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WEAK UNITS IN EPICOMPLETIONS OF ARCHIMEDEAN LATTICE-ORDERED GROUPS

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ABSTRACT. Let κ denote an infinite cardinal number or the symbol ∞. In [**14**] it is shown that in the category of archimedean lattice-ordered groups with l-group homomorphisms, each G has an epicompletion, $\beta^{\kappa}_{\mathcal{A}}(G)$, in which G is κ-completely embedded and which lifts all κ-complete morphisms out of G to epicomplete objects. Since, in general, these epicomplete objects have no concrete realization and since epicomplete objects in the category of archimedean lattice-ordered groups with distinguished weak order unit and unit preserving l-group homomorphisms do have concrete realizations [**4**], it is natural to ask when $\beta^{\kappa}_{\mathcal{A}}(G)$ has a weak unit. In this paper we show that $\beta^{\kappa}_{\mathcal{A}}(G)$ has a weak unit precisely when there is countable $A \subseteq G$ so that the κ-ideal generated by A in G is all of G. Moreover, we show that $\beta^{\omega_0}_{\mathcal{A}}(G)$ has weak unit precisely when every epicompletion of G has weak unit, and we construct an archimedean l-group G with weak unit for which $\beta^{\kappa}_{\mathcal{A}}(G)$ has a weak unit for each κ, $\omega_1 \leq \kappa \leq \infty$, but $\beta^{\omega_0}_{\mathcal{A}}(G)$ has no weak unit.

1. Introduction. Let Arch denote the category of archimedean lattice-ordered groups (l-groups) with l-homomorphisms (that is, homomorphisms which preserve both the group and lattice structure). We note that Arch is closed under products and subobjects. All terms regarding l-groups are standard and can be found in [**8**]. Moreover, some general references for l-groups are [**8**], [**18**], [**2**] and [**12**].

Let A be a subset of an l-group G. The *l-ideal generated by A in G* is the smallest l-ideal of G which contains A and is denoted by (A) or $(A)_G$, and $(\{a\})$ is abbreviated to (a) or $(a)_G$. The set $\{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is denoted by $A^{\perp G}$ or $A^{\perp}$, if the context is clear, and is called the *polar* of A. If A consists of one element a, then one writes $a^{\perp G}$ or $a^{\perp}$ for $\{a\}^{\perp}$. Furthermore, an element $0 \leq u \in G$ is called a *weak unit* if $u^{\perp G} = (0)$.

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In a concrete category $\mathcal{C}$, a morphism $\varepsilon : G \to H$ is an *epimorphism* (epic) if whenever $\alpha, \beta : H \to K$ have $\alpha \circ \varepsilon = \beta \circ \varepsilon$, then $\alpha = \beta$. An object G is *epicomplete* if $\varepsilon : G \to H$ epic and one-to-one implies ε is an isomorphism (i.e., onto). An *epicompletion* of G is a one-to-one epic $\varepsilon : G \to H$ with H epicomplete.

A one-to-one morphism $i : G \to H$ in Arch is called *extremal monic* if whenever $i = \tau \circ \nu$ with ν epic, then ν is an isomorphism. We also say that G is an *extremal subobject* of H.

In Arch, and the related category $\mathcal{W}$, of archimedean l-groups with distinguished weak order unit and unit preserving l-homomorphisms, Ball and Hager have characterized the epimorphisms and the epicomplete objects [**3**], [**4**], [**5**]. Moreover, they have studied $\mathcal{W}$-epicompletions and have shown that these objects can be realized as $D(X)$ for some basically disconnected, compact, Hausdorff space X, where $D(X)$ denotes the set of continuous, extended real-valued functions (with point-wise order) whose domain of reality is dense in X, or as a quotient of real-valued Baire measurable functions modulo some σ-ideal of null functions [**5**]. We build upon their work by focusing on epicomplete objects in Arch and proving precisely when these objects have weak order unit.

There are three motivating or background results which we now state.

Theorem 1.0 [**4**, 4.9]. *The following are equivalent for G in* Arch:

(1) *G is* Arch *epicomplete.*

(2) *G is an extremal subobject of $D(Y)$ for some basically disconnected, compact, Hausdorff space Y.*

(3) *G is divisible and both conditionally and laterally σ-complete.* (*An l-group G is* conditionally σ-complete (*respectively,* laterally σ-complete) *if each countable set in G* (*respectively, countable set in G^+*) *which is bounded above* (*respectively, pairwise disjoint*) *has a supremum.*)

Throughout this paper we shall let κ denote an infinite cardinal number or the symbol ∞. By $|A| < \kappa$, for $k = \infty$, we mean that there is no restriction on the cardinality of A.

Definition 1.1. An l-homomorphism $\phi : G \to H$ is called κ-*complete* if whenever $A \subseteq G$, $|A| < \kappa$, and $\vee A$ exists in G, then $\phi(\vee^G A) = \vee^H \phi(A)$.

An ∞-complete (respectively, ω_1-complete) l-homomorphism is usually called *complete* (respectively, σ-*complete*).

