ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 6, 2006

MINIMAL KERNELS, QUADRATURE IDENTITIES AND PROPORTIONAL HARMONIC MEASURES

JOHN R. AKEROYD, KRISTI KARBER AND ALEXANDER YU. SOLYNIN

ABSTRACT. We describe nonnegative weights on $\mathbb{T}$ that are minimal at a given point and are related to quadrature identities for harmonic functions. The problem has a geometric interpretation in terms of a system of crescent regions carrying proportional harmonic measures. This system occurs as circle domains of a quadratic differential with second order poles. Our results have applications to harmonic polynomial approximation.

1. Introduction and main results. Let $\mathcal{H}$ denote the set of real-valued functions h that are harmonic in $\mathbb{D} = \{z : |z| < 1\}$ and continuous on $\overline{\mathbb{D}}$. By the Poisson formula, for every h in $\mathcal{H}$, and any point a in $\mathbb{D}$,

$$h(a) = \int_{\mathbb{T}} h(z) P_a(z)\, dm(z), \tag{1.1}$$

where $dm = d\theta/2\pi$ is normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$ and $P_a(z)$ denotes the Poisson kernel on $\mathbb{T}$ for evaluation at a:

$$P_a(z) = \frac{1-|a|^2}{|z-a|^2} = \frac{(1-|a|^2)z}{(z-a)(1-\bar{a}z)}.$$

Now (1.1) can be viewed as a first order quadrature identity. A quadrature identity of order $n+1$ on $\mathcal{H}$ has the form

$$\int_{\mathbb{T}} h(z) w(z)\, dm(z) = \sum_{k=0}^{n} (-1)^k c_k h(a_k), \tag{1.2}$$

where the weight w, the distinct reference points a_k are in $\mathbb{D}$, and the nonzero constants c_k are independent of h. The factor $(-1)^k$ in (1.2)

2000 AMS *Mathematics Subject Classification.* Primary 30H05, 47B38, 49J20.
Key words and phrases. Harmonic function, quadrature identity, polynomial approximation, harmonic measure, quadratic differential.
Received by the editors on February 27, 2004.

is chosen to simplify some of our later formulations. The weight w in (1.2) is necessarily of the form:

$$w_{A,C}(z) = \sum_{k=0}^{n} (-1)^k c_k P_{a_k}(z), \tag{1.3}$$

where $A = \{a_0, a_1, \dots, a_n\}$ and $C = \{c_0, c_1, \dots, c_n\}$. Indeed, if w satisfies (1.2), then $[w - w_{A,C}]\, dm \perp C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the set of continuous functions on $\mathbb{T}$, and hence $w(z) = w_{A,C}(z)$ almost everywhere on $\mathbb{T}$. Since harmonic functions are conformally invariant, we may fix one of the reference points. In what follows we always assume that $a_0 = 0$ and $c_0 = 1$. Then $w_{A,C}$ can be represented as

$$w_{A,C}(z) = \frac{R(z)}{\prod_{k=1}^{n} (z - a_k)(z - 1/\bar{a}_k)}, \tag{1.4}$$

where $R(z)$ is a polynomial of degree at most $2n$. And indeed if all of our constants c_k are nonzero, then taking the limit as z tends to infinity in representations (1.3) and (1.4) we find that the degree of $R(z)$ is exactly $2n$ and the coefficient of z^{2n} in $R(z)$ is one. Motivated by applications to the theory of approximation by harmonic polynomials, we are interested in weights $w_{A,C}$ with the maximal possible rate of decay near a given point ζ_0 in $\mathbb{T}$; without loss, we assume that $\zeta_0 = 1$. So we search for nonzero constants $c_0 = 1, c_1, \dots, c_n$ to satisfy:

$$w_{A,C}(z) = O((z-1)^{2n}),$$

and indeed such that $R(z) = (z-1)^{2n}$. The existence of, and formulae for, such constants is easily established in the following way. For $1 \le k \le n$, simply multiply both $\prod_{k=1}^{n} ((z-1)^2)/((z - a_k)(z - 1/\bar{a}_k))$ and $\sum_{k=0}^{n} (-1)^k c_k P_{a_k}(z)$ by $(z - a_k)$, and then substitute $z = a_k$. Equating the results and solving for c_k, we find that

$$c_k = (-1)^k \frac{\bar{a}_k (1 - a_k)^2}{a_k (1 - |a_k|^2)} {\prod}'^{\,n}_{j=1} \frac{\bar{a}_j (1 - a_k)^2}{(a_j - a_k)(1 - a_k \bar{a}_j)}.$$

Here and below $\prod'$ denotes the product over all indices $j \neq k$.

If the reference points a_k are real and positive, then the minimal weight $w_{A,C} = w_A$ is

$$w_A(z) = \frac{(z-1)^{2n}}{\prod_{k=1}^{n} (z - a_k)(z - 1/a_k)} = \prod_{k=1}^{n} \frac{a_k |z-1|^2}{|z - a_k|^2} > 0 \tag{1.5}$$

for all $z \in \mathbb{T} \setminus \{1\}$. If, in addition, $0 = a_0 < a_1 < \cdots < a_n < 1$, then the constants c_k are positive:

$$c_k = (-1)^k \frac{1-a_k}{1+a_k} \prod\nolimits'^{n}_{j=1} \frac{a_j(1-a_k)^2}{(a_j-a_k)(1-a_k a_j)} > 0. \tag{1.6}$$

Combining these observations we obtain the following result.

Theorem 1.1. *For every set A of $n \geq 1$ reference points $0 = a_0 < a_1 < \cdots < a_n < 1$, there is a unique weight w_A minimal at $z = 1$ and a unique set of real constants $C = \{c_0, c_1, \ldots, c_n\}$ with $c_0 = 1$, such that the quadrature identity*

$$\int_{\mathbb{T}} h(\zeta) w_A(\zeta)\, dm(\zeta) = \sum_{k=0}^{n} (-1)^k c_k h(a_k) \tag{1.7}$$

holds for all $h \in \mathcal{H}$. The minimal weight w_A is given by (1.5) *and is positive on $\mathbb{T} \setminus \{1\}$. The constants c_k are positive and defined by* (1.6).

Since $w_A \geq 0$, (1.7) implies the Harnack-type inequality

$$h(0) \geq \sum_{k=1}^{n} (-1)^{k+1} c_k h(a_k),$$

which holds for all $h \geq 0$ in $\mathcal{H}$. For $n = 1$ and $a_1 = a > 0$, this is the classical Harnack's inequality:

$$h(0) \geq \frac{1-a}{1+a}\, h(a).$$

In Section 2, we discuss an application of Theorem 1.1 to harmonic polynomial approximation. This application has a counterpart in the context of analytic polynomial approximation [**2**], which has close ties to Szegö's theorem.

Our problem concerning quadrature identities, cf. Theorem 1.1, has an interesting geometric interpretation. It turns out that any solution corresponds uniquely with a partition of $\mathbb{D}$ into a system of n crescents along with a single Jordan region. To make the picture clear, we first define our terms.

A *crescent* is a bounded, simply connected region of the form $W \setminus \overline{V}$, where V and W are Jordan regions such that $V \subset W$ and $\overline{V} \cap \partial W$ is a single point, the so-called *multiple boundary point* (mbp) of $W \setminus \overline{V}$. In this case, $\gamma^- := \partial W$ and $\gamma^+ := \partial V$ are two Jordan curves that comprise the boundary of $W \setminus \overline{V}$, γ^- and γ^+ have just one point in common (the mbp of $W \setminus \overline{V}$) and γ^+ is *internal* to γ^-.

In this paper we consider systems of Jordan curves of the form: $\gamma_0, \gamma_1, \dots, \gamma_n$, where $\gamma_0 = \mathbb{T}$, $\gamma_i \cap \gamma_j = \{1\}$ whenever $i \neq j$, and γ_j is internal to γ_i whenever $j > i$, see Figure 1. The collection of bounded components of $\mathbb{C} \setminus \cup_{k=0}^n \gamma_k$ forms what we call a *crescent configuration.* For $0 \le k \le n-1$, let Ω_k be the component bounded by γ_k and γ_{k+1}, and let Ω_n be the Jordan region bounded by γ_n. Notice that Ω_k is a crescent for $0 \le k \le n-1$. We further assume that our system of curves is chosen so that $0 \in \Omega_0$. For notational convenience we let $\gamma_k^- = \gamma_k$ and $\gamma_k^+ = \gamma_{k+1}$, $0 \le k \le n-1$, and hence (for such k), $\gamma_k^+ = \gamma_{k+1}^-$.

