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CHARACTERISTIC PAIRS ALONG THE RESOLUTION SEQUENCE

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ABSTRACT. Suppose that f is irreducible in a power series ring in two variables over an algebraically closed field k of characteristic 0. The characteristic pairs of f can be defined from a fractional power series expansion of a solution of f. The singularity of f can be resolved by a finite number of blow ups of points. This subject, which can be traced back to Newton, has been studied extensively. A few references are Abhyankar [**1**], Brieskorn and Knörrer [**2**], Campillo [**3**], Enriques and Chisini [**4**] and Zariski [**7**].

In Sections 1 and 2 we give an exposition of the basic results in the theory of Puiseux series. In Section 3 we give a formula for the characteristic pairs of the transform of f along the sequence of blow ups of points resolving the singularity. As a corollary, we obtain the classical theorem of Enriques and Chisini relating the multiplicity sequence of a resolution and the characteristic pairs of f, and we recover the classical result that the characteristic pairs are an invariant of f. We use an inversion formula of Abhyankar to obtain the results of this paper.

1. The Puiseux series. Let R be a power series ring in two variables over an algebraically closed field k. Then we have the following well-known theorem (see [**2**, pp. 405–406], [**7**, p. 7]).

Theorem 1.1. *Suppose that $f \in R$ is irreducible and (x, y) are regular parameters for R such that the multiplicity $\nu(f) = \nu(f(0, y))$. Then a fractional power series exists* (*called a* Puiseux series) *of y in terms of x. The expansion has the form*

$$y = \sum_{i=1}^{l_1} \alpha_{1,i}\, x^i + b_1\, x^{n_1/m_1}$$

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$$+\sum_{i=1}^{l_2}\alpha_{2,i}\,x^{((n_1+i)/m_1)}+b_2\,x^{(n_2/(m_1m_2))}$$
$$+\cdots \tag{1}$$
$$+\sum_{i=1}^{l_g}\alpha_{g,i}\,x^{((n_{g-1}+i)/(m_1\cdots m_{g-1}))}+b_g\,x^{(n_g/(m_1\cdots m_g))}$$
$$+\sum_{i=1}^{\infty}c_i\,x^{((n_g+i)/(m_1\cdots m_g))},$$

where

$$\begin{aligned} 1<\frac{n_1}{m_1}<\frac{n_2}{m_1m_2}<\cdots<\frac{n_g}{m_1\cdots m_g}, \\ m_j>1,\quad 1\le j\le g, \\ (n_j,m_j)=1,\quad 1\le j\le g, \\ b_j\ne 0,\quad 1\le j\le g, \\ l_j=\left[\frac{n_j-n_{j-1}m_j}{m_j}\right],\quad 1\le j\le g,\ n_0=0 \\ m=m_1m_2\cdots m_g=\nu(f), \end{aligned} \tag{2}$$

where $[t]$ *represents the greatest integer function. Note that the* $\alpha_{j,i}$ *and* c_i *can be* 0. *We define the* characteristic pairs *to be* (m_i,n_i), $1\le i\le g$.

We note that if the power series $p(x^{1/m})$ is the Puiseux series (1), it can be shown that a unit $\varphi\in R$ exists such that

$$f=\varphi\prod_{i=1}^{m}(y-p(\omega^i x^{(1/m)})) \tag{3}$$

where ω is a primitive mth root of unity.

2. An inversion theorem. Suppose that $h\in R$ is irreducible and (x,y) are regular parameters for R. Abhyankar [**1**] writes a fractional power series for y in terms of x in the form

$$\begin{aligned} x&=t^{\bar m} \\ y&=\sum_{j=1}^{\bar g-1}\sum_{i=0}^{s_j}\bar a_{j,i}\,t^{(\bar n_j+i)}(\bar m_{j+1}\cdots\bar m_{\bar g})+\sum_{i=1}^{\infty}\bar a_{\bar g,i}\,t^{\bar n_{\bar g}+i} \end{aligned} \tag{4}$$

where

$$
(5)\qquad
\begin{aligned}
&\bar{g}, \bar{n}_1, \dots, \bar{n}_{\bar{g}}, \bar{m}_1, \dots, \bar{m}_{\bar{g}} \in \mathbf{N}\\
&(\bar{m}_j, \bar{n}_j) = 1, \quad 1 \le j \le \bar{g}\\
&\frac{\bar{n}_{j-1}}{\bar{m}_1 \cdots \bar{m}_{j-1}} < \frac{\bar{n}_j}{\bar{m}_1 \cdots \bar{m}_j}, \quad 1 < j \le \bar{g}\\
&s_j = \left[\frac{\bar{n}_{j+1}}{\bar{m}_{j+1}} - \bar{n}_j\right], \quad 1 \le j < \bar{g}\\
&\bar{a}_{j,0} \neq 0, \quad 1 \le j \le \bar{g}\\
&\bar{m}_j > 1, \quad 1 < j \le \bar{g}\\
&\bar{m} = \bar{m}_1 \cdots \bar{m}_{\bar{g}}
\end{aligned}
$$

Substituting $t = x^{1/\bar{m}}$ in the expression for y, we get

$$
(6)\qquad y = \sum_{j=1}^{\bar{g}-1} \sum_{i=0}^{s_j} \bar{a}_{j,i}\, x^{((\bar{n}_j+i)/(\bar{m}_1\cdots\bar{m}_j))} + \sum_{i=1}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))}.
$$

When $(\bar{n}_1/\bar{m}_1) \ge 1$, it is possible to compare (6) with the expression (1) for the series. We obtain the following

Lemma 2.1. *If $\bar{m}_1 = 1$, then*

$$
\begin{aligned}
&m = \bar{m} = \bar{m}_2 \cdots \bar{m}_{\bar{g}}\\
&g = \bar{g} - 1\\
&m_j = \bar{m}_{j+1}, \quad 1 \le j \le g\\
&n_j = \bar{n}_{j+1}, \quad 1 \le j \le g\\
\Rightarrow\ &(m_j, n_j) = (\bar{m}_{j+1}, \bar{n}_{j+1}), \quad 1 \le j \le g\\
&l_j = s_j, \quad 1 \le j \le g\\
&b_j = a_{j+1,0}, \quad 1 \le j \le g.
\end{aligned}
$$

If $\bar{m}_1 > 1$, then

$$
\begin{aligned}
&m = \bar{m} = \bar{m}_1 \cdots \bar{m}_{\bar{g}}\\
&g = \bar{g}\\
&m_j = \bar{m}_j, \quad 1 \le j \le g
\end{aligned}
$$

$$\begin{aligned} n_j &= \bar{n}_j, \quad 1 \le j \le g \\ \Rightarrow (m_j, n_j) &= (\bar{m}_j, \bar{n}_j), \quad 1 \le j \le g \\ l_{j+1} &= s_j, \quad 1 \le j < g,\ l_1 = 0 \\ b_j &= a_{j,0}, \quad 1 \le j \le g. \end{aligned}$$

