ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 2000

FAITHFULNESS AND CANCELLATION OVER NOETHERIAN DOMAINS

H. PAT GOETERS AND BRUCE OLBERDING

A recent theme in the study of commutative rings is to observe the necessary structure present in a ring in order for a particular theorem in abelian group theory to hold true when interpreted for the ring. A theorem of Warfield [**13**] asserts that when X is a nonzero subgroup of the group of rational integers, then $G \cong_{\text{nat}} \operatorname{Hom}(X, XG)$ for any torsion-free $\operatorname{End}(X)$-module G, here XG has the appropriate meaning as a subgroup of the divisible hull of G. A module M over a ring R is called *faithful*, in the sense of Arnold and Lady [**2**], if $IM = M$ implies $I = R$ for any ideal I of R. In our work we will consider several variations of the faithful concept and relate them to Warfield's result.

Below, all unadorned Hom and $\otimes$ symbols are with respect to the integral domain R, and Q is used to represent the quotient field of R. Given a torsion-free module M over R, we define the *ring of fractions* of M as $E_M = \{t \in Q \mid tM \subseteq M\}$ which is the largest overring of R in Q over which M is a module. The rank of a torsion-free module M is the dimension of the Q-vector space, $Q \otimes M$. In case X is a torsion-free module of rank one, E_X is just the endomorphism ring of X. We note without future reference that if S is an overring of R in Q and X and Y are torsion-free modules with $S \subseteq E_X \cap E_Y$, then $\operatorname{Hom}(X, Y) = \operatorname{Hom}_S(X, Y)$. In particular, E_X is independent of the base ring R. When P is a prime ideal of R, the notation S_P represents the localization of the module S at P and is therefore $S_P = R_P \cdot S \subseteq Q$.

1. Noetherian domains.

Definitions. An integral domain R is called *strongly faithful* if, for any submodule X of Q and torsion-free module G, one has $G \cong_{\text{nat}} \operatorname{Hom}(X, XG)$ when $E_X \subseteq E_G$. Here XG is the submodule of the divisible hull of G, QG, generated by xg such that $x \in X$ and $g \in G$.

Received by the editors on April 24, 1997, and in revised form on September 30, 1998.

We will say that R has *restricted cancellation* if, for any nonzero ideals I, J and K with $E_I \subseteq E_J \cap E_K$, $IJ = IK$ implies $J = K$; as in the proof of Lemma 5 below, this is equivalent to each nonzero ideal I of R being a cancellation ideal of E_I.

An integral domain R is called *HT-faithful* if $G \cong_{\text{nat}} \text{Hom}\,(X, X \otimes_{E_X} G)$ for any rank one module X and torsion-free module G with $E_X \subseteq E_G$.

It follows easily from the definitions that strongly faithful domains have restricted cancellation. Also, it is well known that if, for any ideals I, J and K of R with $I \neq 0$, $IJ = IK$ implies $J = K$, then R is Prüfer, so without the restrictions on the rings of fractions imposed above, the Noetherian domains described are precisely the Dedekind domains.

Recall that an ideal I of an integral domain R is called *stable*, in the sense of Sally and Vasconcelos [**12**], if I is projective as a module over its ring of endomorphisms. The ring R is called *stable* if every nonzero ideal of R is stable. Below we will establish a series of equivalences thereby substantiating our main result.

Main theorem. *Let R be a Noetherian domain. The following are equivalent*:

(1) *R is stable.*

(2) *R is HT-faithful.*

(3) *R is strongly faithful.*

(4) *R has restricted cancellation.*

The following is a standard observation.

Lemma 1. *Let X be a torsion-free module of rank one over an integral domain S. The following are equivalent*:

(a) *X is flat over S.*

(b) *$I \otimes_S X$ is torsion-free for every ideal I of S.*

(c) *$G \otimes_S X$ is torsion-free for every torsion-free S-module G.*

Proof. If X is flat and G is torsion-free, then G embeds in QG, the divisible hull of G. Hence $G \otimes_S X$ embeds in the torsion-free divisible module $QG \otimes_S X$, establishing (a) $\rightarrow$ (c). To finish, we need to show (b) $\rightarrow$ (a). Given an ideal I of S, since $I \otimes_S X$ is torsion-free, then $I \otimes_S X \cong IX$ follows by considering ranks, and therefore X is flat by this well-known criterion [**11**]. $\square$

Since R is Noetherian, R is stable if and only if every ideal of R is flat as a module over its ring of endomorphisms. In [**5**] it is shown that when R is Noetherian, R is stable if and only if every rank one module is faithfully flat as a module over its ring of endomorphisms.