Let $\mathcal{C}_n$ denote the set of all crescent configurations described above.

Problem. For a given set of positive constants $C = \{c_0, c_1, \dots, c_n\}$ with $c_0 = 1$, find all configurations $\{\Omega_0, \dots, \Omega_n\}$ in $\mathcal{C}_n$ and reference points $a_k \in \Omega_k$, which carry harmonic measures proportional with respect to C, i.e., such that for all $k = 0, \dots, n-1$,

$$c_k \, \omega(E, \Omega_k, a_k) = c_{k+1} \, \omega(E, \Omega_{k+1}, a_{k+1}) \tag{1.8}$$

for every Borel set $E \subset \gamma_k^+$.

For two regions this problem was solved in [**1**]. For $n \ge 2$, its solution is given by Theorem 1.3 below. To explain how configurations carrying proportional harmonic measures arise in relation with quadrature identities, let us discuss a theoretical link between them and Theorem 1.1.

Let D be a bounded Dirichlet region in $\mathbb{C}$. We remind the reader that the harmonic measure $\omega(\cdot, D, a)$ is a unique Borel probability measure on ∂D such that

$$h(a) = \int_{\partial D} h(z) \, d\omega(z, D, a)$$

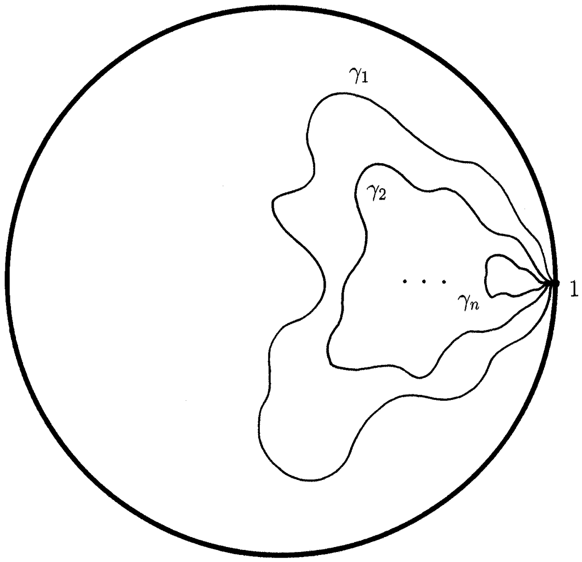

FIGURE 1.

for all functions h harmonic on D and continuous on $\overline{D}$, see [**5**]. Let $g_D(z,a)$ be Green's function of D with singularity at $a \in D$. If ∂D is piecewise smooth, then (cf. [**5**]) for z in ∂D,

$$d\omega(z, D, a) = \frac{1}{2\pi} \frac{\partial g_D(z,a)}{\partial n_z}, \tag{1.9}$$

where $\partial/\partial n_z$ denotes the derivative with respect to the inner normal on ∂D.

Now assume that $\{\Omega_0, \dots, \Omega_n\}$ is a configuration of crescents in $\mathcal{C}_n$ satisfying (1.8). Then, for every $h \in \mathcal{H}$ and $k = 0, \dots, n-1$,

$$c_k \int_{\gamma_k^+} h(z)\, d\omega(z, \Omega_k, a_k) = c_{k+1} \int_{\gamma_{k+1}^-} h(z)\, d\omega(z, \Omega_{k+1}, a_{k+1}). \tag{1.10}$$

Since

$$h(a_k) = \int_{\gamma_k^- \cup \gamma_k^+} h(z)\, d\omega(z, \Omega_k, a_k), \quad k = 0, \dots, n-1,$$

(1.10) implies that

$$c_k h(a_k) = c_k \int_{\gamma_k^-} h(z)\, d\omega(z, \Omega_k, a_k) + c_{k+1} \int_{\gamma_{k+1}^-} h(z)\, d\omega(z, \Omega_{k+1}, a_{k+1}).$$

Multiplying this by $(-1)^k$ and summing over $k = 0, \dots, n$, we get:

$$\int_{\mathbb{T}} h(z)\, d\omega(z, \Omega_0, a_0) = \sum_{k=0}^{n} (-1)^k c_k h(a_k) \tag{1.11}$$

for every $h \in \mathcal{H}$.

Thus, starting with a configuration of crescents carrying proportional harmonic measures, we obtain the same type of quadrature identity as in Theorem 1.1. In particular, if $a_0, \dots, a_n$ and $c_0, \dots, c_n$ in (1.11) are the same as in (1.7), the uniqueness statement of Theorem 1.1 implies that the minimal weight $w(\zeta)$ can be recovered from the equation

$$w(\zeta)\, dm(\zeta) = d\omega(\zeta, \Omega_0, 0).$$

In the opposite direction, Theorem 1.1 allows us to describe configurations of crescents carrying proportional harmonic measures. Let

$$H_{A,C}(z) = \sum_{k=0}^{n} (-1)^k c_k \log \left| \frac{z - a_k}{1 - a_k z} \right|. \tag{1.12}$$

Theorem 1.2. *Let A be a set of reference points $0 = a_0 < a_1 < \cdots < a_n < 1$, and let $C = \{c_0, \dots, c_n\}$, where the constants c_k (dependent on the points $a_1, \dots, a_n$) are given by Theorem* 1.1. *Let*

$$E_A = \{z \in \mathbb{C} :\ H_{A,C}(z) = 0\}.$$

Then E_A partitions $\mathbf{D}$ into a crescent configuration $\{\Omega_0, \dots, \Omega_n\}$, which carries proportional harmonic measures with respect to the set C; $a_k \in \Omega_k$ for $k = 0, 1, \dots, n$.

Figure 2 demonstrates typical configurations with two and three crescents carrying proportional harmonic measures. These figures were generated by Advanced Grapher software.

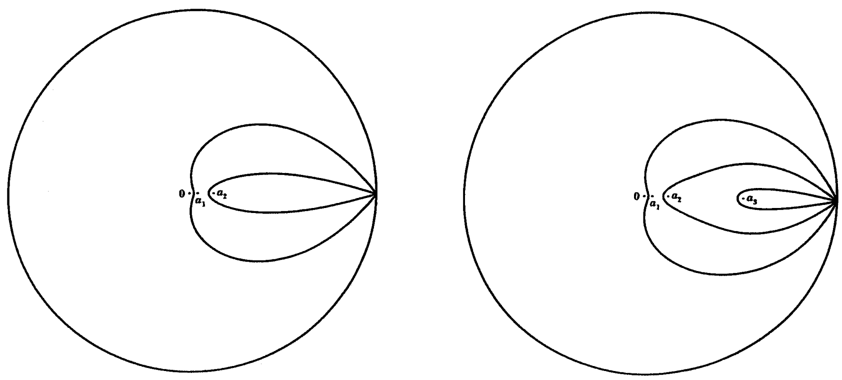

FIGURE 2.

The proof of Theorem 1.2 given in Section 3 uses the fact that E_A is a level set of a linear combination of Green's functions

$$g(z, a_k) = -\log |(z - a_k)/(1 - \bar{a}_k z)|$$

of $\mathbb{D}$ having singularities at $z = a_k$. This approach was used in [**1**].

Note that Green's functions provide an efficient method to study extremal problems on the maximal product of conformal radii of non-overlapping regions (cf. [**3, 4**]), which leads to a description of extremal configurations in terms of quadratic differentials (cf. [**3, 8, 9**]). In Section 4 we show that quadratic differentials are intimately related to the problem on configurations carrying proportional harmonic measures. Our next theorem, whose proof is given in Section 5, solves the problem for crescent configurations.

