}\log\lambda(w)}{\partial\overline{w}\partial w^n} = \frac{1}{[zf'(z)]^n}\,\frac{1}{\overline{zf'(z)}}\sum_{j=0}^{n}\frac{j}{2}\,g_{j,n}(z)\,\frac{1}{[\log(1/|z|)]^{j+1}}. \tag{17}$$

This verifies (16) in the case $m = 1$ with $h_{j,1,1}(z) = j/2$. Suppose that our claim holds for a positive integer m. Differentiation of (16) with respect to $\overline{z}$ yields

$$\frac{\partial^{n+m+1}\log\lambda(w)}{\partial\overline{w}^{m+1}\partial w^n} = \frac{1}{[zf'(z)]^n}\,\frac{1}{[\overline{zf'(z)}]^{m+1}}\sum_{j=0}^{n}\sum_{k=1}^{m+1}\frac{\overline{h_{j,k,m+1}(z)}\,g_{j,n}(z)}{[\log(1/|z|)]^{j+k}}, \tag{18}$$

where

$$h_{j,1,m+1}(z) = -\,m\left[1+\frac{zf''(z)}{f'(z)}\right]\,h_{j,1,m}(z) + zh'_{j,1,m}(z), \tag{19}$$

$$\begin{aligned} h_{j,k,m+1}(z) = &-\,m\left[1+\frac{zf''(z)}{f'(z)}\right]\,h_{j,k,m}(z) + zh'_{j,k,m}(z)\\ &+\frac{j+k-1}{2}\,h_{j,k-1,m}(z), \end{aligned} \tag{20}$$

for $k = 2, 3, \dots, m$, and

$$h_{j,m+1,m+1}(z) = \frac{j+m}{2}\,h_{j,m,m}(z). \tag{21}$$

The inductive hypothesis and $f'(z) \neq 0$ for $|z| < 1$ show that each function $h_{j,k,m+1}$, $j = 0, 1, \dots, n$, $k = 1, 2, \dots, m+1$, is analytic in Δ. This proves our claim.

Equation (16) implies that

$$\begin{aligned} &\lim_{w\to c}(w-c)^n\,(\overline{w-c})^m\,\frac{\partial^{n+m}\log\lambda(w)}{\partial\overline{w}^m\partial w^n}\\ &\quad= \lim_{z\to 0}\left[\frac{f(z)-f(0)}{zf'(z)}\right]^n\left[\overline{\left(\frac{f(z)-f(0)}{zf'(z)}\right)}\right]^m\sum_{j=0}^{n}\sum_{k=1}^{m}\frac{\overline{h_{j,k,m}(z)}\,g_{j,n}(z)}{[\log(1/|z|)]^{j+k}}. \end{aligned}$$

Because $j + k \geqslant 1$ each limit in the sum is zero. This proves (7). □

Theorem 2.2. *Suppose that* Ω *is a hyperbolic domain and* c *is an isolated boundary point of* Ω*. Let* $\lambda = \lambda_\Omega$*. Then for each pair of nonnegative integers* m *and* n

$$\lim_{w\to c}(w-c)^n\overline{(w-c)}^m|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{m+n}\lambda(w)}{\partial\overline{w}^m\partial w^n}=\frac{1}{2}\,c_nc_m, \tag{22}$$

where

$$c_n=\begin{cases}1 & \text{if } n=0\\ \dfrac{(-1)^n}{2^n}\,1\cdot3\cdot5\cdots(2n-1) & \text{if } n=1,2,3,\dots.\end{cases} \tag{23}$$

Proof. An inductive argument, depending only on the existence of derivatives of λ and $\lambda(w)\neq 0$, shows that

$$\frac{\partial^p\lambda(w)}{\partial w^p}=\sum_{j=1}^{p}\binom{p-1}{j-1}\frac{\partial^j\log\lambda(w)}{\partial w^j}\,\frac{\partial^{p-j}\lambda(w)}{\partial w^{p-j}} \tag{24}$$

for $w\in\Omega$ and for every positive integer p. The inductive step from p to $p+1$ is obtained by differentiation of (24) with respect to w, which gives