Proof. Case 1. If $\bar{m}_1 = 1$, then Abhyankar's series (6) is

$$\begin{aligned} \sum_{j=1}^{\bar{g}-1} \sum_{i=0}^{s_j} \bar{a}_{j,i}\, x^{((\bar{n}_j+i)/(\bar{m}_1\cdots\bar{m}_j))} &+ \sum_{i=0}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))} \\ &= \sum_{i=0}^{s_1} \bar{a}_{1,i}\, x^{((\bar{n}_1+i)/\bar{m}_1))} + \bar{a}_{2,0}\, x^{(\bar{n}_2/(\bar{m}_1\bar{m}_2))} \\ &\quad + \sum_{i=1}^{s_2} \bar{a}_{2,i}\, x^{((\bar{n}_2+i)/(\bar{m}_i\bar{m}_2))} + \bar{a}_{3,0}\, x^{(\bar{n}_3/(\bar{m}_1\bar{m}_2\bar{m}_3))} \\ &\quad + \cdots \\ &\quad + \sum_{i=1}^{s_{\bar{g}-1}} \bar{a}_{\bar{g}-1,i}\, x^{((\bar{n}_{\bar{g}-1}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}-1}))} + \bar{a}_{\bar{g},0}\, x^{(\bar{n}_{\bar{g}}/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))} \\ &\quad + \sum_{i=1}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))} \end{aligned}$$

which, since $\bar{m}_1 = 1$, we can write as

$$\begin{aligned} y = \sum_{i=0}^{s_1} \bar{a}_{1,i}\, x^{\bar{n}_1+i} &+ \bar{a}_{2,0}\, x^{(\bar{n}_2/\bar{m}_2)} \\ &+ \sum_{i=1}^{s_2} \bar{a}_{2,i}\, x^{((\bar{n}_2+i)/\bar{m}_2)} + \bar{a}_{3,0}\, x^{(\bar{n}_3/(\bar{m}_2\bar{m}_3))} \\ &+ \cdots \\ &+ \sum_{i=1}^{s_{\bar{g}-1}} \bar{a}_{\bar{g}-1,i}\, x^{((\bar{n}_{\bar{g}-1}+i)/(\bar{m}_2\cdots\bar{m}_{\bar{g}-1}))} + \bar{a}_{\bar{g},0}\, x^{(\bar{n}_{\bar{g}}/(\bar{m}_2\cdots\bar{m}_{\bar{g}}))} \\ &+ \sum_{i=1}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_2\cdots\bar{m}_{\bar{g}}))}, \end{aligned}$$

which is exactly the classical Puiseux series (1) with

$$\begin{aligned} m &= \bar{m} = \bar{m}_2 \cdots \bar{m}_{\bar{g}} \\ g &= \bar{g} - 1 \\ m_j &= \bar{m}_{j+1}, \quad 1 \le j \le g \\ n_j &= \bar{n}_{j+1}, \quad 1 \le j \le g \\ l_j &= s_j, \quad 1 \le j \le g \\ b_j &= \bar{a}_{j+1,0}, \quad 1 \le j \le g. \end{aligned}$$

Hence, $(m_j, n_j) = (\bar{m}_{j+1}, \bar{n}_{j+1})$ for all $1 \le j \le g$ as claimed.

Case 2. If $\bar{m}_1 > 1$, then Abhyankar's series (6) has the form

$$\begin{aligned} \sum_{j=1}^{\bar{g}-1} \sum_{i=0}^{s_j} \bar{a}_{j,i}\, x^{((\bar{n}_j+i)/(\bar{m}_1\cdots\bar{m}_j))} + \sum_{i=0}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_1\cdots\bar{m}_g))} \\ = \bar{a}_{1,0}\, x^{\bar{n}_1/\bar{m}_1} \\ + \sum_{i=1}^{s_1} \bar{a}_{1,i}\, x^{((\bar{n}_1+i)/\bar{m}_1)} + \bar{a}_{2,0}\, x^{(\bar{n}_2/(\bar{m}_1\bar{m}_2))} \\ + \sum_{i=1}^{s_2} \bar{a}_{2,i}\, x^{((\bar{n}_2+i)/(\bar{m}_1\bar{m}_2))} + \bar{a}_{3,0}\, x^{(\bar{n}_3/(\bar{m}_1\bar{m}_2\bar{m}_3))} \\ + \cdots \\ + \sum_{i=1}^{s_{\bar{g}-1}} \bar{a}_{\bar{g}-1,i}\, x^{((\bar{n}_{\bar{g}-1}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}-1}))} + \bar{a}_{\bar{g},0}\, x^{(\bar{n}_{\bar{g}}/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))} \\ + \sum_{i=1}^{\infty} \bar{a}_{\bar{g},i}\, x^{((\bar{n}_{\bar{g}}+i)/(\bar{m}_1\cdots\bar{m}_{\bar{g}}))}. \end{aligned}$$

This is the classical Puiseux series (1) with $\alpha_{1,i} = 0$ for $1 \le i \le l_1$. Comparing terms, we have

$$\begin{aligned} m &= \bar{m} = \bar{m}_1 \cdots \bar{m}_{\bar{g}} \\ g &= \bar{g} \\ m_j &= \bar{m}_j, \quad 1 \le j \le g \\ n_j &= \bar{n}_j, \quad 1 \le j \le g \\ l_{j+1} &= s_j, \quad 1 \le j < g, l_1 = 0 \\ b_j &= \bar{a}_{j,0}, \quad 1 \le j \le g, \end{aligned}$$

giving $(m_j, n_j) = (\bar{m}_j, \bar{n}_j)$ for all $1 \le j \le g$. $\square$

For a fractional power series of the form (6) we define $g(y, x) = g$, $m_j(y, x) = m_j$ and $n_j(y, x) = n_j$ for $1 \le j \le g$. Abhyankar [**1**, Theorem 1] proves the following inversion theorem.

Theorem 2.2 (Abhyankar). *Given a fractional power series of the form* (6)*, we can express the inversion of this series using* $g(x, y) = g$, $n_1(x, y) = m_1$, $m_1(x, y) = n_1$, $n_j(x, y) = n_j - (n_1 - m_1)m_2 \cdots m_j$ *for* $1 < j \le g$ *and* $m_j(x, y) = m_j$ *for* $1 < j \le g$.

3. The characteristic pairs. Let R be a power series ring in two variables over an algebraically closed field k of characteristic 0. A *quadratic transform* of R, $R \to R_1$, is defined as follows. Let (x, y) be regular parameters in R, $x = x_1$, $y = x_1 y_1$. Set $R_1 = k[[x_1, y_1]]$.

Suppose that $f \in R$ is irreducible of multiplicity $\nu(f) = r$ and $R \to R_1$ is a quadratic transform. Then $f = x_1^r f_1$ in R_1 where $x_1 \nmid f_1$. There is a unique quadratic transform $R \to R_1$ such that f_1 is not a unit in R_1. The multiplicity $\nu(f_1) \le r$. We call x_1 the *exceptional divisor* of $R \to R_1$, and we call f_1 the *strict transform* of f in R_1.

After a finite sequence of quadratic transforms, the strict transform of f becomes nonsingular (it has multiplicity 1).

There is a unique sequence of quadratic transforms

$$R \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_n \tag{7}$$

such that the strict transform of f in R_n has multiplicity 1 and $fR_n = (x_n^a y_n^b)$ where (x_n, y_n) are regular parameters in R_n (fR_n has *simple normal crossings*), and for $m < n$, fR_m does not have simple normal crossings. This is proved in [**2**] or [**6**] and will follow from Theorem 3.1. We will call (7) the *resolution sequence* of f.

Using the notation of (1), define $r_{1,1} = m$, $k_0 = 0$ and $k_j = n_j m_{j+1} \cdots m_g$ for $1 \le j \le g$. We consider the following chain of g

Euclidean algorithms:

$$
\begin{aligned}
k_j - k_{j-1} &= \mu_{j,1}\, r_{j,1} + r_{j,2} \\
r_{j,1} &= \mu_{j,2}\, r_{j,2} + r_{j,3} \\
&\ \vdots \\
r_{j,w(j)-1} &= \mu_{j,w(j)}\, r_{j,w(j)}
\end{aligned} \tag{8}
$$

where $1 \le j \le g$, with $0 \le r_{j,q+1} < r_{j,q}$, and we define $r_{j,1} = r_{j-1,w(j-1)}$ for $1 < j \le g$.