Theorem 1.3. *For every set of positive constants* $C = \{c_0, \dots, c_n\}$ *with* $c_0 = 1$ *such that*

$$(1.13) \qquad c_k - c_{k+1} + \cdots + (-1)^{n-k} \cdot c_n > 0 \quad \textit{for} \quad k = 0, \dots, n,$$

there is a unique system $\widetilde{\Omega} \in \mathcal{C}_n$ *of crescents* Ω_k*, each of which is symmetric with respect to* $\mathbb{R}$*, and a unique set* $A = (a_0, \dots, a_n)$ *with*

$a_0 = 0$ *of reference points* $a_k \in \Omega_k$*, carrying harmonic measures proportional with respect to* C*. The regions* Ω_k *are the circle domains of the quadratic differential*

$$Q_A(z)\,dz^2 = -\frac{(z-1)^{4n}}{z^2\prod_{k=1}^{n}(z-a_k)^2(z-1/a_k)^2}\,dz^2. \tag{1.14}$$

The reference points a_k *are solutions, unique up to ordering, to the equations*

$$c_{k+1}F_k(a_1,\dots,a_n) = c_kF_{k+1}(a_1,\dots,a_n), \quad k=0,\dots,n-1, \tag{1.15}$$

where $F_0 = 1$ *and for* $k = 1,\dots,n$*,* F_k *denotes the right-hand side of* (1.6) *considered as a function of* $a_1,\dots,a_n$*.*

We should emphasize that all problems studied in this paper are conformally invariant, i.e., they can be reformulated for any simply connected region instead of the unit disk. Often such a transplantation leads to an essential simplification of computations.

2. Harmonic polynomial approximation. We begin this section with a famous result in the context of analytic polynomial approximation.

Theorem 2.1 (G. Szegö). *Let* $\mathcal{P}$ *denote the collection of analytic polynomials in* z*, and let* μ *be a finite, positive Borel measure with support in* $\mathbb{T}$*. Let* $\mu = \mu_a + \mu_s$ *be the Lebesgue decomposition of* μ *with respect to* m (*normalized Lebesgue measure on* $\mathbb{T}$); $\mu_a \ll m$ *and* $\mu_s \perp m$*. Then, for* $0 < t < \infty$*,*

$$\inf_{p\in\mathcal{P}}\int_{\mathbb{T}}\left|\frac{1}{z}-p(z)\right|^t d\mu(z) = \exp\left\{\int_{\mathbb{T}}\log\left(\frac{d\mu_a}{dm}\right)\,dm\right\}.$$

Corollary 2.2. *Under the hypothesis of Szegö's theorem,* $\mathcal{P}$ *is dense in* $L^t(\mu)$ *if and only if* $\int_{\mathbb{T}}\log(d\mu_a/dm)\,dm = -\infty$*.*

In general terms, Corollary 2.2 indicates that, in order for the analytic polynomials to be dense in $L^t(\mu)$, where $0 \le \mu \ll m$, there must be a "location" (in $\mathbb{T}$) where μ is *weak* relative to m. In the context of the (real) harmonic polynomials, there is no requirement of this sort since such functions are in fact uniformly dense in the continuous real-valued functions on $\mathbb{T}$. If, however, μ has some additional mass in $\mathbb{D}$, then the question of density of the harmonic polynomials in $L^t(\mu)$ is usually nontrivial. Our main result of this section, Theorem 2.4, addresses a problem of this type. It has a close relative in the context of analytic polynomial approximation; cf. [**2**], where a "plugging" phenomenon is described. Specifically, in [**2**], the authors consider measures of the form $\mu = \eta+\sigma$, where $d\eta = w\, dm$, $0 \leq w \in L^\infty(m)$, $\int_{\mathbb{T}} \log(w)\, dm = -\infty$, and σ is a series of weighted point masses in $\mathbb{D}$ such that, for t sufficiently large, $\mathcal{P}$ is not dense in $L^t(\mu)$. One can thus say that, for sufficiently large t, the series of weighted point masses σ plugs the weakness in μ. We find a similar phenomenon quite commonplace in our work here. Since $\mathcal{H}$ contains the harmonic polynomials and any function in $\mathcal{H}$ can be uniformly approximated (on $\overline{\mathbb{D}}$) by harmonic polynomials, without loss we work with $\mathcal{H}$ as if it were the set of harmonic polynomials. The next two results address the case $t = 1$. We later investigate the less tractable setting of t in the range $1 < t < \infty$. We begin by noting that, if $a \in \mathbb{D}$ and $\mu = m + \delta_a$, where δ_a is the (unit) point mass at a, then $\mathcal{H}$ is not dense in $L^t(\mu)$ for any t, $1 \leq t < \infty$. This simple fact follows from Harnack's inequality. Indeed, for any h in $\mathcal{H}$ that is nonnegative on $\overline{\mathbb{D}}$,

$$h(a) \leq \frac{1+|a|}{1-|a|} \int_{\mathbb{T}} h\, dm.$$

And so, the characteristic function χ_a is not in $\mathcal{H}^1(\mu)$, the closure of $\mathcal{H}$ in $L^1(\mu)$.

Lemma 2.3. *Suppose* $0 \leq v \in L^\infty(m)$ *and yet* $1/v \notin L^\infty(m)$. *For some fixed* a *in* $\mathbb{D}$, *define* μ *by*

$$\mu = \eta + \delta_a,$$

where $d\eta = v\, dm$. *Then* $\mathcal{H}$ *is dense in* $L^1(\mu)$.

Proof. Now, by our assumption concerning v, for any $\varepsilon > 0$ and any value λ in $\mathbb{R}$, we can find a continuous real-valued function g on $\mathbb{T}$ such that $\int_{\mathbb{T}} |g| v\, dm < \varepsilon$ and yet $\hat{g}(a) = \lambda$; where $\hat{g}$ denotes the solution to the Dirichlet problem on $\mathbb{D}$ with boundary values g. So, functions of the form $\hat{h} + \hat{g}$, where h is continuous and real-valued on $\mathbb{T}$ and g is as above, are dense in $L^1(\mu)$. Since functions of this form are also in $\mathcal{H}^1(\mu)$, the result follows. □

In Section 1 we have shown that, for any set A of reference points, $0 = a_0 < a_1 < \cdots < a_n < 1$, there are uniquely determined positive constants $c_0 = 1, c_1, \dots, c_n$ and a unique corresponding L^∞ weight w_A on $\mathbb{T}$ that has a zero of order $2n$ at $z = 1$ such that

$$\int_{\mathbb{T}} h w_A \, dm = h(0) - c_1 h(a_1) + \cdots + (-1)^n c_n h(a_n),$$

whenever $h \in \mathcal{H}$. And hence, for such h,

$$|h(0)| \le \int_{\mathbb{T}} |h| w_n \, dm + c_1 |h(a_1)| + \cdots + c_n |h(a_n)|. \tag{2.1}$$

Evidently, $\mathcal{H}$ is not dense in $L^1(\mu)$, where $\mu = \eta + \sigma$ and $\sigma = \delta_0 + c_1 \delta_{a_1} + \cdots + c_n \delta_{a_n}$ and $d\eta = w_A \, dm$. Our next result shows that this is sharp.

Theorem 2.4. *With the terms of the above discussion, suppose* $0 \le v \in L^\infty(m)$, $v \le w_A$ *and yet* $(w_A/v) \notin L^\infty(m)$. *Define* ν *by* $d\nu = v \, dm + d\sigma$. *Then* $\mathcal{H}$ *is dense in* $L^1(\nu)$.

Proof. For $k = 0, 1, \dots, n$, let

$$f_k(\zeta) = \prod_{j=1, j \ne k}^{n} \left(1 - \left(\frac{1 - a_j}{1 + a_j} \right)^2 \left(\frac{1 + z}{1 - z} \right)^2 \right).$$

Notice that f_k is analytic in $\mathbb{D}$, f_k is real-valued on $\mathbb{T} \cup [0, 1]$ which contains the support of μ, and $f_k \in L^1(\mu)$. Consider the measure ν_k defined by $d\nu_k = f_k \, d\nu$. Now ν_k has the form $\eta_k + \sigma_k$, where $\sigma_k = c\delta_{a_k}$, $0 \ne c \in \mathbb{R}$, $d\eta_k = v_k dm$, $0 \le v_k|_{\mathbb{T}} \in L^\infty(m)$ and yet $(1/v_k) \notin L^\infty(m)$. So, as in the proof of Lemma 2.3, for any $\varepsilon > 0$ and any λ in $\mathbb{R}$, we can find g real-valued and continuous on $\mathbb{T}$ such that $\int_{\mathbb{T}} |g| \, d\nu_k < \varepsilon$ and yet $\hat{g}(a_k) = \lambda$, where $\hat{g}$ denotes the solution to the Dirichlet problem on $\mathbb{D}$ with boundary values g. In fact, we may assume that g is Dini continuous on $\mathbb{T}$, and so g has a continuous harmonic conjugate g^* on $\overline{\mathbb{D}}$; cf. [**7**]. We let $G = g + ig^*$. We then have $h_k = \Re\{Gf_k\} \in \mathcal{H}^1(\nu)$, $\int_{\mathbb{T}} |h_k|\, d\nu < \varepsilon$, $h_k(a_i) = 0$ for $i \neq k$ and $h_k(a_k)$ can be prescribed in $\mathbb{R}$. Therefore, linear combinations of functions of the type h_k along with summands from $\mathcal{H}$ are dense in $L^1(\nu)$. Since $h_k \in \mathcal{H}^1(\nu)$ for $0 \leq k \leq n$, we conclude that $\mathcal{H}$ is dense in $L^1(\nu)$. □