$$\begin{aligned}\frac{\partial^{p+1}\lambda(w)}{\partial w^{p+1}}&=\sum_{j=1}^{p}\binom{p-1}{j-1}\left[\frac{\partial^j\log\lambda(w)}{\partial w^j}\,\frac{\partial^{p-j+1}\lambda(w)}{\partial w^{p-j+1}}\right.\\&\qquad\qquad\left.+\frac{\partial^{j+1}\log\lambda(w)}{\partial w^{j+1}}\,\frac{\partial^{p-j}\lambda(w)}{\partial w^{p-j}}\right]\\&=\frac{\partial\log\lambda(w)}{\partial w}\,\frac{\partial^p\lambda(w)}{\partial w^p}+\frac{\partial^{p+1}\log\lambda(w)}{\partial w^{p+1}}\,\lambda(w)\\&\quad+\sum_{j=2}^{p}\left\{\left[\binom{p-1}{j-1}+\binom{p-1}{j-2}\right]\frac{\partial^j\log\lambda(w)}{\partial w^j}\,\frac{\partial^{p-j+1}\lambda(w)}{\partial w^{p-j+1}}\right\}.\end{aligned}$$

Then we use $\binom{p-1}{j-1}+\binom{p-1}{j-2}=\binom{p}{j-1}$.

The case $n=m=0$ of the theorem corresponds to (1). Assume that $m=0$ and (22) holds for $n=0,1,2,\dots,k$ where k is a nonnegative integer. From (5), (24) and our assumption, we obtain

$$\begin{aligned}
&\lim_{w\to c}(w-c)^{k+1}|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{k+1}\lambda(w)}{\partial w^{k+1}}\\
&\quad=\sum_{j=1}^{k+1}\left\{\binom{k}{j-1}\left[\lim_{w\to c}(w-c)^{j}\frac{\partial^{j}\log\lambda(w)}{\partial w^{j}}\right]\right.\\
&\qquad\qquad\left.\times\left[\lim_{w\to c}(w-c)^{k+1-j}|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{k+1-j}\lambda(w)}{\partial w^{k+1-j}}\right]\right\}\\
&\quad=\sum_{j=1}^{k+1}\binom{k}{j-1}\frac{(-1)^{j}(j-1)!}{2}\,\frac{c_{k+1-j}}{2}\\
&\quad=\frac{1}{4}\sum_{j=1}^{k+1}\frac{k!}{(k+1-j)!}\,(-1)^{j}c_{k+1-j}.
\end{aligned}$$

A straightforward inductive argument shows that

$$\sum_{j=1}^{l}\frac{(l-1)!}{(l-j)!}\,(-1)^{j}c_{l-j}=2c_{l} \tag{25}$$

for every positive integer l. Therefore

$$\lim_{w\to c}(w-c)^{k+1}|w-c|\log\frac{1}{|w-c|}\,\frac{\partial^{k+1}\lambda(w)}{\partial w^{k+1}}=\frac{c_{k+1}}{2}.$$

This completes the inductive argument that (22) holds when $m=0$ and n is any nonnegative integer.

If p is a positive integer and q is a nonnegative integer, then we will show that

$$\frac{\partial^{p+q}\lambda(w)}{\partial\overline{w}^{p}\partial w^{q}}=\sum_{j=1}^{p}\sum_{k=0}^{q}\binom{q}{k}\binom{p-1}{j-1}\frac{\partial^{j+k}\log\lambda(w)}{\partial\overline{w}^{j}\partial w^{k}}\,\frac{\partial^{q-k+p-j}\lambda(w)}{\partial\overline{w}^{p-j}\partial w^{q-k}} \tag{26}$$

for $w\in\Omega$. Since λ is real-valued and infinitely differentiable,

$$\frac{\partial^{p}\lambda(w)}{\partial\overline{w}^{p}}=\overline{\left(\frac{\partial^{p}\lambda(w)}{\partial w^{p}}\right)}$$

and hence (24) yields

$$\frac{\partial^p \lambda(w)}{\partial \overline{w}^p} = \sum_{j=1}^{p} \binom{p-1}{j-1} \frac{\partial^j \log \lambda(w)}{\partial \overline{w}^j} \frac{\partial^{p-j} \lambda(w)}{\partial \overline{w}^{p-j}}$$

for all positive integers p. This proves that (26) holds for all positive integers p when $q = 0$. We complete the proof of (26) by induction on q with p fixed. The inductive step from q to $q+1$ follows by differentiation of (26) with respect to w. The remaining details are similar to those used to prove (24).