In (8) we have

1. $\gcd(k_j - k_{j-1}, r_{j,1}) = r_{j,w(j)} = m_{j+1} \cdots m_g$ for $1 \le j \le g$, note that $r_{g,w(g)} = 1$,
2. $\mu_{1,1} > 0$ but $\mu_{j,1}$ can be zero for $j > 1$,
3. $r_{j,2} > 0$ for all j.

As convention we use

$$\prod_{i=n+1}^{n} \beta_i = 1 \quad \text{and} \quad \sum_{i=1}^{0} \alpha_i = 0.$$

Theorem 3.1. *Let $f \in R$ be irreducible and $R \to R_1 \to R_2 \to \cdots \to R_n$ be the resolution sequence of f. Let f_k be the strict transform of f in R_k, let $g(k)$ be the genus of f_k and $(m_1(k), n_1(k)), \dots, (m_{g(k)}(k), n_{g(k)}(k))$ be the characteristic pairs of the Puiseux expansion of f_k. We have the following*

1. *If $(\sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}) \le k < (\sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}) + \mu_{j,1}$ with $1 \le j \le g$, set $l = k - \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}$. Then*

$$
\begin{aligned}
g(k) &= g - j + 1 \\
m_1(k) &= \frac{r_{j,1}}{m_{j+1} \cdots m_g} = m_j \\
m_i(k) &= m_{i+j-1}, \quad 1 < i \le g - j + 1 \\
n_i(k) &= \frac{k_{i+j-1} - k_{j-1} - l r_{j,1}}{m_{i+j} \cdots m_g}, \quad 1 \le i \le g - j + 1 \\
m(k) &= m_j \cdots m_g = r_{j,1}.
\end{aligned} \tag{9}
$$

2. *If* $\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{q-1}\mu_{j,i}\le k<\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{q}\mu_{j,i}$ *with* $1\le j\le g$, $2\le q\le w(j)-1$, *set* $l=k-\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}-\sum_{i=1}^{q-1}\mu_{j,i}$. *Then*

$$\begin{aligned} g(k) &= g-j+1\\ m_1(k) &= \frac{r_{j,q}}{m_{j+1}\cdots m_g}\\ m_i(k) &= m_{i+j-1}, \quad 1<i\le g-j+1\\ n_i(k) &= \frac{k_{i+j-1}-k_j+r_{j,q-1}-lr_{j,q}}{m_{i+j}\cdots m_g}, \quad 1\le i\le g-j+1\\ m(k) &= r_{j,q}. \end{aligned} \tag{10}$$

3. *If* $\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{w(j)-1}\mu_{j,i}\le k<\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{w(j)}\mu_{j,i}$ *with* $1\le j\le g-1$, *set* $l=k-\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}-\sum_{i=1}^{w(j)-1}\mu_{j,i}$. *Then*

$$\begin{aligned} g(k) &= g-j\\ m_1(k) &= \frac{r_{j,w(j)}}{m_{j+2}\cdots m_g}=m_{j+1}\\ m_i(k) &= m_{i+j}, \quad 1\le i\le g-j\\ n_i(k) &= \frac{k_{i+j}-k_j+r_{j,w(j)-1}-lr_{j,w(j)}}{m_{i+j+1}\cdots m_g}, \quad 1\le i\le g-j,\\ m(k) &= r_{j,w(j)}=m_{j+1}\cdots m_g. \end{aligned} \tag{11}$$

4. *If* $\sum_{h=1}^{g-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{w(g)-1}\mu_{j,i}\le k$, *then* $g(k)=0$.

We will find coordinates (x_k,y_k) in R_k such that f_k has a Puiseux series expansion $y_k(x_k^{(1/m(k))})$:

$$\begin{aligned} y_k = & \sum_{i=1}^{l_1(k)} \alpha_{1,i}(k)\, x_k^i + b_1(k)\, x_k^{(n_1(k)/m_1(k))} \\ & + \sum_{i=1}^{l_2(k)} \alpha_{2,i}(k)\, x_k^{((n_1(k)+i)/m_1(k))} + b_2(k)\, x_k^{(n_2(k)/(m_1(k)m_2(k)))} \\ & + \cdots \\ & + \sum_{i=1}^{l_{g(k)}(k)} \alpha_{g(k),i}(k)\, x_k^{((n_{g-1}(k)+i)/(m_1(k)\cdots m_{g(k)-1}(k)))} \\ & + b_{g(k)}(k)\, x_k^{(n_g(k)/(m_1(k)\cdots m_{g(k)}(k)))} \\ & + \sum_{i=1}^{\infty} c_i(k)\, x_k^{((n_g(k)+i)/(m_1(k)\cdots m_{g(k)}(k)))}. \end{aligned}$$

Before proving the theorem we state and prove

Lemma 3.2. *Suppose that f_k has the Puiseux expansion $y_k(x_k)$.*

1. *Suppose that $2 \leq (n_1(k)/m_1(k))$. Set $x_{k+1} = x_k$, $y_{k+1} = (y_k/x_k) - \alpha_{1,1}(k)$. Then f_{k+1} has the Puiseux expansion $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ with*

$$\begin{aligned} g(k+1) &= g(k), \\ m_i(k+1) &= m_i(k), \quad 1 \leq i \leq g(k), \\ n_i(k+1) &= n_i(k) - m_1(k)\cdots m_i(k), \quad 1 \leq i \leq g(k). \end{aligned} \tag{12}$$

2. *Suppose that $1 < (n_1(k)/m_1(k)) < 2$ and $n_1(k) - m_1(k) > 1$. (Note that this forces $l_1(k) = 0$.) Set $x_{k+1} = (y_k/x_k)$, $y_{k+1} = x_k$. Then f_{k+1} has the Puiseux expansion $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ with*

$$\begin{aligned} g(k+1) &= g(k), \\ m_1(k+1) &= n_1(k) - m_1(k), \\ m_i(k+1) &= m_i(k), \quad 1 < i \leq g(k), \\ n_1(k+1) &= m_1(k), \\ n_i(k+1) &= n_i(k) + m_1(k)\cdots m_i(k) - n_1(k)\, m_2(k)\cdots m_i(k), \\ & \qquad 1 < i \leq g(k+1). \end{aligned} \tag{13}$$

3. *Suppose that* $1 < (n_1(k)/m_1(k)) < 2$ *and that* $n_1(k) - m_1(k) = 1$. (*This also forces* $l_1(k) = 0$). *Then* f_{k+1} *has the Puiseux series expansion* $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ *with*

$$\begin{aligned} g(k+1) &= g(k) - 1, \\ m_i(k+1) &= m_{i+1}(k), \quad 1 \le i \le g(k+1), \\ n_i(k+1) &= n_{i+1}(k) + m_1(k) \cdots m_{i+1}(k) \\ &\quad - n_1(k) m_2(k) \cdots m_{i+1}(k). \end{aligned} \tag{14}$$

Proof. Suppose we are in Case 1. Then a blow up of f_k at 0 gives the fractional power series expansion for y_{k+1} as

$$\begin{aligned} y_{k+1} = &\sum_{i=2}^{l_1(k)} \alpha_{1,i}\,(k)\, x_{k+1}^{i} + b_1\,(k)\, x_{k+1}^{((n_1(k)-m_1(k))/m_1(k))} \\ &+ \sum_{i=1}^{l_2(k)} \alpha_{2,i}\,(k)\, x_{k+1}^{((n_1(k)-m_1(k)+i)/m_1(k))} \\ &+ b_2(k)\, x_{k+1}^{((n_2(k)-m_1(k)m_2(k))/(m_1(k)m_2(k)))} \\ &+ \cdots \\ &+ \sum_{i=1}^{l_{g(k)}(k)} \alpha_{g(k),i}\,(k)\, x_{k+1}^{((n_{g-1}(k)-m_1(k)\cdots m_{g(k)}(k)+i)/(m_1(k)\cdots m_{g(k)-1}(k)))} \\ &+ b_{g(k)}\,(k)\, x_{k+1}^{((n_g(k)-m_1(k)\cdots m_{g(k)}(k))/(m_1(k)\cdots m_{g(k)}(k)))} \\ &+ \sum_{i=1}^{\infty} c_i\,(k)\, x_{k+1}^{((n_g(k)-m_1(k)\cdots m_{g(k)}(k)+i)/(m_1(k)\cdots m_{g(k)}(k)))}. \end{aligned}$$

We will show that this $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ is a Puiseux series.