Remark. Let us now return to the inequality given in (2.1) and apply Jensen's inequality for t in the range $1 < t < \infty$ to get

$$|h(0)|^t \leq 2^{t-1} \int_{\mathbb{T}} |h|^t w_A^t \, dm + M_t \left[c_1 |h(a_1)|^t + \cdots + c_n |h(a_n)|^t\right],$$

whenever $h \in \mathcal{H}$; $M_t = \left(2 \sum_{k=1}^n c_k\right)^{t-1}$. And therefore $\mathcal{H}$ fails to be dense in $L^t(\mu_t)$, where $\mu_t = \eta_t + \sigma$, $d\eta_t = w_A^t \, dm$ and $\sigma = \delta_0 + c_1 \delta_{a_1} + \cdots + c_n \delta_{a_n}$. Since the zero of w_A^t at $z = 1$ is of higher order than that of w_A, provided $1 < t < \infty$, we see, by Theorem 2.4, that plugging for $t > 1$ is a more common occurrence than for $t = 1$. However, we have not yet been able to obtain "sharpness" for t in the range $1 < t < \infty$. The best analogue of Theorem 2.4 that we have been able to achieve for $1 < t < \infty$ and the measures μ_t is for weights v satisfying

$$v(z) \leq |1 - z|^{t-1} w_A^t(z).$$

3. Proof of Theorem 1.2. Now E_A is a level set of the function $H_{A,C}$ defined by (1.12), and $H_{A,C}$ is harmonic on $\mathbf{C}$ except poles at the points $a_0, a_1, \ldots, a_n$ and $a_1^{-1}, \ldots, a_n^{-1}$. Therefore, E_A consists of a finite number of Jordan analytic curves or analytic arcs, each of which "terminates" on the set of critical points of $H_{A,C}$. The critical points of $H_{A,C}$ are zeros of $(\partial/\partial z) H_{A,C}$. Since

$$\begin{aligned} 2 \frac{\partial}{\partial z} H_{A,C}(z) &= \sum_{k=0}^n (-1)^k c_k \frac{1 - a_k^2}{(z - a_k)(1 - a_k z)} \\ &= z^{-1} \sum_{k=0}^n (-1)^k c_k P_{a_k}(z) = z^{-1} w_A(z), \end{aligned}$$

there is only one critical point, of order $2n$, at $z = 1$.

Note that $\mathbb{T} \subset E_A$. Let $\widetilde{E}_A$ be the collection of arcs γ_j of E_A, which lie in $\mathbb{D}$ and terminate at $z = 1$. Since $H_{A,C}(\bar{z}) = H_{A,C}(z)$, the set $\widetilde{E}_A$ is symmetric with respect to $\mathbb{R}$. Since there are no critical points of $H_{A,C}$ in $\mathbb{D}$, each γ_j is either symmetric with respect to $\mathbb{R}$ or it does not intersect $\mathbb{R}$ except at $z = 1$. The latter case never happens. Indeed, if it does, then $\gamma_j \cup \{1\}$ bounds a simply connected region D on $\mathbb{D} \setminus \mathbb{R}$. Therefore, $H_{A,C}$ is harmonic on D. Since $H_{A,C}(z) = 0$ on $\gamma_j \cup \{1\}$, the maximum principle implies that $H_{A,C} \equiv 0$ contradicting (1.12).

Since $z = 1$ is a critical point of $H_{A,C}$ of order $2n$, our analysis shows that $\widetilde{E}_A$ consists of n analytic arcs γ_j symmetric with respect to $\mathbb{R}$, which split $\mathbb{D}$ into $n + 1$ simply connected regions Ω_k, each of which contains at least one of the points $a_0, a_1, \dots, a_n$. Since the regions are pairwise disjoint, each of them contains exactly one of these points. We numerate the domains such that $a_k \in \Omega_k$, $k = 0, \dots, n$.

The maximum principle also implies that $E_A \cap \mathbb{D} = \widetilde{E}_A$. If not, we can find a Jordan analytic curve γ, which belongs to one of the regions, say Ω_k, and separates a_k from $\partial\Omega_k$. Let D be a doubly connected region bounded by γ and $\partial\Omega_k$. Then $H_{A,C}$ is harmonic on D and $H_{A,C} \equiv 0$ on ∂D. By the maximum principle, $H_{A,C} \equiv 0$ on $\mathbb{C}$ contradicting (1.12).

Summarizing, we have shown that $\mathbb{D} \setminus E_A$ consists of n crescents Ω_k and a Jordan region Ω_n, such that $a_k \in \Omega_k$ for $0 \le k \le n$. Let $g_k(z)$ be the restriction of $(-1)^k c_k^{-1} H_{A,C}(z)$ to Ω_k. Then g_k is harmonic on Ω_k except for a logarithmic singularity at $z = a_k$, and $g_k \equiv 0$ on $\partial\Omega_k$. Therefore g_k is Green's function of Ω_k having a pole at $z = a_k$. Then, by (1.9), for $z \in \partial\Omega_k$,

$$d\omega(z, \Omega_k, a_k) = \frac{1}{2\pi}\,\frac{\partial}{\partial n}\, g_k(z)\,|dz| = \frac{(-1)^k}{2\pi c_k}\,\frac{\partial}{\partial n}\, H_{A,C}(z)\,|dz|,$$

where $\partial/\partial n$ denotes differentiation in the direction of the inner normal on $\partial\Omega_k$. This implies that for $k = 0, \dots, n-1$ and $z \in \partial\Omega_k \cap \partial\Omega_{k+1}$,

$$c_k\, d\omega(z, \Omega_k, a_k) = c_{k+1}\, d\omega(z, \Omega_{k+1}, a_{k+1}).$$

Therefore, the crescent configuration Ω_k carries proportional harmonic measures. The proof is complete. □

4. Harmonic measure on trajectories of quadratic differentials. The expression $Q(z)\,dz^2$ with the function Q meromorphic in a region $\Omega \subset \overline{\mathbb{C}}$ is called a quadratic differential on Ω. If $z = z(\zeta)$ is a conformal mapping from a region Ω_ζ onto Ω, then $Q(z)\,dz^2$ can be represented in terms of ζ as

$$Q_1(\zeta)\,d\zeta^2 = Q(z(\zeta))(z'(\zeta))^2\,d\zeta^2. \tag{4.1}$$

Equation (4.1), which is a part of the definition of a quadratic differential, describes how a conformal change of variables affects quadratic differentials; see [**6, 8, 9**] for properties and applications of quadratic differentials. A maximal curve or arc γ such that $Q(z)\,dz^2 > 0$ (respectively, $Q(z)\,dz^2 < 0$) along γ is called a trajectory (respectively, orthogonal trajectory) of $Q(z)\,dz^2$. Now $Q(z)\,dz^2$ is called real (respectively, positive) on Ω if the expression $Q(z)\,dz^2$ is real on $\partial\Omega$ (respectively, positive on $\partial\Omega$ except possibly for a finite number of points where Q vanishes).

Zeros and poles of Q are its critical points. Any trajectory or orthogonal trajectory having at least one of its terminal points at a zero or simple pole of Q is called a critical trajectory or a critical orthogonal trajectory, respectively. By Φ_Q we denote the set of points of all critical trajectories of $Q(z)\,dz^2$.

A simply connected region D is called a circle domain of $Q(z)\,dz^2$ if the following properties hold. The meromorphic function Q has a second order pole at some point a in D, which is the only critical point in D, and if γ is a trajectory of $Q(z)\,dz^2$ intersecting D, then γ is a closed Jordan curve in D that separates a from ∂D. The maximal circle domain containing a pole a is necessarily bounded by a finite number of critical trajectories or boundary arcs.