Let n be a fixed nonnegative integer. For each nonnegative integer l let P_l denote the statement that

$$\lim_{w \to c} (w-c)^r \, (\overline{w-c})^l \, |w-c| \log \frac{1}{|w-c|} \, \frac{\partial^{r+l} \lambda(w)}{\partial \overline{w}^l \partial w^r} = \frac{1}{2} \, c_r c_l$$

for $r = 0, 1, \ldots, n$. We already showed that P_0 holds. Suppose that m is a nonnegative integer and assume P_l for $l = 0, 1, \ldots, m$. Let s be an integer satisfying $0 \leqslant s \leqslant n$. With $q = s$ and $p = m+1$, (26) yields

$$\begin{aligned}
&(w-c)^s \, (\overline{w-c})^{m+1} \, |w-c| \log \frac{1}{|w-c|} \, \frac{\partial^{s+m+1} \lambda(w)}{\partial \overline{w}^{m+1} \partial w^s} \\
&\quad = \sum_{j=1}^{m+1} \sum_{k=0}^{s} \binom{s}{k} \binom{m}{j-1} \left\{ (w-c)^k \, (\overline{w-c})^j \, \frac{\partial^{j+k} \log \lambda(w)}{\partial \overline{w}^j \partial w^k} \right\} \\
&\quad \times \left\{ (w-c)^{s-k} \, (\overline{w-c})^{m+1-j} \, |w-c| \log \frac{1}{|w-c|} \, \frac{\partial^{s-k+m+1-j} \lambda(w)}{\partial \overline{w}^{m+1-j} \partial w^{s-k}} \right\}.
\end{aligned}$$

By using (7) and then (6) and our inductive assumption we obtain

$$\begin{aligned}
\lim_{w \to c} (w-c)^s \, (\overline{w-c})^{m+1} \, |w-c| \log \frac{1}{|w-c|} \, \frac{\partial^{s+m+1} \lambda(w)}{\partial \overline{w}^{m+1} \partial w^s} \\
= \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{(-1)^j (j-1)!}{2} \, \frac{1}{2} \, c_s c_{m+1-j} \\
= \frac{1}{4} \, c_s \sum_{j=1}^{m+1} \frac{m!}{(m+1-j)!} \, (-1)^j c_{m+1-j}.
\end{aligned}$$

Equation (25) implies that this last expression equals $(1/2)\, c_s c_{m+1}$. This yields the statement P_{m+1}. Therefore P_m holds for all nonnegative integers m. □

3. Asymptotic limits at infinity.

Lemma 3.1. *Suppose that* Ω *is a hyperbolic domain and* $w_0 \in \mathcal{C} \setminus \Omega$. *For* $w \in \Omega$ *let* $g(w) = 1/(w - w_0)$. *Let* $\lambda = \lambda_\Omega$, $\widetilde{\Omega} = g(\Omega)$, *and* $\widetilde{\lambda} = \lambda_{\widetilde{\Omega}}$. *Then, for each pair of nonnegative integers* m *and* n,

$$\frac{\partial^{n+m}\lambda(w)}{\partial \overline{w}^m \partial w^n} = (-1)^{n+m} \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \frac{n!}{k!} \binom{m}{j} \frac{m!}{j!}\, \zeta^{n+k+1}\, \overline{\zeta}^{m+j+1}\, \frac{\partial^{j+k}\widetilde{\lambda}(\zeta)}{\partial \overline{\zeta}^j \partial \zeta^k}, \tag{27}$$

where $\zeta = 1/(w - w_0)$ *and* $w \in \Omega$.

Proof. Since g maps open sets onto open sets and connected sets onto connected sets, $\widetilde{\Omega}$ is a domain. Because Ω is hyperbolic, there exists w_1 such that $w_1 \in \mathcal{C} \setminus \Omega$ and $w_1 \neq w_0$. Thus $1/(w_1 - w_0) \in \mathcal{C} \setminus \widetilde{\Omega}$ and $0 \in \mathcal{C} \setminus \widetilde{\Omega}$ and therefore $\widetilde{\Omega}$ is hyperbolic.