Recalling (3), we can write

$$f_k = \varphi_k \prod_{r=1}^{m(k)} (y_k - y_k(\omega^r x_k^{(1/m(k))}))$$

for some unit φ_k in R_k and where ω is a primitive $m(k)$th root of unity. Making the change of variables $x_{k+1} = x_k$, $y_{k+1} = (y_k/x_k) - \alpha_{1,1}(k)$

and solving the second of these for y_k, we get $y_k = x_k(y_{k+1} + \alpha_{1,1}(k))$. Substituting for y_k, we have

$$f_k = \varphi_k \prod_{r=1}^{m(k)} ((x_k(y_{k+1} + \alpha_{1,1}(k)) - x_k(y_{k+1}(\omega^r x_{k+1}^{(1/m(k))}) + \alpha_{1,1}(k)).$$

Then, substituting for x_k, we get

$$\begin{aligned} f_k &= \varphi_k \prod_{r=1}^{m(k)} ((x_{k+1}(y_{k+1} + \alpha_{1,1}(k)) \\ &\qquad - x_{k+1}(y_{k+1}(\omega^r x_{k+1}^{(1/m(k))}) + \alpha_{1,1}(k)))) \\ &= \varphi_k x_{k+1}^{m(k)} \prod_{r=1}^{m(k)} (y_{k+1} - y_{k+1}(\omega^r x_{k+1}^{(1/m(k))})). \end{aligned}$$

Thus, $f_{k+1} = \varphi_k \prod_{r=1}^{m(k)} (y_{k+1} - y_{k+1}(\omega^r x_{k+1}^{(1/m(k)}))$ is y-general of order $m(k)$ and $y_{k+1}(x_{k+1})$ is a fractional power series of f_{k+1}. Further, we note that

$$\frac{n_1(k) - m_1(k)}{m_1(k)} \geq 1$$

and

$$\begin{aligned} \frac{n_{i+1}(k) - m_1(k) \cdots m_{i+1}(k)}{m_1(k) \cdots m_{i+1}(k)} &= \frac{n_{i+1}(k)}{m_1(k) \cdots m_{i+1}(k)} - 1 \\ &\geq \frac{n_i(k)}{m_1(k) \cdots m_i(k)} - 1 \\ &= \frac{n_i(k) - m_1(k) \cdots m_i(k)}{m_1(k) \cdots m_i(k)} \end{aligned}$$

for all $1 \leq i \leq g-1$ and

$$(n_i(k) - m_1(k) \cdots m_i(k), m_i(k)) = (n_i(k), m_i(k)) = 1$$

for all i. Thus, (2) is satisfied and we see that $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ is a Puiseux series.

Now suppose we are in Case 2 or Case 3, then f_{k+1} has a fractional power series in the coordinates $\tilde{x}_{k+1} = x_k$, $\tilde{y}_{k+1} = (y_k/x_k)$:

$$\begin{aligned}
&\tilde{y}_{k+1}\\
&= b_1(k)\,\tilde{x}_{k+1}^{((n_1(k)-m_1(k))/m_1(k))}\\
&\quad + \sum_{i=1}^{l_2(k)} \alpha_{2,i}(k)\,\tilde{x}_{k+1}^{((n_1(k)-m_1(k)+i)/m_1(k))}\\
&\quad + b_2(k)\,\tilde{x}_{k+1}^{((n_2(k)-m_1(k)m_2(k)+i)/(m_1(k)m_2(k)))}\\
&\quad + \cdots\\
&\quad + \sum_{i=1}^{l_{g(k)}(k)} \alpha_{g(k),i}(k)\,\tilde{x}_{k+1}^{((n_{g(k)-1}(k)-m_1(k)\cdots m_{g(k)-1}(k)+i)/(m_1(k)\cdots m_{g(k)-1}))}\\
&\quad + b_{g(k)}(k)\,\tilde{x}_{k+1}^{((n_{g(k)}(k)-m_1(k)\cdots m_{g(k)}(k))/(m_1(k)\cdots m_{g(k)}(k)))}\\
&\quad + \sum_{i=1}^{\infty} c_i\,\tilde{x}_{k+1}^{((n_{g(k)}(k)-m_1(k)\cdots m_{g(k)}(k)+i)/(m_1(k)\cdots m_{g(k)}(k)))}.
\end{aligned}$$

In the notation of (5) we have, for the series $\tilde{y}_{k+1}(\tilde{x}_{k+1}^{(1/m(k))})$, $\tilde{m}_i = m_i(k)$, $1 \le i \le g(k)$ and $\tilde{n}_i = n_i(k) - m_1(k)\cdots m_i(k)$, $1 \le i \le g(k)$. By hypothesis, this expansion has $(\tilde{n}_1/\tilde{m}_1) = ((n_1(k) - m_1(k))/m_1(k)) < 1$, so we must perform the inversion $x_{k+1} = \tilde{y}_{k+1}$, $y_{k+1} = \tilde{x}_{k+1}$ to construct the Puiseux series.

There are two possibilities. Firstly, suppose that $n_1(k) - m_1(k) > 1$. Then we are in Case 1. By Abhyankar's inversion theorem $y_{k+1}(x_{k+1}^{(1/m(k))})$ is a Puiseux series with

$$\begin{aligned}
g(k+1) &= g(k)\\
m_1(k+1) &= n_1(k) - m_1(k)\\
m_i(k+1) &= m_i(k) \qquad (15)\\
n_1(k+1) &= m_1(k)\\
n_i(k+1) &= n_i(k) + m_1(k)\cdots m_i(k) - n_1(k)m_2(k)\cdots m_i(k),\\
&\qquad 1 < i \le g(k+1).
\end{aligned}$$

Let f_{k+1} be the strict transform of f_k in $R_k \cong k[[x_{k+1}, y_{k+1}]]$. We have

$$R_{k+1}/(f_{k+1}) \hookrightarrow T \cong k[[t]]$$

where in T the relations (15) and (4) hold. Set

$$\Lambda = \prod_{r=1}^{m(k+1)} (y_{k+1} - y_{k+1}(\omega^r x_{k+1}^{(1/m(k+1))}))$$

where ω is a primitive root $m(k+1)$th root of unity. Now $\Lambda = 0$ in T and $\Lambda \in R_{k+1}$. We note that Λ is irreducible in R_{k+1} since its irreducible factors in $R_{k+1}[x_{k+1}^{(1/m(k+1))}]$ are the terms $y_{k+1} - y_{k+1}(\omega^r x_{k+1}^{(1/m(k+1))})$, and the only product of these terms which is invariant under the action $x^{(1/m(k+1))} \mapsto \omega x^{(1/m(k+1))}$ is Λ.

Hence, $(\Lambda) = (f_{k+1})$, f_{k+1} is y-general of order $m(k+1)$ and $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ is a Puiseux expansion of f_{k+1}.