Proposition 2. *Let R be a Noetherian domain. Then R is stable if and only if R is HT-faithful.*

Proof. Assume that R is HT-faithful but some ideal I of R is not flat over its ring of endomorphisms. A moment's reflection reveals that every overring of R in Q is HT-faithful. Also it is easy to see that E_I is Noetherian, so we may assume that $R = E_I$ to simplify notation. By Lemma 1, $I \otimes J$ has a nonzero torsion submodule T for some ideal J of R.

Again, since R is Noetherian, it is easy to see that the torsion module T has an element x whose annihilator is a nonzero prime ideal P of R, [**7**, p. 387]. Consequently, the torsion submodule of $(I \otimes J)_P = I_P \otimes_{R_P} J_P$, namely T_P, has a submodule T' isomorphic to $(R/P)_P = R_P/PR_P$. By Nakayama's lemma, $PI_P \neq I_P$ and consequently I_P/PI_P is a finite direct sum of copies of the field $\kappa = R_P/PR_P$. Therefore, because $T' \subseteq T_P$ and I_P/PI_P contains κ as a direct factor, $0 \neq \mathrm{Hom}\,(\kappa, T') \subseteq \mathrm{Hom}\,(I_P/PI_P, T_P) \subseteq \mathrm{Hom}\,(I_P, T_P) \subseteq \mathrm{Hom}\,(I_P, I_P \otimes_{R_P} J_P)$. It is a well-known result that, because I is finitely presented, $\mathrm{Hom}\,(I_P, I_P) = \mathrm{Hom}\,(I, I)_P = R_P$ [**9**]. But R is HT-faithful which implies that $\mathrm{Hom}\,(I_P, I_P \otimes_{R_P} J_P) \cong_{\mathrm{nat}} J_P$, thus leading to the conclusion that $\mathrm{Hom}\,(\kappa, T') \cong \kappa$ is a submodule of the torsion-free module J_P. This contradiction shows that every ideal is flat hence projective over its ring of endomorphisms.

Now suppose that R is stable, and let X be a rank one module. In order to show that R is HT-faithful, we may assume that X is a submodule of Q containing R. Set $S = E_X$ and let G be torsion-free such that $E_X \subseteq E_G$. As mentioned prior to the statement of this proposition, X is a faithfully flat S-module. In particular, $X \otimes_S G \cong_{\text{nat}} XG$. This is due to the fact that tensoring the epimorphism $X \otimes_S G \to XG$ with Q produces the isomorphism $Q \otimes_S X \otimes_S G = Q \otimes_S Q \otimes_S G \cong Q \otimes_S XG \cong QG$ implying that $\operatorname{Ker}(X \otimes_S G \to XG)$ is torsion. However, $X \otimes_S G$ is torsion-free by Lemma 1. So, in order to conclude that R is HT-faithful, it is enough to argue that $G \cong_{\text{nat}} \operatorname{Hom}(X, XG)$.

Identify G with its canonical image in $\operatorname{Hom}(X, XG)$, and let $T = \operatorname{Hom}(X, XG)/G$. Our aim is to show that $T = 0$. In order to do this we first need to show that T is a torsion module. Let $f : X \to XG$ and write $f(1) = \Sigma_j x_j g_j$ with $x_j \in X$ and $g_j \in G$. Since $X \subseteq Q$, there is an element $0 \neq r \in R$ with $rx_j \in R$ for each j. Then $rf(1) = \Sigma_j r x_j g_j = g \in G$. Thus, given $x = c/d \in X$ with $c, d \in R$, $rdf(x) = cg$ so that $rf(x) = (c/d)g = xg$. That is, rf belongs to the image of G in $\operatorname{Hom}(X, XG)$ and T is torsion.