Here we consider only quadratic differentials $Q(z)\,dz^2$, without density structures, for which $\overline{\mathbb{C}} \setminus \bar{\Phi}_Q$ consists of a finite number of circle domains. In general, $\overline{\mathbb{C}} \setminus \Phi_Q$ may also contain ring domains, strip domains, and end domains; see [**8**, Chapter 3].

A point a is a second order pole of $Q(z)\,dz^2$ in a circle domain D if and only if there exists $c > 0$ such that

$$Q(z) = -\frac{c^2}{4\pi^2}\frac{1}{(z-a)^2} + \frac{a_1}{z-a} + \cdots \tag{4.2}$$

near $z = a$.

The metric $|Q(z)|^{1/2}\,|dz|$ is called the Q-metric. If γ is a trajectory of $Q(z)\,dz^2$ in a circle domain D, then

$$|\gamma|_Q := \int_\gamma |Q(z)|^{1/2}\,|dz| = \int_\gamma Q^{1/2}(z)\,dz$$

is the Q-length of γ. Here and later on we always assume that $Q^{1/2}\,dz > 0$ along the corresponding trajectory. Using (4.2) we easily get

$$|\gamma|_Q = c.$$

Let $\zeta = f(z)$ map D conformally onto the unit disk $\mathbb{D}$ such that $f(a) = 0$ and $f(b) = 1$ for some $b \in \partial D$. Then

$$f(z) = \exp\left\{\frac{2\pi i}{c}\int_b Q^{1/2}(z)\,dz\right\}; \tag{4.3}$$

see [**8**, Chapter 3.3].

Lemma 4.1. *Let D be a circle domain of a quadratic differential $Q(z)\,dz^2$ having expansion* (4.2) *at $z = a \in D$. Then*

$$d\omega(z, D, a) = c^{-1}|Q(z)|^{1/2}\,|dz| \quad \textit{for all} \quad z \in \partial D. \tag{4.4}$$

Proof. Let $\zeta = f(z)$ map D conformally onto $\mathbb{D}$ such that $f(a) = 0$. Then using (4.3), we get

$$d\omega(z, D, a) = \frac{1}{2\pi}\,|d\zeta| = \frac{1}{2\pi}\,|f'(z)|\,|dz| = c^{-1}|Q(z)|^{1/2}\,|dz|. \qquad \square$$

Corollary 4.2. *Let D_1 and D_2 be circle domains of $Q(z)\,dz^2$ having an open arc L on $\partial D_1 \cap \partial D_2$, and choose a_1 in D_1 and a_2 in D_2. Let c_1 and c_2 be Q-lengths of trajectories of $Q(z)\,dz^2$ in D_1 and D_2, respectively. If Q is meromorphic on L, then for every Borel set $E \subset L$,*

$$c_1\omega(E, D_1, a_1) = c_2\omega(E, D_2, a_2). \tag{4.5}$$

Equality (4.5), which is an immediate consequence of (4.4), reveals a role played by quadratic differentials in problems on regions carrying proportional harmonic measures on their boundaries.

5. Proportional harmonic measures on crescents. The proof of Theorem 1.3 will be given after two lemmas. First we show that inequalities (1.13) are necessary for the existence of crescent configurations carrying proportional harmonic measures.

Lemma 5.1. *Assume there is a crescent configuration* $\widetilde{\Omega} \in \mathcal{C}_n$ *of* $\Omega_0, \dots, \Omega_n$ *with reference points* $a_k \in \Omega_k$ *satisfying* (1.8) *with* $c_0 = 1$ *and positive* $c_1, \dots, c_n$. *Then inequalities* (1.13) *hold true for all* $k = 0, \dots, n$.

Proof. Let $\omega_k^+ = \omega(\gamma_k^+, \Omega_k, a_k)$ and $\omega_k^- = \omega(\gamma_k^-, \Omega_k, a_k)$. Then, by (1.8),

$$\omega_k^- = 1 - \omega_k^+ = 1 - \frac{c_{k+1}}{c_k}\,\omega_{k+1}^-, \quad k = 0, \dots, n-1. \tag{5.1}$$

Using (5.1) and proceeding by induction, we get

$$\omega_k^- = (-1)^k c_k^{-1} \sum_{j=k}^{n} (-1)^j c_j, \quad k = 0, \dots, n. \tag{5.2}$$

Indeed, (5.2) is trivial for $k = n$. Assume that (5.2) holds true for $k = n, n-1, \dots, s$. Then using (5.1) and our induction hypothesis, we obtain

$$\begin{aligned} \omega_{s-1}^- &= 1 - \frac{c_s}{c_{s-1}}\,\omega_s^- = 1 - \frac{c_s}{c_{s-1}} \left[(-1)^s c_s^{-1} \sum_{j=s}^{n} (-1)^j c_j \right] \\ &= (-1)^{s-1} c_{s-1}^{-1} \sum_{j=s-1}^{n} (-1)^j c_j, \end{aligned} \tag{5.3}$$

which proves (5.2). Since the harmonic measures ω_k^- and constants c_k are positive, (5.2) implies (1.13). □

Lemma 5.2. *For every set of positive constants* $C = \{c_0, \dots, c_n\}$ *with* $c_0 = 1$*, there is at most one crescent configuration* $\widetilde{\Omega} \in \mathcal{C}_n$ *carrying harmonic measures proportional with respect to* C.

Proof. Assume there are two configurations $\widetilde{\Omega}_1 = \{\Omega_{k,1}\}_{k=0}^n$ and $\widetilde{\Omega}_2 = \{\Omega_{k,2}\}_{k=0}^n$ carrying harmonic measures proportional with respect to C. Let $a_{k,m}$ be a reference point in $\Omega_{k,m}$, and let $\gamma_{k,m}^+$ and $\gamma_{k,m}^-$ be the corresponding boundary arcs of $\Omega_{k,m}$. Then, for $k = 0, \dots, n-1$ and $m = 1, 2$, and for any Borel set $E \subset \gamma_{k,m}^+$,

$$c_k \omega(E, \Omega_{k,m}, a_{k,m}) = c_{k+1} \omega(E, \Omega_{k+1,m}, a_{k+1,m}).$$

For notational convenience, let

$$\omega_{k,m}^+ = \omega(\gamma_{k,m}^+, \Omega_{k,m}, a_{k,m}) \quad \text{and} \quad \omega_{k,m}^- = \omega(\gamma_{k,m}^-, \Omega_{k,m}, a_{k,m}).$$

Since $\omega_{k,1}^-$ and $\omega_{k,2}^-$ satisfy (5.2) for the same set of constants c_k, (5.3) implies that

$$\omega_{k,1}^+ = \omega_{k,2}^+, \quad \omega_{k,1}^- = \omega_{k,2}^-, \quad k = 0, 1, \dots, n. \tag{5.4}$$

Let $\zeta = f_{k,m}(z)$ map $\Omega_{k,m}$ conformally onto the unit disk if k is even and onto the exterior of the unit disk $\mathbb{D}^* = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ if k is odd. For even k, we normalize $f_{k,m}$ by the conditions $f_{k,m}(a_{k,m}) = 0$, $f_{k,m}(\gamma_{k,m}^+) = L(\varphi_{k,m})$, where $0 < \varphi_{k,m} \le 2\pi$ and $L(\varphi) := \{e^{i\theta} : 0 < \theta < \varphi\}$. For odd k we assume that $f_{k,m}(a_{k,m}) = \infty$ and $f_{k,m}(\gamma_{k,m}^-) = L(\varphi_{k,m})$. Note that (5.4) implies that $\varphi_{k,1} = \varphi_{k,2}$ for all $k = 0, \dots, n$.

Consider the function $\Phi : \mathbb{D} \mapsto \mathbb{D}$ defined by:

$$\Phi(z) = f_{k,2}^{-1}(f_{k,1}(z)) \quad \text{if} \quad z \in \Omega_{k,1} \cup \gamma_{k,1}^+.$$

Then $\Phi(a_{k,1}) = a_{k,2}$ for all $k = 0, 1, \dots, n$, $\Phi(1) = 1$, and Φ is analytic in $\cup_{k=0}^n \Omega_{k,1}$. Since $\gamma_{k,m}^+$ and $\gamma_{k,m}^-$ are Jordan arcs, the conformal mapping $f_{k,m}$ is one-to-one on $\bar{\Omega}_{k,m} \setminus \{1\}$. Therefore Φ maps $\overline{\mathbb{D}}$ one-to-one and onto $\overline{\mathbb{D}}$.