Secondly, suppose that $n_1(k) - m_1(k) = 1$. Then we are in Case 2. By Abhyankar's inversion theorem, $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ is a Puiseux series with

$$\begin{aligned} g(k+1) &= g(k) - 1 \\ m_i(k+1) &= m_{i+1}(k) \quad 1 \le i \le g(k+1) \\ n_i(k+1) &= n_{i+1}(k) + m_1(k) \cdots m_{i+1}(k) - n_1(k) m_2(k) \cdots m_{i+1}(k) \\ &\qquad 1 \le i \le g(k+1). \end{aligned}$$

As in the previous case, $y_{k+1}(x_{k+1}^{(1/m(k+1))})$ is a Puiseux expansion of f_{k+1}, the strict transform of f_k in R_{k+1}. □

We now offer an inductive proof of Theorem 3.1.

Proof. Suppose that $k = 0$. We have $\mu_{1,1} > 0$, so we are in Case 1 with $j = 1$, $l = 0$, $g(0) = g$, $m_1(0) = (r_{1,1}/(m_2 \cdots m_g)) = m_1$, $m_i(0) = m_i$ for $1 < i \le g-j+1$, $n_i(k) = (k_i/(m_{i+1} \cdots m_g)) = n_i$ for $1 \le i \le g-j+1$ in agreement with the formula.

Now suppose that the theorem is true for $k = n$. We will verify the theorem for $k = n+1$. There are six cases to consider:

C1 $$\sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i} \le n = \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i} + \mu_{j,1} - 1 \quad 1 \le j \le g$$

C2 $$\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} \le n < \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \mu_{j,1} - 1 \quad 1 \le j \le g$$

C3 $$\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{q-1}\mu_{j,i} \le n = \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{q}\mu_{j,i} - 1$$
$$1 \le j \le g, \quad 2 \le q \le w(j) - 1$$

C4 $$\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{q-1}\mu_{j,i} \le n < \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{q}\mu_{j,i} - 1$$
$$1 \le j \le g, \quad 2 \le q \le w(j) - 1$$

C5 $$\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{w(j)-1}\mu_{j,i} \le n = \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{w(j)}\mu_{j,i} - 1$$
$$1 \le j \le g - 1$$

C6 $$\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{w(j)-1}\mu_{j,i} \le n < \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \sum_{i=1}^{w(j)}\mu_{j,i} - 1.$$
$$1 \le j \le g - 1$$

Suppose we are in Case C1. Then $n + 1 = \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} + \mu_{j,1}$. There are three subcases to consider.

C1.1. $2 < w(j)$. Then $n + 1$ is in Case 2 of the statement of the theorem (with $q = 2$ and $l = 0$).

C1.2. $2 = w(j)$, $j \le g - 1$. Then $n + 1$ is in Case 3 of the statement of the theorem (with $l = 0$).

C1.3. $2 = w(j)$, $j = g$. Then $n + 1$ is in Case 4 of the statement of the theorem.

We begin with subcase C1.1. Here n is in Case 1 of the statement of the theorem. So, by the inductive hypothesis and (9) we have

$$\begin{aligned}\frac{n_1(n)}{m_1(n)} &= \frac{k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1}}{m_{j+1}\cdots m_g} \cdot \frac{m_{j+1}\cdots m_g}{r_{j,1}} \\ &= \frac{k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1}}{r_{j,1}}\end{aligned}$$

$$= \frac{r_{j,1} + (k_j - k_{j-1} - \mu_{j,1}\, r_{j,1})}{r_{j,1}}.$$

Applying the identities from the Euclidean algorithms (8) we get

$$\frac{n_1(n)}{m_1(n)} = \frac{r_{j,1} + r_{j,2}}{r_{j,1}} < 2$$

since $r_{j,2} < r_{j,1}$. Similarly,

$$\begin{aligned} n_1(n) - m_1(n) &= \frac{k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1} - r_{j,1}}{m_{j+1} \cdots m_g} \\ &= \frac{k_j - k_{j-1} - \mu_{j,1} r_{j,1}}{m_{j+1} \cdots m_g} \\ &= \frac{r_{j,2}}{r_{j,w(j)}} \\ &> 1, \end{aligned}$$

since $r_{j,w(j)} \mid r_{j,2}$ and $2 < w(j)$. We are thus in Case 2 of Lemma 3.2. So we have, after applying (13) and (9),

$$\begin{aligned} g(n{+}1) &= g(n) = g - j + 1 \\ m_1(n{+}1) &= n_1(n) - m_1(n) \\ &= \frac{k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1} - r_{j,1}}{m_{j+1} \cdots m_g} \\ &= \frac{r_{j,2}}{m_{j+1} \cdots m_g} \\ m_i(n{+}1) &= m_i(n) = m_{i+j-1}, \quad 1 < i \le g - j + 1 \\ n_1(n{+}1) &= m_1(n) = \frac{r_{j,1}}{m_{j+1} \cdots m_g} \\ n_i(n{+}1) &= n_i(n) + m_1(n) \cdots m_i(n) - n_1(n) m_2(n) \cdots m_i(n) \\ &= \frac{(k_{i+j-1} - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1}) + r_{j,1} - (k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1})}{m_{i+j} \cdots m_g} \\ &= \frac{k_{i+j-1} - k_{j-1} + r_{j,1}}{m_{i+j} \cdots m_g}, \quad 1 < i \le g - j + 1, \end{aligned}$$

which is in agreement with the conclusion of the theorem.

Now consider subcase C1.2. Here we have n in Case 1 of the statement of the theorem. Therefore,

$$\begin{aligned}\frac{n_1(n)}{m_1(n)} &= \frac{k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1}}{r_{j,1}} \\ &= \frac{r_{j,1} + r_{j,2}}{r_{j,1}} \\ &< 2\end{aligned}$$

and

$$n_1(n) - m_1(n) = \frac{r_{j,2}}{r_{j,w(j)}} = 1,$$

so we are in Case 3 of Lemma 3.2. Accordingly, we apply (14) and (9) to get

$$\begin{aligned}g(n+1) &= g(n) - 1 = g - j \\ m_i(n+1) &= m_{i+1}(n) = m_{i+j}, \quad 1 \le i \le g - j \\ n_i(n+1) &= n_{i+1}(n) + m_i(n) \cdots m_{i+1}(n) - n_1(n) m_2(n) \cdots m_{i+1}(n) \\ &= \frac{(k_{i+j} - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1}) + r_{j,1} - (k_j - k_{j-1} - (\mu_{j,1} - 1)\, r_{j,1})}{m_{i+j} \cdots m_g} \\ &= \frac{k_{i+j} - k_j + r_{j,1}}{m_{i+j} \cdots m_g}, \quad 1 \le i \le g - j,\end{aligned}$$

again, in agreement with the conclusion of the theorem.

Next we consider subcase C1.3. In this case n is again in Case 1 of the statement of the theorem so we can apply (9) to get

$$\frac{n_1(n)}{m_1(n)} = \frac{r_{j,1} + r_{j,2}}{r_{j,1}} < 2$$

and

$$n_1(n) - m_1(n) = \frac{r_{j,2}}{r_{j,w(j)}} = 1$$

which implies we are again in Case 3 of Lemma 3.2. Therefore, (14) and (9) imply $g(n+1) = g(n) - 1 = (g - j + 1) - 1 = 1 - 1 = 0$ as claimed in the theorem.