Theorem 1.2 [**14**, 1.4, 2.5]. *In* Arch*, given an object G, there is an epicompletion $\beta^\kappa_{\mathcal{A}}(G)$ of G and a κ-complete embedding $\varepsilon : G \to \beta^\kappa_{\mathcal{A}}(G)$ with this universal mapping property*: *If $\theta : G \to H$ is a κ-complete morphism with H epicomplete, then a unique morphism $\tau : \beta^\kappa_{\mathcal{A}}(G) \to H$ exists with $\tau \circ \varepsilon = \theta$. If θ is epic, then τ is a surjection. Moreover, $\beta^\kappa_{\mathcal{A}}(G)$ is unique up to isomorphism over G; that is, if (ε', K) is another epicompletion of G, in which G is κ-completely embedded and which has the aforementioned universal mapping property, then there is an isomorphism $h : \beta^\kappa_{\mathcal{A}}(G) \to K$ with $h \circ \varepsilon = \varepsilon'$.*

The construction provided of $\beta^\kappa_{\mathcal{A}}(G)$ in [**14**, 1.4], however, does not give a concrete realization of this epicomplete object. Since Arch epicomplete objects with a weak unit are epicomplete in $\mathcal{W}$ and $\mathcal{W}$-epicomplete objects have concrete realizations [**4**, 3.9, 4.9], which make them easier to work with, it is natural to ask when any epicompletion H of an archimedean l-group G has a weak unit.

We provide a general answer to this question in Section 3, but the result is not very useful for it depends on the epicompletion H. However, if we restrict to epicompletions of the form $\beta^\kappa_{\mathcal{A}}(G)$, we have a nice answer to this question which can be stated completely in terms of G. The analysis relies heavily on archimedean kernels and κ-ideals.

Before we discuss archimedean kernels, we first present the main result of [**6**] which gives a partial answer to our main question.

Theorem 1.3 [**6**, 2.1]. *In the category of archimedean l-groups, the following are equivalent about G*:

(1) *There is H with a weak unit and an epic embedding $G \leq H$.*

(2) *There is H with a weak unit and a coessential embedding $G \leq H$. (In* Arch*, an embedding $G \leq H$ is called* coessential *if whenever a*

morphism $\alpha : H \to K$ *has* $\alpha|_G$ *identically* 0*, then* α *is identically* 0*.*)

(3) *There is countable* $A \subseteq G$ *with* $A^{\perp G} = (0)$*.*

(4) *There is epicomplete* H *with a weak unit and an epic and essential embedding* $G \leq H$*.* (*An embedding* $G \leq H$ *is called* essential *if every nontrivial* l*-ideal of* H *intersects* G *nontrivially.*)

We note that the "H" in statement (4) of Theorem 1.3 is $\beta^{\infty}_{\mathcal{A}}(G)$, by the uniqueness of this object [**14**, 7.4]. Hence, $\beta^{\infty}_{\mathcal{A}}(G)$ has a weak unit precisely when there is a countable subset $A \subseteq G$ with $A^{\perp G} = (0)$ or, equivalently, when $A^{\perp\perp G} = G$.

Definition 1.4. An l-ideal I in an archimedean l-group H is called an *archimedean kernel* if H/I is archimedean or, equivalently, if I is the kernel of an l-homomorphism in Arch.

The following discussion of archimedean kernels comes from [**6**].

Given a subset A of an archimedean l-group H, there is a smallest archimedean kernel containing A, denoted $ak_H(A)$, because Arch is closed under the formation of products and sub-l-groups. However, since this "outside-in" construction is not very useful in understanding archimedean kernels, we present an "inside-out" construction due to [**17**]. (See [**17**, 3.3].)

In an l-group H, a sequence of elements $\{h_n : n \in N\}$ *converges relatively uniformly* to h, denoted $h_n \to h\,$r.u., if there is u, called the *regulator*, for which: if $k \in N$, there is n_0 such that $n \geq n_0$ implies $k|h_n - h| \leq u$. For $A \subseteq H$, let $A' = \{h \in H \mid \text{there is } \{h_n : n \in N\} \text{ in } A \text{ with } h_n \to h \text{ relatively uniformly}\}$.

Then $A = ak_H(A)$ if and only if A is an l-ideal with $A = A'$. (This appears in [**18**, pp. 85, 427] for vector lattices, and the proof appears to use just divisibility of H, but even that is easily eliminated.) It then follows that for general $A \subseteq H$, $ak_H(A) = \cup_{\alpha<\omega_1} A_\alpha$, where A_0 is the l-ideal generated by A in H, $A_{\alpha+1} = A'_\alpha$, and for a limit ordinal β, $A_\beta = \cup_{\alpha<\beta} A_\alpha$.

We now note that an embedding $G \leq H$ is coessential if $ak_H(G) = H$.

Definition 1.5. A κ*-ideal* K in an archimedean l-group G is an archimedean kernel with the property that whenever $A \subseteq K$, $|A| < \kappa$, and $\vee A$ exists in G, then $\vee A$ belongs to K.