Tensoring $0 \to G \to \operatorname{Hom}(X, XG) \to T \to 0$ with X yields a commutative rectangle

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G \otimes_S X & \longrightarrow & \operatorname{Hom}(X, XG) \otimes_S X & \longrightarrow & T \otimes_S X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & XG & \longrightarrow & XG & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

The map α was shown above to be an isomorphism. It follows from the Snake Lemma [**11**] that $\operatorname{Im}\alpha = \operatorname{Ker}\beta = T \otimes_S X$, with $T \otimes_S X$ torsion. But because X is flat, $\operatorname{Hom}(X, XG) \otimes_S X$ is torsion-free by Lemma 1. Consequently, the map σ must be an isomorphism and therefore $T \otimes_S X = 0$. However, X is faithfully flat over S, as mentioned above, forcing $T = 0$. Thus $G = \operatorname{Hom}(X, XG)$ canonically. $\square$

We note for later reference that the proof above contains an argument that Noetherian stable domains are strongly faithful.

Lemma 3. *Let R be an integral domain with nonzero ideals I and J. If I is a cancellation ideal of E_I and $S = E_J$, $E_{SI} = SE_I$.*

Proof. Clearly $SE_I \subseteq E_{SI}$, from which it follows that $IE_{SI} = SI$. Thus $IE_{SI} = I(SE_I)$. Let $0 \neq t \in J \cap I$. Then $(t^2)E_{SI} \subseteq tSI \subseteq$

$JI \subseteq E_I$ and $t^2(SE_I) \subseteq JI$. Since I is a cancellation ideal of E_I, $(t^2)SE_I = (t^2)E_{SI}$ and so $E_{SI} = SE_I$. $\square$

As mentioned in the proof of Proposition 2, $\mathrm{Hom}\,(I, I)_P = \mathrm{Hom}\,(I_P, I_P)$ when I is a finitely presented ideal of R and $P \in \max(R)$. We require a more general version of this fact.

Lemma 4. *Let S be an overring of the integral domain R inside the quotient field Q of R. If I and J are ideals of S with I finitely generated, then for any $P \in \max(R)$, $\mathrm{Hom}\,(I, J)_P = \mathrm{Hom}\,(I_P, J_P)$. In particular, $(E_I)_P = E_{I_P}$.*

Proof. One has $\mathrm{Hom}\,(I, J)_P \subseteq \mathrm{Hom}\,(I_P, J_P)$ canonically. Let $x_1, \dots, x_n$ generate I. If $f \in \mathrm{Hom}\,(I_P, J_P)$, then for each j there exists $t_j \in R \backslash P$ such that $t_j \cdot f(x_j) \in J$. Then for $t = t_1 \cdots t_n$, $t \cdot f \in \mathrm{Hom}\,(I, J)$ and so $\mathrm{Hom}\,(I_P, J_P) \subseteq \mathrm{Hom}\,(I, J)_P$. $\square$

Lemma 5. *If I is a nonzero ideal of an integral domain R with restricted cancellation, then E_I has restricted cancellation.*

Proof. Since I is an ideal of $S = E_I$, for any $0 \neq r \in I$, $rS \subseteq I$. Thus if I', J and K are ideals of S with $I'J = I'K$ and $E_{I'} \subseteq E_J \cap E_K$, then $(rI')(rJ) = (rI')(rK)$ over R, implying $J = K$. $\square$

Lemma 6. *Let R be an integral domain and I a nonzero finitely generated ideal of R. Given any $P \in \max(R)$, there are only finitely many maximal ideals of E_I containing P.*

Proof. Assume that I can be generated by n elements over R and suppose to the contrary that there are maximal ideals $P_1, \dots, P_{n+1}$ of $S = E_I$ containing P. By Nakayama's lemma, $PI \neq I$, so that I/PI is a nonzero vector space over R/P of dimension at most n.

Set $J = \cap_j P_j$. By the Chinese remainder theorem, $I/JI \cong I/P_1 I \oplus \cdots \oplus I/P_{n+1}I$. But I is finitely generated over S so by Nakayama's lemma again, $P_j I \neq I$ for all j. This means that $I/P_j I$ is a nonzero R/P-module and implies that I/JI has dimension at least $n+1$. As

I/JI is an epimorphic image of I/PI, we are left with a contradiction. $\square$

The next observation undoubtedly appears elsewhere but as it follows directly from Lemmas 4 and 6 and is required below, a proof is included.