Suppose we have Case C2. Then $n+1$ is in Case 1 of the statement of the theorem. This puts n in Case 1 of the statement of the theorem. Thus, by (9), we have

$$\begin{aligned}\frac{n_1(n)}{m_1(n)} &= \frac{k_j - k_{j-1} - lr_{j,1}}{m_{j+1}\cdots m_g} \cdot \frac{m_{j+1}\cdots m_g}{r_{j,1}} \\ &= \frac{\mu_{j,1} r_{j,1} - \left(n - \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}\right) r_{j,1}}{r_{j,1}} \\ &\geq \frac{2r_{j,1}}{r_{j,1}} = 2\end{aligned}$$

which is in Case 1 of Lemma 3.2. Applying (12) and (9) we will have

$$\begin{aligned}g(n+1) &= g(n) = g - j + 1 \\ m_1(n+1) &= m_1(n) = \frac{r_{j,1}}{m_{j+1}\cdots m_g} \\ m_i(n+1) &= m_i(n) = m_{i+j-1}, \quad 1 < i \leq g - j + 1 \\ n_i(n+1) &= n_i(n) - m_1(n)\cdots m_i(n) \\ &= \frac{k_{i+j-1} - k_{j-1} - \left(n - \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}\right) r_{j,1} - r_{j,1}}{m_{i+j}\cdots m_g} \\ &= \frac{k_{i+j-1} - k_{j-1} - \left((n+1) - \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i}\right) r_{j,1}}{m_{i+j}\cdots m_g} \\ &= \frac{k_{i+j-1} - k_{j-1} - lr_{j,1}}{m_{i+j}\cdots m_g}\end{aligned}$$

as desired.

Now we consider case C3. Here we have

$$n+1 = \sum_{h=1}^{j-1} \sum_{i=1}^{w(h)} \mu_{h,i} + \sum_{i=1}^{q(n)} \mu_{j,i}$$

with $1 \leq j \leq g$ and $2 \leq q(n) \leq w(j) - 1$. There are three subcases.

C3.1. $q(n) + 1 \leq w(j) - 1$. Then $n+1$ is in Case 2 of Theorem 3.1 with $q(n+1) = q(n) + 1$ and $l = 0$.

C3.2. $q(n) = w(j) - 1$, $j \leq g - 1$. Then $n+1$ is in Case 3 of Theorem 3.1 with $l = 0$.

C3.3. $q(n) = w(j) - 1$, $j = g$.

Suppose we are in subcase C3.1. Then we have that n is in Case 2 of the statement of the theorem, so by (10):

$$\begin{aligned}\frac{n_1(n)}{m_1(n)} &= \frac{r_{j,q(n)-1} - (\mu_{j,q(n)} - 1)\, r_{j,q(n)}}{m_{j+1}\cdots m_g} \cdot \frac{m_{j+1}\cdots m_g}{r_{j,q(n)}} \\ &= \frac{r_{j,q(n)} + r_{j,q(n)+1}}{r_{j,q(n)}} \\ &< 2\end{aligned}$$

and

$$n_1(n) - m_1(n) = \frac{r_{j,q(n)+1}}{m_{j+1}\cdots m_g} = \frac{r_{j,q(n)+1}}{r_{j,w(j)}} > 1$$

since $r_{j,w(j)} \mid r_{j,q(n)+1}$ and $w(j) > q(n) + 1$. This places us in Case 2 of Lemma 3.2. We use (13) and (10) to conclude

$$\begin{aligned}g(n+1) &= g(n) = g - j + 1 \\ m_1(n+1) &= n_1(n) - m_1(n) \\ &= \frac{r_{j,q(n)-1} - (\mu_{j,q(n)} - 1)\, r_{j,q(n)} - r_{j,q(n)}}{m_{j+1}\cdots m_g} \\ &= \frac{r_{j,q(n)+1}}{m_{j+1}\cdots m_g} \\ &= \frac{r_{j,q(n+1)}}{m_{j+1}\cdots m_g} \\ m_i(n+1) &= m_i(n) = m_{i+j-1}, \quad 1 < i \leq g - j + 1 \\ n_1(n+1) &= m_1(n) = \frac{r_{j,q(n)}}{m_{j+1}\cdots m_g} \\ &= \frac{r_{j,q(n)+1} - 1}{m_{j+1}\cdots m_g} \\ n_i(n+1) &= n_i(n) + m_1(n)\cdots m_i(n) - n_1(n) m_2(n)\cdots m_i(n) \\ &= \frac{k_{i+j-1} - k_j + r_{j,q(n)-1} - (\mu_{j,q(n)} - 1)\, r_{j,q(n)}}{m_{j+1}\cdots m_g} \\ &\quad + \frac{r_{j,q(n)} - (r_{j,q(n)-1} - (\mu_{j,q(n)} - 1)\, r_{j,q(n)})}{m_{j+1}\cdots m_g}\end{aligned}$$

$$= \frac{k_{i+j-1} - k_j + r_{j,q(n)}}{m_{j+1} \cdots m_g}$$
$$= \frac{k_{i+j-1} - k_j + r_{j,q(n+1)-1}}{m_{j+1} \cdots m_g}, \quad 1 < i \le g - j + 1,$$

as desired.

Now consider subcase C3.2. In this case we have that n is in Case 2 of the statement of the theorem. From (10) we get:

$$1 < \frac{n_1(n)}{m_1(n)} = \frac{r_{j,q} + r_{j,q+1}}{r_{j,q}} < 2$$

as in C3.1 above. Also

$$n_1(n) - m_1(n) = \frac{r_{j,q(n)+1}}{r_{j,w(j)}} = 1$$

since $q(n) + 1 = w(j)$. This places us in Case 3 of Lemma 3.2. Hence we invoke (14) and (10) to write

$$\begin{aligned}
g(n+1) &= g(n) - 1 = g - j \\
m_i(n+1) &= m_{i+1}(n) = m_{i+j}, \quad 1 \le i \le g - j \\
n_i(n+1) &= n_{i+1}(n) + m_1(n) \cdots m_{i+1}(n) \\
&\quad - n_1(n) m_2(n) \cdots m_{i+1}(n) \\
&= \frac{k_{i+j} - k_j + r_{j,q-1} - (\mu_{j,q} - 1)\, r_{j,q}}{m_{i+j} \cdots m_g} \\
&\quad + \frac{r_{j,q} - (r_{j,q-1} - (\mu_{j,q} - 1)\, r_{j,q})}{m_{i+j} \cdots m_g} \\
&= \frac{k_{i+j} - k_j + r_{j,q}}{m_{i+j} \cdots m_g} \\
&= \frac{k_{i+j} - k_j + r_{j,w(j)-1}}{m_{i+j} \cdots m_g}
\end{aligned}$$

as claimed in the theorem.

In subcase C3.3, $n + 1$ and n are each in Case 2 of the statement of the theorem. By (10), we have

$$\begin{aligned}
\frac{n_1(n)}{m_1(n)} &= \frac{r_{j,q-1} - (\mu_{j,q} - 1)\, r_{j,q}}{m_{j+1} \cdots m_g} \cdot \frac{m_{j+1} \cdots m_g}{r_{j,q}} \\
&= \frac{r_{j,q} + r_{j,q+1}}{r_{j,q}} \\
&< 2
\end{aligned}$$

and

$$n_1(n) - m_1(n) = \frac{r_{j,q(n)+1}}{r_{j,w(j)}} = 1.$$

Again we are in Case 3 of Lemma 3.2 so (14) and (10) imply $g(n+1) = g(n) - 1 = 1 - 1 = 0$.