The conformal invariance of the hyperbolic metric implies $\lambda(w) = |g'(w)|\widetilde{\lambda}(g(w))$. Let $\zeta = g(w)$. Then $(\partial\zeta/\partial w) = -\zeta^2$ and hence $\lambda(w) = \zeta\overline{\zeta}\widetilde{\lambda}(\zeta)$. This verifies (27) when $n = m = 0$. Assume that (27) holds when $m = 0$ and n is some nonnegative integer. Since $(\partial^{n+1}\lambda(w))/(\partial w^{n+1}) = -\,\zeta^2(\partial/\partial\zeta)(\partial^n\lambda(w)/\partial w^n)$, this yields

$$\frac{\partial^{n+1}\lambda(w)}{\partial w^{n+1}} = (-1)^{n+1}\Bigg\{(n+1)!\zeta^{n+2}\overline{\zeta}\widetilde{\lambda}(\zeta) + \sum_{k=1}^{n}\left[\binom{n}{k}\frac{n!}{k!}(n+k+1) + \binom{n}{k-1}\frac{n!}{(k-1)!}\right] \times \zeta^{n+k+2}\,\overline{\zeta}\,\frac{\partial^k\widetilde{\lambda}(\zeta)}{\partial\zeta^k} + \zeta^{2n+3}\,\overline{\zeta}\,\frac{\partial^{n+1}\widetilde{\lambda}(\zeta)}{\partial\zeta^{n+1}}\Bigg\}.$$

Because $\binom{n}{k}n!/k!(n+k+1) + \binom{n}{k-1}n!/(k-1)! = (n+1)!/k!\binom{n+1}{k}$, this gives (27) with $m = 0$ and n replaced by $n+1$. Thus (27) holds when $m = 0$ and n is any nonnegative integer.

Let n be a fixed nonnegative integer. Assume that (27) holds for some nonnegative integer m. Differentiating (27) with respect to $\overline{w}$ we find that

$$\begin{aligned}
&\frac{\partial^{m+n+1}\lambda(w)}{\partial\overline{w}^{m+1}\partial w^n}\\
&\quad= (-1)^{n+m+1}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}\,\zeta^{n+k+1}\Bigg\{(m+1)!\,\overline{\zeta}^{m+2}\,\frac{\partial^k\widetilde{\lambda}(\zeta)}{\partial\zeta^k}\\
&\qquad+\sum_{j=1}^{m}\left[\binom{m}{j}\frac{m!}{j!}\,(m+1+j)+\binom{m}{j-1}\frac{m!}{(j-1)!}\right]\\
&\qquad\times\overline{\zeta}^{m+j+2}\,\frac{\partial^{j+k}\widetilde{\lambda}(\zeta)}{\partial\overline{\zeta}^j\partial\zeta^k}+\overline{\zeta}^{2m+3}\,\frac{\partial^{k+m+1}\widetilde{\lambda}(\zeta)}{\partial\overline{\zeta}^{m+1}\partial\zeta^k}\Bigg\}\\
&\quad= (-1)^{n+m+1}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}\,\zeta^{n+k+1}\\
&\qquad\times\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{(m+1)!}{j!}\,\overline{\zeta}^{m+j+2}\,\frac{\partial^{j+k}\widetilde{\lambda}(\zeta)}{\partial\overline{\zeta}^j\partial\zeta^k}.
\end{aligned}$$

This provides the inductive step. □

Lemma 3.2. *Let c_n be defined by* (23)*. For each nonnegative integer n,*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}\,c_k=\frac{(-1)^n}{n!}\,c_n. \tag{28}$$

Proof. Let Γ denote the Gamma function. Then $\Gamma(z+1)=z\Gamma(z)$ implies that $\Gamma(k+(1/2)) = (k-(1/2))(k-(3/2))\cdots(3/2)\cdot(1/2)\cdot\Gamma(1/2)$ for each nonnegative integer k. Hence (23) can be expressed

$$c_k=(-1)^k\,\frac{\Gamma(k+(1/2))}{\Gamma(1/2)} \tag{29}$$

for $k=0,1,2,\dots$. The Beta function is defined by $B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$ for $\mathbf{R}z>0$ and $\mathbf{R}w>0$. Then $B(z,w)=$