Corollary 7. *Let I be a finitely generated ideal of an integral domain R. Then I is stable if and only if for every $P \in \max(R)$, I_P is a principal ideal in $(E_I)_P$.*

Proof. By Lemma 4, $\mathrm{Hom}\,(I, E_I)_P = \mathrm{Hom}\,(I_P, (E_I)_P)$ for each $P \in \max(R)$. Therefore, I is invertible in E_I if and only if I_P is invertible in $(E_I)_P$ for each maximal ideal P. Now Lemma 6 shows that $(E_I)_P$ is semi-local for each $P \in \max(R)$, and a direct application of the Chinese remainder theorem shows that a projective ideal in a semi-local ring is principal. Therefore, I is projective over E_I if and only if I_P is a principal ideal in $(E_I)_P$ for each $P \in \max(R)$. $\square$

Lemma 8. *If R is an integral domain with restricted cancellation, then every ideal which is generated by two elements over its ring of endomorphisms is stable.*

Proof. Let I be an ideal 2-generated over its ring E_I of endomorphisms. By Lemma 5, E_I has restricted cancellation and we may assume without loss of generality that $R = E_I$. Let $P \in \max(R)$ and represent I as (a, b).

It is easy to see that the cancellation assumption on I is equivalent to the property that whenever $IJ \subseteq IK$ for ideals J and K, then $J \subseteq K$ [**8**]. Therefore, $(a, b)(ab) \subseteq (a, b)(a^2, b^2)$ implies $(ab) \subseteq (a^2, b^2)$. Write $ab = ca^2 + db^2$. Then $(a, b)(db) \subseteq (a, b)(a)$ and therefore $(db) \subseteq (a)$. Write $db = ua$. If $u \in P$, then $ab = ca^2 + uab$ implies $b = ca + ub$ and consequently that $b = c(1-u)^{-1}a$ in R_P. If $u \notin P$, then $a = u^{-1}db$ in R_P. In either case, I_P is principal. Since P was arbitrary and I is finitely generated, I is projective over E_I by Corollary 7. $\square$

Theorem 9. *In a domain with restricted cancellation, every ideal that is finitely generated over its ring of endomorphisms is stable.*

Proof. Let R be an integral domain with restricted cancellation, and let I be an ideal of R finitely generated over its ring of endomorphisms. We will show that I is stable by induction on n, the minimal number of generators required to generate I over E_I. Clearly we may pass to the inductive step and, by Lemma 8, assume that $n > 2$. In order to show that I is stable over E_I when I is n-generated over E_I, we may assume that $E_I = R$ since, by Lemma 5, E_I has restricted cancellation. By the induction hypothesis, there exist $x_1, x_2, \ldots, x_n \in I$ such that $I = Rx_1 + \cdots + Rx_n$, and any ideal of R generated by less than n elements is stable.

By Corollary 7 we need to show that I_P is principal in R_P for every maximal ideal P of R, so let us fix $P \in \max(R)$. The ideals $I_1 = Rx_2 + \cdots + Rx_n$ and $I_2 = R(x_2 - x_1) + \cdots + R(x_n - x_1)$ of R are stable. Set $S_j = E_{I_j}$. By Corollary 7, the localization of I_j at P, $(I_j)_P$, is a principal ideal of $(S_j)_P$. Write $(I_j)_P = (S_j)_P a_j$ for some $a_j \in I_j \subseteq I$. By device, $I_1 + I_2 = I$. From this it is easy to see that I is a module over $S_1 \cap S_2$ and, because $R = E_I$, we must have $R = S_1 \cap S_2$.