If we are in Case C4 we see that n is in Case 2 of the statement of the theorem. Thus, (10) yields

$$\begin{aligned}
\frac{n_1(n)}{m_1(n)} &= \frac{r_{j,q-1} - \left(n - \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} - \sum_{i=1}^{q-1}\mu_{j,i}\right) r_{j,q}}{m_{i+j}\cdots m_g} \cdot \frac{m_{i+j}\cdots m_g}{r_{j,q}} \\
&= \frac{\mu_{j,q}r_{j,q} - \left(n - \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} - \sum_{i=1}^{q-1}\mu_{j,i}\right) r_{j,q}}{r_{j,q}} \\
&\geq 2,
\end{aligned}$$

and we are in Case 1 of Lemma 3.2. Whence, by (12) and (10),

$$\begin{aligned}
g(n+1) &= g(n) = g - j + 1 \\
m_1(n+1) &= m_1(n) = \frac{r_{j,q}}{m_{j+1}\cdots m_g} \\
m_i(n+1) &= m_i(n) = m_{i+j-1}, \quad 1 < i \leq g - j + 1 \\
n_i(n+1) &= n_i(n) - m_1(n)\cdots m_i(n), \quad 1 \leq i \leq g - j + 1 \\
&= \frac{k_{i+j-1} - k_j + r_{j,q-1}}{m_{i+j}\cdots m_g} \\
&\quad + \frac{-\left(n - \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} - \sum_{i=1}^{q-1}\mu_{j,i}\right) r_{j,q} - r_{j,q}}{m_{i+j}\cdots m_g} \\
&= \frac{k_{i+j-1} - k_j + r_{j,q-1}}{m_{i+j}\cdots m_g} \\
&\quad - \frac{\left((n+1) - \sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i} - \sum_{i=1}^{q-1}\mu_{j,i}\right) r_{j,q}}{m_{i+j}\cdots m_g} \\
&= \frac{k_{i+j-1} - k_j + r_{j,q-1} - lr_{j,q}}{m_{i+j}\cdots m_g}
\end{aligned}$$

in agreement with the statement of the theorem.

Turning to Case C5 we see that

$$n+1=\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}+\sum_{i=1}^{w(j)}\mu_{j,i}=\sum_{h=1}^{j}\sum_{i=1}^{w(h)}\mu_{h,i}$$

when $1\leq j\leq g-1$. This generates the following subcases.

C5.1. $\mu_{j+1,1}>0$. Here $n+1$ is in Case 1 of the statement of the theorem with $l=0$ and $j(n+1)=j(n)+1$.

C5.2. $\mu_{j+1,1}=0$ and $2<w(j+1)$. Then $n+1$ is in Case 2 of the statement of the theorem with $l=0$, $j(n+1)=j(n)+1$ and $q(n+1)=2$.

C5.3. $\mu_{j+1,1}=0$, $w(j+1)=2$ and $j+1\leq g-1$. Then $n+1$ is in Case 3 of the statement of the theorem and we have $l=0$ and $j(n+1)=j(n)+1$.

C5.4. $\mu_{j+1,1}=0$, $w(j+1)=2$ and $j+1=g$. In this case $n+1$ is in Case 4 of the statement of the theorem.

In subcase C5.1 we have that n is in Case 3 of the statement of the theorem. Using (11), we compute

$$\begin{aligned}\frac{n_1(n)}{m_1(n)}&=\frac{k_{j+1}-k_j+r_{j,w(j)-1}-(\mu_{j,w(j)}-1)\,r_{j,w(j)}}{r_{j,w(j)}}\\&=\frac{k_{j+1}-k_j+r_{j,w(j)}}{r_{j,w(j)}}\end{aligned}$$

since $r_{j,w(j)-1}-\mu_{j,w(j)}\,r_{j,w(j)}=0$. Now, by the Euclidean algorithms (8) and the fact that $r_{j,w(j)}=r_{j+1,1}$, we can write this as

$$\begin{aligned}\frac{n_1(n)}{m_1(n)}&=\frac{\mu_{j+1,1}\,r_{j+1,1}+r_{j+1,2}+r_{j+1,1}}{r_{j+1,1}}\\&=\frac{(\mu_{j+1,1}+1)\,r_{j+1,1}+r_{j+1,2}}{r_{j+1,1}}\\&\geq 2,\end{aligned}$$

and we are in Case 1 of Lemma 3.2. Thus, by (12) and (11),

$$g(n+1)=g(n)=g-j(n)=g-j(n+1)+1$$

$$
\begin{aligned}
m_1(n+1) &= m_1(n) = m_{j(n)+1} = m_{j(n+1)} \\
m_i(n+1) &= m_i(n) = m_{i+j(n)} = m_{i+j(n+1)-1} \quad 1 < i \leq g - j(n{+}1) + 1 \\
n_i(n+1) &= n_i(n) - m_1(n) \cdots m_i(n) \\
&= \frac{k_{i+j} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)} - r_{j,w(j)}}{m_{i+j+1} \cdots m_g} \\
&= \frac{k_{i+j} - k_j}{m_{i+j+1} \cdots m_g} \\
&= \frac{k_{i+j(n+1)-1} - k_j(n+1) - 1}{m_{i+j(n+1)} \cdots m_g} \quad 1 \leq i \leq g - j(n{+}1) + 1,
\end{aligned}
$$

as claimed in the theorem.

In the event that we are in subcase C5.2, we would have that n is in Case 3 of the statement of the theorem. Hence, by (11):

$$
\begin{aligned}
\frac{n_1(n)}{m_1(n)} &= \frac{k_{j+1} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)}}{r_{j,w(j)}} \\
&= \frac{(k_{j+1} - k_j) + (r_{j,w(j)-1} - \mu_{j,w(j)}\, r_{j,w(j)}) + r_{j,w(j)}}{r_{j,w(j)}} \\
&= \frac{(\mu_{j+1,1} r_{j+1,1} + r_{j+1,2}) + (0) + r_{j+1,1}}{r_{j,w(j)}} \\
&= \frac{r_{j+1,1} + r_{j+1,2}}{r_{j+1,1}} \\
&< 2
\end{aligned}
$$

and

$$
\begin{aligned}
n_1(n) - m_1(n) &= \frac{(k_{j+1} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)}) - r_{j,w(j)}}{m_{j+2} \cdots m_g} \\
&= \frac{(r_{j+1,1} + r_{j+1,2}) - r_{j,w(j)}}{m_{j+2} \cdots m_g} \\
&= \frac{r_{j+1,2}}{m_{j+2} \cdots m_g} \\
&> 1
\end{aligned}
$$

since $2 < w(j+1)$. This places us in Case 2 of Lemma 3.2. We then apply (13) and (11) to get

$$
g(n+1) = g(n) = g - j(n) = g - j(n+1) + 1
$$

$$\begin{aligned}
m_1(n+1) &= n_1(n) - m_1(n) \\
&= \frac{k_{i+j} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)} - r_{j,w(j)}}{m_{j+1} \cdots m_g} \\
&= \frac{r_{j+1,2}}{m_{j+2} \cdots m_g} \\
&= \frac{r_{j(n+1),q(n+1)}}{m_{j(n+1)+1} \cdots m_g} \\
m_i(n+1) &= m_i(n) = m_{i+j(n)} = m_{i+j(n+1)-1} \quad 1 < i \leq g - j(n+1) + 1 \\
n_1(n+1) &= m_1(n) = \frac{r_{j,w(j)}}{m_{j+1} \cdots m_g} \\
&= \frac{r_{j+1,1}}{m_{j+2} \cdots m_g} = \frac{r_{j(n+1),\, q(n+1)-1}}{m_{j(n+1)+1} \cdots m_g} \\
n_i(n+1) &= n_i(n) + m_1(n) \cdots m_i(n) - n_1(n)\, m_2(n) \cdots m_i(n) \\
&= \frac{k_{i+j} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)} + r_{j,w(j)}}{m_{i+j+1} \cdots m_g} \\
&\quad - \frac{k_{j+1} - k_j + r_{j,w(j)-1} - (\mu_{j,w(j)} - 1)\, r_{j,w(j)}}{m_{i+j+1} \cdots m_g} \\
&= \frac{k_{i+j} - k_{j+1} + r_{j,w(j)}}{m_{i+j+1} \cdots m_g} \\
&= \frac{k_{i+j(n+1)-1} - k_{j(n+1)} + r_{j(n+1),\, q(n+1)-1}}{m_{i+j(n+1)} \cdots m_g} \\
&\quad 1 < i \leq g - j(n+1) + 1
\end{aligned}$$

in agreement with the conclusion of the theorem.