Let us prove that Φ is analytic in $\mathbb{D}$. We have to show that Φ is analytic across $\gamma_{k,1}^+$ for every $0 \le k < n$. To be specific, we fix an even

k. If k is odd, the argument is similar. Let $\Omega'_{k,m} = f_{k,m}^{-1}(\mathbb{D} \setminus [0,1])$, $\Omega'_{k+1,m} = f_{k+1,m}^{-1}(\mathbb{D}^* \setminus [1,\infty])$, let $g_{k,m} = f_{k,m}^{1/c_{k+1}}$ and $g_{k+1,m} = f_{k+1,m}^{1/c_k}$. The functions $g_{k,m}$ and $g_{k+1,m}$ are single-valued on $\Omega'_{k,m}$ and $\Omega'_{k+1,m}$, respectively. For $m = 1,2$, let

$$\Phi_m(z) = \begin{cases} g_{k,m}(z) & \text{if } z \in \Omega'_{k,m} \cup \gamma^+_{k,m} \\ g_{k+1,m}(z) & \text{if } z \in \Omega'_{k+1,m}. \end{cases}$$

Notice that $\Phi(z) = \Phi_2^{-1}(\Phi_1(z))$ for $z \in \Omega'_{k,1} \cup \Omega'_{k+1,m} \cup \gamma^+_{k,1}$.

For any point $\tau \in \gamma^+_{k,m}$, let $l_m(\tau)$ denote the arc of $\gamma^+_{k,m}$ with ends at $z = 1$ and $z = \tau$ such that $f_{k,m}(l_m(\tau)) = L(\varphi_m)$ for some $\varphi_m = \varphi_m(\tau)$, $0 < \varphi_m \le 2\pi$. Since

$$c_k \omega(l_m(\tau), \Omega_{k,m}, a_{k,m}) = c_{k+1}\omega(l_m(\tau), \Omega_{k+1,m}, a_{k+1,m}),$$

we have

$$g_{k,m}(\tau) = g_{k+1,m}(\tau),$$

for $m = 1,2$ and every τ in $\gamma^+_{k,m}$. This implies that Φ_m is continuous on $\Omega'_{k,m} \cup \Omega'_{k+1,m} \cup \gamma^+_{k,m}$. Since $|\Phi_m(z)| = 1$ for z in $\gamma^+_{k,m}$, it follows that $\Phi_m(z)$ is analytic on $\gamma^+_{k,m}$, $m = 1,2$. This implies that $\Phi = \Phi_2^{-1} \circ \Phi_1$ is analytic and one-to-one on $\overline{\mathbb{D}}$. Hence Φ is a Möbius mapping from $\mathbb{D}$ onto $\mathbb{D}$. Since $\Phi(0) = 0$ and $\Phi(1) = 1$, we have $\Phi(z) = z$. Therefore, $\Omega_{k,2} = \Phi(\Omega_{k,1}) = \Omega_{k,1}$ and $a_{k,2} = \Phi(a_{k,1}) = a_{k,1}$. This completes the proof. □

For any set A, let $A^* = \{\bar{z} : z \in A\}$. Note that a configuration $\widetilde{\Omega} = \{\Omega_k\}_{k=0}^n$ in $\mathcal{C}_n$, with reference points a_k in Ω_k, carries proportional harmonic measures if and only if the configuration of symmetric regions $\widetilde{\Omega}^* = \{\Omega_k^*\}_{k=0}^n$ with the reference points $\bar{a}_k \in \Omega_k^*$ carries proportional harmonic measures. Therefore, by the uniqueness result of Lemma 5.2, we obtain the following.

Corollary 5.3. *If a crescent configuration $\widetilde{\Omega} = \{\Omega_0, \dots, \Omega_n\}$, with reference points $a_k \in \Omega_k$, carries proportional harmonic measures, then Ω_k is symmetric with respect to $\mathbb{R}$ and*

$$0 = a_0 < a_1 < \cdots < a_n < 1. \tag{5.5}$$

Proof of Theorem 1.3. The uniqueness of a crescent configuration $\widetilde{\Omega} = \{\Omega_0, \dots, \Omega_n\}$, the symmetry of Ω_k with respect to $\mathbb{R}$, for $k = 0, \dots, n$, and inequalities (5.5) follow from Lemma 5.2 and its corollary. Let M_1 be the set of all points $(a_1, \dots, a_n) \in \mathbb{R}^n$, whose coordinates satisfy (5.5), and let M_2 be the set of points $(c_1, \dots, c_n) \in \mathbb{R}^n$ with positive coordinates satisfying (1.13). Let $F : M_1 \mapsto \mathbb{R}^n$ be the mapping with components F_k, $k = 1, \dots, n$, defined by (1.6).

The proof will be complete if we show that F is a diffeomorphism from M_1 onto M_2. Indeed, for every set of constants c_k satisfying (1.13), there is a unique solution $a_1, \dots, a_n$ of equations (1.15), which satisfies (5.5). Therefore the crescent configuration defined by Theorem 1.2 is a unique system carrying proportional harmonic measures with respect to $c_0, \dots, c_n$.

The same crescent configuration arises as a system of circle domains of quadratic differential (1.14). To show this, we note that $Q_A(z)$ has a second order pole at $z = a_k$ in the Laurent expansion

$$Q_A(z) = \frac{-c_k^2}{(z - a_k)^2} + \cdots ,$$

where $c_k = F_k(a_1, \dots, a_n)$ and F_k is defined as in Theorem 1.3. Therefore, there is a maximal circle domain D_k of $Q_A(z)\, dz^2$ centered at a_k.

Notice that the trajectory structure of $Q_A(z)\, dz^2$ is symmetric with respect to the real axis and with respect to the unit circle. The punctured circle $\gamma_0 = \mathbb{T} \setminus \{1\}$ is a critical trajectory of $Q_A(z)\, dz^2$. Inside the unit disk, $Q_A(z)\, dz^2$ has n critical trajectories $\gamma_1, \dots, \gamma_n$, each of which terminates at the point $z = 1$; where $z = 1$ is a zero of order $4n$ of $Q_A(z)\, dz^2$. We enumerate the trajectories such that γ_{k+1} lies inside γ_k. The complementary set $\mathbb{D} \setminus \cup_{k=1}^n \gamma_k$ consists of $n+1$ maximal circle domains D_k of $Q_A(z)\, dz^2$, where $a_k \in D_k$ and $\partial D_k = \gamma_k \cup \gamma_{k+1} \cup \{1\}$. Hence, for every $k = 0, \dots, n-1$, D_k is a crescent in $\mathbb{D}$, D_n is a Jordan region and the system $\widetilde{D}_A = \{D_0, \dots, D_n\}$ is in $\mathcal{C}_n$. And this holds for every set $A = \{a_0, \dots, a_n\}$ satisfying (5.5). In our standard notation for boundary arcs of crescents, $\gamma_k^- = \gamma_k$, $\gamma_k^+ = \gamma_{k+1}$.

By Corollary 4.2,

$$c_k \omega(E, D_k, a_k) = c_{k+1} \omega(E, D_{k+1}, a_{k+1})$$

for all $k = 1, \dots, n$ and for every Borel set $E \subset \gamma_k^- = \gamma_k$. Therefore, the crescent configuration $\widetilde{D}_A$ with the set of reference points A carries harmonic measures proportional with respect to the constants $c_0, c_1, \dots, c_n$ defined by (1.6). Now the uniqueness result of Lemma 5.2 shows that $D_k = \Omega_k$ for all $k = 0, \dots, n$.

To prove that F is a diffeomorphism from M_1 onto M_2, we represent F as $F = \Phi \circ L$ with $L = (\psi, \dots, \psi)$, where $\psi = \psi(x) = (1+x)/(1-x)$. Then $\Phi = F \circ L^{-1}$ is a mapping from the set $N = \{(b_1, \dots, b_n) : 1 < b_1 < \cdots < b_n\}$ into M_2. Let $(\partial L/\partial A)$ denote the Jacobian matrix of L. We then have

$$\|(\partial L/\partial A)\| = 2^n \prod_{k=1}^{n} (1 - a_k)^{-2} \neq 0. \tag{5.6}$$

This easily implies that L is a diffeomorphism from M_1 onto N.