For each $i = 1, 2$, consider the ideals $J_i = Rx_1 + Ra_i \subseteq I$. By assumption, each J_i is stable, and therefore by Corollary 7, $(J_i)_P$ is a principal ideal of $(T_i)_P$ where $T_i = E_{J_i}$. Write $(J_i)_P = (T_i)_P b_i$ for some $b_i \in J_i \subseteq I$. Applying Lemma 3, $E_{S_i J_i} = S_i E_{J_i} = T_i S_i$. Recall that $Rx_1 + I_i = I$ for $i = 1, 2$. Since $S_i J_i = S_i x_1 + S_i a_i \subseteq S_i I$, if we localize at P, $(S_i J_i)_P = (S_i)_P x_1 + (S_i)_P a_i = (S_i)_P x_1 + (I_i)_P = (S_i I)_P$. By Lemma 3, again, $S_i = E_{S_i I}$ and so, by Lemma 4 and the remarks above, $(S_i)_P = E_{(S_i I)_P} = E_{(S_i J_i)_P} = (S_i)_P (E_{J_i})_P = (S_i)_P (T_i)_P$. This shows that $(T_i)_P \subseteq (S_i)_P$ for $i = 1, 2$. Also, from $J_i + I_i = I$, it follows that I is a module over $T_i \cap S_i$, so $T_i \cap S_i = R$. Assimilating these observations, $(T_i)_P = (T_i \cap S_i)_P = R_P$ for $i = 1, 2$.

Now consider the ideal $K = Rb_1 + Rb_2$. By Lemma 8, K is stable, so by Corollary 7 again, K_P is principal in S_P where $S = E_K$. Write $K_P = S_P c$ where c belongs to $K \subseteq I$. Recall that $(S_i J_i)_P = (S_i)_P I_P = (S_i)_P b_i$ because $(T_i)_P \subseteq (S_i)_P$. Then $(S_I)_P K_P = (S_i)_P I_P$ for $i = 1, 2$. Applying Lemma 3, $S_i S = E_{S_i K}$. Hence, using Lemma 4 as above, $(S_i)_P = (E_{S_i I})_P = E_{(S_i K)_P} = (S_i)_P (E_K)_P = (S_i)_P S_P$ for $i = 1, 2$.

From this we notice that $S_P \subseteq (S_1)_P \cap (S_2)_P = R_P$ and so $K_P = R_P c$. Finally, $K_P = R_P c \subseteq I_P \subseteq (S_1 I)_P \cap (S_2 I)_P = (S_1 K)_P \cap (S_2 K)_P = (S_1)_P c \cap (S_2)_P c = (S_1 \cap S_2)_P c = R_P c = K_P$, so I_P is principal. This induction argument shows that every ideal finitely generated over its endomorphism ring is stable. $\square$

Thus, if R is Noetherian with restricted cancellation, then R is stable. As mentioned at the start of the section, strongly faithful domains have restricted cancellation, and the observation that Noetherian stable domains are strongly faithful was made during the proof of Proposition 2. This completes the cycle in our main theorem. Using a result in [**1**], it is not hard to give a different proof of Theorem 9, and we would like to thank Professor Dan Anderson for providing a preprint of [**1**] subsequent to our development here.

2. A strongly faithful integral domain that is not stable. In this section we show that the main theorem does not generalize to arbitrary domains. In particular, we give an example of a strongly faithful integrally closed domain that is not stable. We state only the results necessary to justify our example, postponing a more systematic treatment of the integrally closed case until [**6**].

A Prüfer domain R is defined to be *strongly discrete* if, for each nonzero prime ideal P of R, $P \neq P^2$. An integral domain R is said to satisfy (#) provided

$$\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{N \in \Delta_2} R_N$$

whenever Δ_1 and Δ_2 are nonempty disjoint subsets of $\max(R)$. If every overring of R satisfies (#), then R is said to satisfy (##). A *generalized Dedekind domain* is a strongly discrete Prüfer domain satisfying (##) [**3**].

Proposition 10. *Generalized Dedekind domains are strongly faithful.*

Proof. Let R be a generalized Dedekind domain with quotient field Q. We first show that each submodule X of Q is locally free as an