$(\Gamma(z)\Gamma(w))/(\Gamma(z+w))$ [**1**, p. 213] and $B(z,w) = B(w,z)$. Hence (29) yields $c_k = ((-1)^k k!)/(\Gamma^2(1/2))B(k+(1/2),(1/2))$. Therefore

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}\,c_k &= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\Gamma^2(1/2)}\int_0^1 t^{k-1/2}(1-t)^{-1/2}\,dt\\ &= \frac{1}{\Gamma^2(1/2)}\int_0^1\left\{\sum_{k=0}^{n}\binom{n}{k}(-t)^k\right\}t^{-1/2}(1-t)^{-1/2}\,dt\\ &= \frac{1}{\Gamma^2(1/2)}\int_0^1 (1-t)^n t^{-1/2}(1-t)^{-1/2}\,dt\\ &= \frac{1}{\Gamma^2(1/2)}\,B((1/2),n+(1/2))\\ &= \frac{1}{\Gamma^2(1/2)}\,B(n+(1/2),(1/2)) = \frac{(-1)^n}{n!}\,c_n. \qquad \square\end{aligned}$$

Theorem 3.3. *Suppose that* Ω *is a hyperbolic domain which contains a neighborhood of* ∞. *Let* $\lambda = \lambda_\Omega$. *Then, for each pair of nonnegative integers* m *and* n,

$$\lim_{w\to\infty} w^n\overline{w}^m|w|\log|w|\,\frac{\partial^{n+m}\lambda(w)}{\partial\overline{w}^m\partial w^n} = \frac{1}{2}\,c_n c_m, \tag{30}$$

where the sequence $\{c_n\}$ *is defined by* (23).

Proof. Choose $w_0 \in \mathcal{C}\setminus\Omega$. Let $\zeta = g(w) = 1/(w-w_0)$ and $\widetilde{\Omega} = g(\Omega)$. Then $\widetilde{\Omega}$ is a hyperbolic domain and 0 is an isolated boundary point of $\widetilde{\Omega}$. Lemma 3.1 yields

$$\begin{aligned} w^n\overline{w}^m|w|\log|w|\,&\frac{\partial^{n+m}\lambda(w)}{\partial\overline{w}^m\partial w^n}\\ &= (-1)^{n+m}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\frac{n!}{k!}\binom{m}{j}\frac{m!}{j!}\,[1+w_0\zeta]^n\,[\overline{1+w_0\zeta}]^m\\ &\quad\times\left|\frac{1}{\zeta}\,(1+w_0\zeta)\right|\log\left|\frac{1}{\zeta}\,(1+w_0\zeta)\right|\zeta^{k+1}\,\overline{\zeta}^{j+1}\,\frac{\partial^{j+k}\widetilde{\lambda}(\zeta)}{\partial\overline{\zeta}^j\partial\zeta^k}.\end{aligned}$$

Since $\zeta \to 0$ as $w \to \infty$, an application of Theorem 2.2 to $\widetilde{\Omega}$ with $c = 0$ implies that

$$\begin{aligned}
&\lim_{w\to\infty} w^n \overline{w}^m |w| \log|w| \, \frac{\partial^{n+m}\lambda(w)}{\partial\overline{w}^m \partial w^n} \\
&\quad = (-1)^{n+m} \sum_{k=0}^{n}\sum_{j=0}^{m} \binom{n}{k}\frac{n!}{k!}\binom{m}{j}\frac{m!}{j!}\left\{\lim_{\zeta\to 0} \zeta^k \overline{\zeta}^j |\zeta| \log\frac{1}{|\zeta|}\, \frac{\partial^{j+k}\widetilde{\lambda}(\zeta)}{\partial\overline{\zeta}^j \partial\zeta^k}\right\} \\
&\quad = (-1)^{n+m} \sum_{k=0}^{n}\sum_{j=0}^{m} \binom{n}{k}\frac{n!}{k!}\binom{m}{j}\frac{m!}{j!}\,\frac{1}{2}\, c_k c_j \\
&\quad = \frac{1}{2}\left\{ n!(-1)^n \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}\, c_k\right\}\left\{ m!(-1)^m \sum_{j=0}^{m}\binom{m}{j}\frac{1}{j!}\, c_j\right\}.
\end{aligned}$$

Lemma 3.2 yields (30). □

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