To show that Φ is a local diffeomorphism, we change variables in (1.3) via $z = (i-\zeta)/(i+\zeta)$. Then (1.3), with $w_{A,C}$ defined by (1.5), becomes

$$\sum_{k=0}^{n} (-1)^k c_k \frac{b_k}{1 + b_k^2 \zeta^2} = \frac{\zeta^{2n}}{1 + \zeta^2} \prod_{k=1}^{n} \frac{b_k^2 - 1}{1 + b_k^2 \zeta^2}, \tag{5.7}$$

where $b_k = (1 + a_k)/(1 - a_k)$, $k = 0, \dots, n$. Developing both sides of (5.7) into power series at $\zeta = 0$ and equating corresponding coefficients, we get a system of linear equations in c_k:

$$\sum_{k=1}^{n} (-1)^k c_k b_k^{2j+1} = -1, \quad j = 0, \dots, n-1. \tag{5.8}$$

We know from Theorem 1.1 that (5.8) has a unique solution. Therefore, the determinant $\|((-1)^k b_k^{2j+1})\| \neq 0$ for considered values of b_k's.

Differentiating equations (5.8) with respect to the b_l's and using matrix notation, we obtain:

$$(\partial c_k/\partial b_l) \left((-1)^k b_k^{2j+1}\right) = -\left((2j+1)(-1)^l c_l b_l^{2j}\right), \tag{5.9}$$

where $k = 1, \dots, n$, $l = 1, \dots, n$ and $j = 0, \dots, n-1$. Finding the determinant of the matrix on the right-hand side of (5.9), we get:

(5.10)

$$\left\|\left((2j+1)(-1)^l c_l b_l^{2j}\right)\right\| = (2n-1)!! \prod_{k=1}^{n} (c_k/b_k) \left\|\left((-1)^l b_l^{2j+1}\right)\right\| \neq 0.$$

Now (5.10) and (5.6) imply that $\|(\partial c_k/\partial b_l)\| \neq 0$ for considered values of the parameters. Thus, Φ is a local diffeomorphism and therefore $F = \Phi \circ L$ is a diffeomorphism from M_1 into M_2.

To finish the proof, we have to show that F maps ∂M_1 into ∂M_2. Since $F = \Phi \circ L$ and L is a diffeomorphism from M_1 onto N, it is enough to show that Φ maps ∂N into ∂M_2.

Suppose that $\partial\Phi(N) \not\subset \partial M_2$. Then we can find a sequence $(b_{1,m}, \dots, b_{n,m})$ in N, which converges to $(\beta_1, \dots, \beta_n)$ in ∂N as $m \to \infty$, such that, for $k = 1, \dots, n$, $c_{k,m} := c_k(b_{1,m}, \dots, b_{n,m}) \to \lambda_k$ as $m \to \infty$, where $(c_{1,m}, \dots, c_{n,m}) \in M_2$ for all $m = 1, 2, \dots$ and indeed $(\lambda_1, \dots, \lambda_n) \in M_2$. Since $(\beta_1, \dots, \beta_n) \in \partial N$, we have:

$$\beta_0 = 1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq +\infty \tag{5.11}$$

with at least one additional sign of equality in the inequalities (5.11).

Substituting $a_j = (b_j - 1)/(b_j + 1)$ into (1.6), we express the functions c_k in terms of parameters b_j's:

$$c_k = (-1)^k \frac{1}{b_k} {\prod_{j=1}^{n}}' \frac{b_j^2 - 1}{b_j^2 - b_k^2}, \quad k = 1, \dots, n. \tag{5.12}$$

Note that $\{\lambda_k\}_{k=1}^{\infty}$ are the limit values of functions (5.12) with $b_j = b_{j,m}$, where $b_{j,m} \to \beta_j$ as $m \to \infty$. By our assumption, these limit values λ_k are finite positive numbers, which satisfy inequalities

$$\lambda_k - \lambda_{k+1} + \cdots + (-1)^{n-k}\lambda_n > 0, \quad k = 0, \dots, n. \tag{5.13}$$

If $\beta_1 > 1$ and $\beta_n < \infty$, then $\beta_k \neq \beta_j$ for $k \neq j$ since all the limit values $c_{k,\infty}$ are finite. Thus, we cannot have the sign of equality in relations (5.11) in this case.

Consider the case $\beta_n = \infty$. Since λ_n is positive, it follows from (5.12) that $b_{n,m}^2 - b_{j,m}^2 \to 0$ and $b_{n,m}^2 - b_{s-1,m}^2 \not\to 0$ as $m \to \infty$ for $j = n-1, \dots, s$ and some s, $1 \leq s \leq n-1$.

Considering the sum (5.13) for $k = s$, we have:

$$\begin{aligned}
\lambda_s - &\lambda_{s+1} + \cdots + (-1)^{n-s}\lambda_n \\
&= \lambda_s \sum_{j=s}^{n} (-1)^{j-s} \frac{\lambda_j}{\lambda_s} \\
&= (-1)^{n-s}\lambda_s \lim_{m\to\infty} \sum_{k=s}^{n} \frac{b_{s,m}}{b_{j,m}} {\prod}'^{n}_{j=1} \frac{b_{s,m}^2 - b_{j,m}^2}{b_{k,m}^2 - b_{j,m}^2} \qquad (5.14)\\
&= (-1)^{n-s}\lambda_s \lim_{m\to\infty} \sum_{k=s}^{n} {\prod}'^{n}_{j=s} \frac{b_{s,m}^2 - b_{j,m}^2}{b_{k,m}^2 - b_{j,m}^2} = 0,
\end{aligned}$$

which contradicts (5.13). The third equality in this chain follows from the limit relation $b_{j,m}/b_{s,m} \to 1$ as $m \to \infty$, and the last one follows from the identity

$$\sum_{k=1}^{n} {\prod}'^{n}_{j=1} \frac{1}{z_k - z_j} = 0,$$

which holds for any distinct complex numbers $z_1, \ldots, z_n$.

Now suppose $\beta_n < \infty$. Then, as we noted above, $\beta_1 = 1$. As in the previous case, since $\lambda_n > 0$, there exists s, $0 \le s < n$, such that $\beta_j = \beta_n$ for all $j = s, \ldots, n$ and, in addition, $\beta_{s-1} < \beta_s$ if $s > 0$. Calculating the sum (5.13) with $k = s$, we get the same chain of relations (5.14).

Since (5.14) contradicts (5.11), the assumption $\partial\Phi(N) \not\subset \partial M_2$ is false in all cases. The proof is now complete. □

Remark. In the proof above we use the fact that $\|((-1)^k b_k^{2j+1})\| \neq 0$, which we derive from the uniqueness assertion of Theorem 1.1. This relation can be verified directly from the well-known property of the Vandermonde determinant. Indeed, we have:

$$\|((-1)^k b_k^{2j+1})\| = (-1)^{[n/2]} \Delta \prod_{k=1}^{n} b_k,$$

where $[n/2]$ denotes the integer part of $n/2$ and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ b_1^2 & b_2^2 & b_3^2 & \ldots & b_n^2 \\ b_1^4 & b_2^4 & b_3^4 & \ldots & b_n^4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_1^{2(n-1)} & b_2^{2(n-1)} & b_3^{2(n-1)} & \ldots & b_n^{2(n-1)} \end{vmatrix}$$

is the Vandermonde determinant for $b_1^2, \dots, b_n^2$. The well-known identity for the Vandermonde determinant, see [**10**, p. 3]:

$$\Delta = \prod_{i>j} (b_i^2 - b_j^2)$$

shows that $\Delta > 0$ (and therefore $\Delta' \neq 0$) since $b_i > b_j$ if $i > j$.