Subcase C5.3 puts both n and $n+1$ into Case 3 of the statement of the theorem. So, by (11), we have

$$\frac{n_1(n)}{m_1(n)} = \frac{r_{j+1,1} + r_{j+1,2}}{r_{j+1,1}} < 2$$

which we computed in C5.2 above. But now,

$$n_1(n) - m_1(n) = \frac{r_{j+1,2}}{m_{j+2} \cdots m_g} = 1$$

since $w(j+1)=2$. We are thus in Case 3 of Lemma 3.2 which, via (14) and (11), gives

$$\begin{aligned}
g(n+1) &= g(n)-1 = g-j-1 = g-j(n+1)\\
m_i(n+1) &= m_{i+1}(n) = m_{i+j+1} = m_{i+j(n+1)}, \quad 1\le i\le g-j(n{+}1)\\
n_i(n+1) &= n_{i+1}(n){+}m_1(n)\cdots m_{i+1}(n){-}n_1(n)\, m_2(n)\cdots m_{i+1}(n)\\
&= \frac{k_{i+j+1}-k_j+r_{j,w(j)-1}-(\mu_{j,w(j)}-1)\, r_{j,w(j)}+r_{j,w(j)}}{m_{i+j+2}\cdots m_g}\\
&\quad - \frac{k_{i+j}-k_j+r_{j,w(j)-1}-(\mu_{j,w(j)}-1)\, r_{j,w(j)}}{m_{i+j+2}\cdots m_g}\\
&= \frac{k_{i+j+1}-k_{j+1}+r_{j,w(j)}}{m_{i+j+2}\cdots m_g}\\
&= \frac{k_{i+j(n+1)}-k_{j(n+1)}+r_{j(n+1),w(j((n+1))-1)}}{m_{i+j(n+1)+1}\cdots m_g}\\
&\quad 1\le i\le g-j(n+1),
\end{aligned}$$

as desired.

If we are in subcase C5.4, we note that n is in Case 3 of the statement of the theorem. Thus, as in C5.3, we have

$$\frac{n_1(n)}{m_1(n)} = \frac{r_{j+1,1}+r_{j+1,2}}{r_{j+1,1}} = \frac{r_{g,1}+r_{g,2}}{r_{g,1}} < 2$$

and

$$n_1(n)-m_1(n) = \frac{r_{j+1,2}}{m_{j+2}\cdots m_g}.$$

Now by our convention

$$\prod_{n=i}^{i-1}\beta_n = 1.$$

Using this, the fact that $j+1=g$, $w(g)=2 \Rightarrow r_{j+1,2}=r_{g,w(g)}$, and recalling from (8) that $r_{g,w(g)}=1$, we get

$$n_1(n)-m_1(n) = r_{g,w(g)} = 1,$$

which implies that we are in Case 3 of Lemma 3.2 and we have from (14) and (11) that $g(n+1)=g(n)-1=g-j-1=g-(j+1)=0$ as claimed.

Lastly, suppose we are in Case C6. Then n and $n+1$ are each in part 3 of the statement of the theorem. So by (11) we have

$$\begin{aligned}
\frac{n_1(n)}{m_1(n)} &= \frac{k_{j+1}-k_j+r_{j,w(j)-1}}{m_{j+2}\cdots m_g} \\
&\quad - \frac{\left(n-\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}-\sum_{i=1}^{w(j)-1}\mu_{j,i}\right) r_{j,w(j)}}{m_{j+2}\cdots m_g}\cdot\frac{m_{j+2}\cdots m_g}{r_{j,w(j)}} \\
&\geq \frac{k_{j+1}-k_j+r_{j,w(j)-1}-(\mu_{j,w(j)}-2)\,r_{j,w(j)}}{r_{j,w(j)}} \\
&= \frac{k_{j+1}-k_j+\mu_{j,w(j)}r_{j,w(j)}-(\mu_{j,w(j)}-2)\,r_{j,w(j)}}{r_{j,w(j)}} \\
&= \frac{k_{j+1}-k_j+2\,r_{j,w(j)}}{r_{j,w(j)}} \\
&\geq 2,
\end{aligned}$$

which places us in Case 1 of Lemma 3.2. By (12) and (11) we have

$$\begin{aligned}
g(n+1) &= g(n) = g-j \\
m_i(n+1) &= m_i(n) = m_{i+j}, \quad 1\leq i\leq g-j \\
n_i(n+1) &= n_i(n) - m_1(n)\cdots m_i(n) \\
&= \frac{k_{i+j}-k_j+r_{j,w(j)-1}-\left(n-\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}\right.}{m_{i+j+1}\cdots m_g} \\
&\quad + \frac{\left.-\sum_{i=1}^{w(j)-1}\mu_{j,i}\right) r_{j,w(j)} - r_{j,w(j)}}{m_{i+j+1}\cdots m_g} \\
&= \frac{k_{i+j}-k_j+r_{j,w(j)-1}}{m_{i+j+1}\cdots m_g} \\
&\quad \frac{-\left((n+1)-\sum_{h=1}^{j-1}\sum_{i=1}^{w(h)}\mu_{h,i}-\sum_{i=1}^{w(j)-1}\mu_{j,i}\right) r_{j,w(j)}}{m_{i+j+1}\cdots m_g} \\
&= \frac{k_{i+j}-k_j+r_{j,w(j)-1}-l\,r_{j,w(j)}}{m_{i+j+1}\cdots m_g}, \quad 1\leq i\leq g-j,
\end{aligned}$$

in accord with the theorem.

Thus, we've shown the theorem true by induction. □

Let

$$R \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_n$$

be the resolution sequence of an irreducible $f \in R$. Let $\nu_i = \nu(f_i)$ where f_i is the strict transform of f in R_i and $\nu_0 = \nu(f)$. We define the *multiplicity sequence* of f to be the sequence $(\nu_0, \nu_1, \dots, \nu_{n-1})$.

A classical theorem of Enriques and Chisini [**4**] follows from Theorem 3.1 since $\nu_k = m(k)$ for all k.

Corollary 3.3 (Enriques-Chisini). *Let the notation be as in the statement of Theorem* 3.1.

1. *The multiplicity sequence of f is completely determined by the characteristic pairs of f. In fact, the multiplicity sequence is determined by the chain of Euclidean algorithms* (8). *In the multiplicity sequence, the multiplicity $r_{i,j}$ appears $\mu_{i,j}$ times where $i = 1, \dots, g$; $j = 1, \dots, w(i)$, i.e., the multiplicity sequence is*

$$\underbrace{r_{1,1}, \dots, r_{1,1}}_{\mu_{1,1}} \underbrace{r_{1,2}, \dots, r_{1,2}}_{\mu_{1,2}}, \dots, \underbrace{r_{1,w(1)}, \dots, r_{1,w(1)}}_{\mu_{1,w(1)}}, \underbrace{r_{2,1}, \dots, r_{2,1}}_{\mu_{2,1}}, \dots$$

2. *Conversely, one can reconstruct the characteristic pairs of a Puiseux expansion of f from the multiplicity sequence by the chain of Euclidean algorithms* (8).

An immediate consequence of this corollary is the fact that the characteristic pairs are an invariant of f.

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