E_X-module. Let X be a proper submodule of Q, $S = E_X$ and $M \in \max(S)$. Then $S_M = \mathrm{Hom}\,(X, \cap_{N\in\max(S)} X_N)_M = \mathrm{Hom}\,(X_M, X_M) \cap (\cap_{N\neq M} \mathrm{Hom}\,(X_N, X_N))_M$. Since S is a Prüfer domain, S_M is a valuation domain, which means the S_M-submodules of Q are linearly ordered. Therefore, either $\mathrm{Hom}\,(X, X)_M = \mathrm{Hom}\,(X_M, X_M)$ or $\mathrm{Hom}\,(X, X)_M = (\cap_{N\neq M} \mathrm{Hom}\,(X_N, X_M))_M$. If we assume the latter, then $\cap_{N\neq M} S_N \subseteq (\cap_{N\neq M} \mathrm{Hom}\,(X_N, X_N))_M = \mathrm{Hom}\,(X, X)_M = S_M$, which contradicts (##). It follows that $\mathrm{Hom}\,(X, X)_M = \mathrm{Hom}\,(X_M, X_M)$. Moreover, $X_M \neq Q$. Since S_M is a valuation domain, X_M is a fractional ideal of S_M. Strongly discrete valuation domains are stable [**3**, Proposition 5.3.8]. Therefore, $X_M \cong \mathrm{Hom}\,(X_M, X_M) = \mathrm{Hom}\,(X, X)_M$, proving the claim.

Assuming still that X is a proper submodule of Q, let G be a torsion-free R-module such that $S = E_X \subseteq E_G$. Viewing G as contained in $\mathrm{Hom}\,(X, XG)$ and $\mathrm{Hom}\,(X, XG)$ as contained in $\mathrm{Hom}\,(X, QG)$, we then have

$$\begin{aligned}\mathrm{Hom}\,(X, XG) &= \bigcap_{M\in\max(S)} \mathrm{Hom}\,(X_M, X_M G_M)\\ &= \bigcap_{M\in\max(S)} \mathrm{Hom}\,(S_M, S_M G_M)\\ &= \bigcap_{M\in\max(S)} G_M\\ &= G,\end{aligned}$$

as desired. □

Gabelli has given an example of a generalized Dedekind domain that is not stable [**4**]. In [**10**], it is shown that other examples can be constructed in the following way: Let R be a generalized Dedekind domain having infinitely many maximal ideals. If Q is the quotient field of R and $Q[X]$ is the polynomial ring of Q in X, then $R + XQ[X]$ is a generalized Dedekind domain [**3**, Corollary 5.7.3] that is not stable, since every nonzero ideal of an integrally closed stable domain must be contained in at most finitely many maximal ideals [**10**, Theorem 4.5]. In particular, $\mathbf{Z} + X\mathbf{Q}[X]$ is a strongly faithful domain that is not stable.

REFERENCES

1. D.D. Anderson and M. Roitman, *A characterization of cancellation ideals*, Proc. Amer. Math. Soc., to appear.

2. D.M. Arnold and E.L. Lady, *Endomorphism rings and direct sums of torsion-free abelian groups*, Trans. Amer. Math. Soc. **211** (1975), 225–236.

3. M. Fontana, J. Huckaba and I. Papick, *Prüfer domains*, Marcel Dekker, New York, 1997.

4. S. Gabelli, *A class of integral domains with nice divisorial ideals*, Proc. Second Intl. Conf. on Commutative Ring Theory, Fès 1995, Lecture Notes in Pure and Appl. Math., Dekker, 1995.

5. H. Pat Goeters, *Noetherian stable domains*, J. Algebra **202** (1998), 343–356.

6. H. Pat Goeters and Bruce Olberding, *On the multiplicative properties of submodules of the quotient field of an integral domain*, Houston J. Math., to appear.

7. Thomas W. Hungerford, *Algebra*, Graduate Texts in Math., **73**, Springer, New York, 1974.

8. Max D. Larsen and Paul J. McCarthy, *Multiplicative ideal theory*, Academic Press, New York, 1971.

9. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1980.

10. B. Olberding, *Globalizing local properties of Prüfer domains*, J. Algebra **205** (1998), 480–504.

11. Joseph J. Rotman, *An introduction to homological algebra*, Academic Press, New York, 1979.

12. J.D. Sally and W.V. Vasconcelos, *Stable rings*, J. Pure Appl. Algebra **4** (1974), 319–336.

13. R.B. Warfield, *Homomorphisms and duality for abelian groups*, Math. Z. **107** (1968), 189–212.

Department of Mathematics, Auburn University, Auburn, AL 36849-5310
E-mail address: `goetehp@mail.auburn.edu`

Department of Mathematics, Northeast Louisiana University, Monroe, LA 71209
E-mail address: `maolberding@alpha.nlu.edu`