6. The case of infinitely many reference points. In this section we consider the case of a sequence of reference points $\{a_j\}_{j=1}^{\infty}$, $0 = a_0 < a_1 < \cdots < a_{j-1} < a_j \to 1$ as $n \to \infty$, and related quadrature identities. Once again we transplant the problem to $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ via the Möbius transformation $\zeta(z) = i(1-z)/(1+z)$, and let $b_j = (1+a_j)/(1-a_j)$. Our analysis is based on the finite case. In the finite case, and under the transformation $z(\zeta) = (i-\zeta)/(i+\zeta)$, the minimal weight v_n in $\mathbb{H}$ is given by $v_n(\zeta) = w_n(z(\zeta))|z'(\zeta)|$ and has the form:

$$v_n(\zeta) = \frac{1}{1+\zeta^2} \prod_{j=1}^{n} \frac{(b_j^2 - 1)\zeta^2}{1 + b_j^2 \zeta^2}.$$

And, for $n \geq 2$ and $1 \leq k \leq n$, the constants c_k are given by:

$$c_k = \frac{1}{b_k} \prod_{1 \leq j \neq k}^{n} \frac{b_j^2 - 1}{|b_j^2 - b_k^2|}.$$

Now $1 - ((b_j^2 - 1)\zeta^2)/(1 + b_j^2 \zeta^2) = (1+\zeta^2)/(1 + b_j^2 \zeta^2)$ and so, by [**11**, Theorem 15.6],

$$v(\zeta) = \frac{1}{1+\zeta^2} \prod_{j=1}^{\infty} \frac{(b_j^2 - 1)\zeta^2}{1 + b_j^2 \zeta^2}$$

converges to a positive weight on $\mathbb{R}$ if and only if

$$\sum_{j=1}^{\infty} b_j^{-2}$$

converges. The convergence of this series also assures us of the convergence of

$$c_k = \frac{1}{b_k} \prod_{1 \leq j \neq k}^{\infty} \frac{b_j^2 - 1}{|b_j^2 - b_k^2|}.$$

Notice that, if $v(\zeta)$ converges, then

$$0 < v(x) < \frac{1}{1+x^2}$$

on $\mathbb{R} \setminus \{0\}$ and so $v \in L^1(\mathbb{R})$. Suppose, additionally, that the sequence of positive constants $\{c_n\}_{n=1}^\infty$ were summable, i.e., the series $\sum_{n=1}^\infty c_n$ converges, and define measures η (on $\mathbb{R}$) and σ (on $\mathbb{H}$) by:

$$d\eta = \frac{1}{\pi} v(x)\, dx$$

and

$$\sigma = \delta_i + \sum_{j=1}^\infty c_j \delta_{i/b_j}.$$

Then the signed measure $\mu = \eta - \sigma$ has finite total variation and, by our earlier quadrature identities, satisfies

$$\int h\, d\mu = 0$$

whenever h is bounded and continuous on $\overline{\mathbb{H}}$ and harmonic on $\mathbb{H}$. In what follows we give a sufficient condition on $\{b_n\}_{n=1}^\infty$, and hence on $\{a_n\}_{n=1}^\infty$, that ensures that the sequence $\{c_n\}_{n=1}^\infty$ is indeed summable. We later examine two cases that are instructive. We have not yet obtained a condition on $\{b_n\}_{n=1}^\infty$ that is necessary and sufficient for the summability of $\{c_n\}_{n=1}^\infty$.

Theorem 6.1. *If there exist $\lambda > 1$ and a positive integer N such that*

$$\frac{b_{n+1}}{b_n} \geq \lambda$$

for $n \geq N$, then $\{c_n\}_{n=1}^\infty$ is summable.

Proof. Without loss of generality, we may assume that $N = 1$. To find an upper bound for c_k, we first estimate the product of the initial

$k-1$ factors in its infinite product representation. Observe that

$$\begin{aligned}\prod_{j=1}^{k-1}\frac{b_j-1}{b_k-b_j} &\le \prod_{j=1}^{k-1}\frac{1}{(b_k/b_j)-1}\\ &\le \prod_{j=1}^{k-1}\frac{1}{\lambda^j-1}\\ &\le \frac{1}{\lambda^{k(k-1)/2}}\prod_{j=1}^{\infty}\frac{1}{1-(1/\lambda)^j}\\ &\le \frac{C}{\lambda^k},\end{aligned}$$

where C is some positive constant, independent of k. And, concerning the tail of our infinite product, notice that

$$\begin{aligned}\log\left(\prod_{j=k+1}^{\infty}\frac{b_j-1}{b_k-b_j}\right) &= \sum_{j=k+1}^{\infty}\log\left(\frac{b_j-1}{b_j-b_k}\right)\\ &\le \sum_{j=k+1}^{\infty}\frac{b_k}{b_j-b_k}\\ &\le \sum_{j=k+1}^{\infty}\frac{1}{\lambda^{j-k}-1}\\ &\le \frac{1}{\lambda-1}\sum_{j=0}^{\infty}\frac{1}{\lambda^j}\\ &= \frac{\lambda}{(\lambda-1)^2}.\end{aligned}$$

Combining these estimates, we obtain an upper bound for c_k of the form:

$$c_k \le \frac{C^*}{\lambda^k},$$

where $C^* > 0$ is independent of k. Evidently, $\{c_k\}_{k=1}^{\infty}$ is summable. □

Theorem 6.2. *In the case that* $b_j = (j+1)^2$, $j = 1, 2, 3, \ldots$, *the sequence* $\{c_k\}_{k=1}^{\infty}$ *is summable. Yet, in the case that* $b_j = j+1$, $j = 1, 2, 3, \ldots$, *the sequence* $\{c_k\}_{k=1}^{\infty}$ *is not even bounded.*

Proof. Now, for $k = 2, 3, \dots,$

$$f_k(z) = \prod_{2\le j\ne k}^{\infty}\left(1-\frac{z^2}{j^2}\right) = \frac{\sin \pi z}{\pi z(1-z^2)(1-(z^2/k^2))}.$$

And therefore,

$$\prod_{2\le j\ne k+1}^{\infty}\frac{j^2-1}{|j^2-(k+1)^2|} = \frac{|f_{k+1}(1)|}{|f_{k+1}(k+1)|} = (k+1)^2.$$

So, in the case that $b_j = (j+1)^2$,

$$\begin{aligned} c_k &= \frac{1}{(k+1)^2}\cdot\prod_{1\le j\ne k}^{\infty}\frac{(j+1)^4-1}{|(j+1)^4-(k+1)^4|} \\ &= \frac{1}{(k+1)^2}\cdot\prod_{2\le j\ne k+1}^{\infty}\frac{j^4-1}{|j^4-(k+1)^4|} \\ &= \frac{1}{(k+1)^2}\cdot\prod_{2\le j\ne k+1}^{\infty}\frac{j^2-1}{|j^2-(k+1)^2|}\cdot\prod_{2\le j\ne k+1}^{\infty}\frac{j^2+1}{|j^2+(k+1)^2|} \\ &\le \frac{5}{4+(k+1)^2}. \end{aligned}$$

Thus, $\{c_k\}_{k=1}^{\infty}$ is summable in the case that $b_j = (j+1)^2$. However, if $b_j = j+1$, then

$$c_k := \frac{1}{k+1}\cdot\prod_{2\le j\ne k+1}^{\infty}\frac{j^2-1}{|j^2-(k+1)^2|} = k+1,$$

which is unbounded as k ranges over the positive integers. □

REFERENCES

1. J. Akeroyd, *Harmonic measures on complementary subregions of the disk*, Complex Variables Theory Appl. **36** (1998), 183–187.

2. J. Akeroyd and E.G. Saleeby, *A class of $P^t(d\mu)$ spaces whose point evaluations vary with t*, Proc. Amer. Math. Soc. **127** (1999), 537–542.

3. V.N. Dubinin, *Symmetrization in the geometric theory of a complex variable*, Uspekhi Mat. Nauk **49** (1994), 3–76; Russian Math. Surveys **49** (1994), 1–79 (in English).

4. V.N. Dubinin and L.V. Kovalev, *Reduced modulus of the complex sphere*, Zap. Nauchn. Sem. POMI **258** (2001), 76–94; J. Math. Sci. **105** (2001), 2165–2179 (in English).

5. P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

6. F.P. Gardiner, *Teichmüller theory and quadratic differentials*, John Wiley & Sons, New York, 1987.

7. J.B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1982.

8. J.A. Jenkins, *Univalent functions and conformal mapping*, Ergeb. Math. Grenzgeb. **18**, 1958.

9. G.V. Kuz'mina, *Moduli of families of curves and quadratic differentials*, Proc. Steklov Inst. Math., Trudy Mat. Inst. Steklov **139**, no. 1, 1982.

10. V.V. Prasolov, *Problems and theorems in linear algebra*, Transl. Math. Monographs **134**, Amer. Math. Soc., Providence, RI, 1994.

11. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987.

Dept. of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701
E-mail address: jakeroyd@uark.edu

Dept. of Mathematics and Statistics, University of Central Oklahoma, Edmond, Oklahoma 73034
E-mail address: kkarber1@ucok.edu

Steklov Institute of Mathematics at St. Petersburg, Russian Academy of Sciences, Fontanka 27, St. Petersburg, 191023, Russia
Current address: Dept. of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409
E-mail address: alex.solynin@ttu.edu