An ∞-ideal (respectively, ω_1-ideal) is usually called a *closed ideal* or *complete ideal* (respectively, *σ-ideal*), in the literature. In an archimedean l-group, the complete ideals are precisely the polars. Actually this condition characterizes the archimedean l-groups [**8**, 11.1.10].

We note that if K is an ω_1-ideal in an archimedean l-group G, then G/K is archimedean [**8**, 11.1.9]. Hence, every κ-ideal is an archimedean kernel.

Proposition 1.6. [**7**, 3.2] *In* Arch*, κ-ideals are precisely the kernels of κ-complete l-homomorphisms.* (*This actually holds in the more general context of l-groups.*)

Definition 1.7. An embedding $G \leq H$ in Arch is called *κ-coessential* if whenever $\alpha : H \to K$ is κ-complete and $\alpha|_G = 0$, then $\alpha = 0$.

Now since Arch is closed under products and sub-l-groups, given any subset A of an object H in Arch and any κ, there is a smallest κ-ideal in H containing A, which we shall denote by $\kappa_H(A)$, or also by $ak_H(A)$ if $\kappa = \omega_0$ and $A^{\perp\perp H}$ if $\kappa = \infty$.

Then note that $G \leq H$ is κ-coessential if and only if $\kappa_H(G) = H$. For $\kappa = \omega_0$, ω_0-coessential is what we have been calling coessential (since an ω_0-complete l-homomorphism is simply an l-homomorphism).

Definition 1.8. An l-group G is called *conditionally κ-complete* if whenever $A \subseteq G$, $|A| < \kappa$, and A is bounded above in G, then $\vee A$ exists in G.

A conditionally ∞-complete (respectively, conditionally ω_1-complete) l-group is usually called *conditionally complete* (respectively, *conditionally σ-complete*).

2. Morphisms which preserve weak order unit. This section is preliminary to the main results in Section 3. Throughout this section, G is an archimedean l-group and A is a subset of G.

Proposition 2.0. *Let $\phi : G \to H$ be κ-complete, and let $A \subseteq G$. Then $\phi(\kappa_G(A)) \subseteq \kappa_H(\phi(A))$.*

Proof. Since $\kappa_H(\phi(A))$ is a κ-ideal in H, $\kappa_H(\phi(A)) = \ker(\psi)$ for some κ-complete morphism ψ out of H in Arch (Proposition 1.6). Then $\ker(\psi \circ \phi)$ is a κ-ideal of G which contains A. Hence, $\kappa_G(A) \subseteq \ker(\psi \circ \phi)$; that is, $\phi(\kappa_G(A)) \subseteq \ker(\psi) = \kappa_H(\phi(A))$.

Corollary 2.1. *Let $\phi : G \to H$ be a κ-complete surjection. If there is $A \subseteq G$ with $\kappa_G(A) = G$, then $\kappa_H(\phi(A)) = H$.*

Proof. By Proposition 2.0, we have that $\phi(\kappa_G(A)) \subseteq \kappa_H(\phi(A))$. The hypothesis then gives that $H = \phi(G) = \phi(\kappa_G(A)) \subseteq \kappa_H(\phi(A)) \subseteq H$.

Proposition 2.2. *Let $\omega_1 \leq \kappa \leq \infty$. If $A \subseteq G$ and $|A| < \kappa$, then $\kappa_G(A) = A^{\perp\perp G}$.*

Proof. Let $A \subseteq G$, $|A| < \kappa$, and suppose $0 \leq g \in A^{\perp\perp G}$. Then by [**8**, 11.1.6], we have that $g = \vee\{g \wedge n|a| : n \in N, a \in A\}$. It then follows that $g \in \kappa_G(A)$ since $\kappa_G(A)$ is a κ-ideal which contains $g \wedge n|a|$ for each $n \in N$, $a \in A$.

This improves [**18**, 24.7] which has $|A| = 1$, $\kappa = \omega_1$ and also [**7**, 3.3] for any κ.

Corollary 2.3. *Let $\phi : G \to H$ be a κ-complete surjection for any $\kappa > \omega_0$. If there is $A \subseteq G$, $|A| < \kappa$ with $A^{\perp\perp G} = G$, then $\phi(A)^{\perp\perp H} = H$.*

Proof. By Proposition 2.2, we have that $\kappa_G(A) = A^{\perp\perp G}$ and $\kappa_H(\phi(A)) = \phi(A)^{\perp\perp H}$. Now apply Corollary 2.1.

In particular, Corollary 2.3 shows that surjective σ-complete morphisms preserve weak order units. Moreover, σ-complete cannot be dropped in this hypothesis.

Example. Consider $\phi : C[0,1] \to R$ defined by $\phi(f) = f(0)$ for every $f \in C[0,1]$. Let $e(x) = x$ for each $x \in [0,1]$. Then e is a weak unit in $C[0,1]$ and ϕ is a surjection, but $\phi(e) = e(0) = 0$ is not a weak unit in R.

Definition 2.4. An l-group G has the *principal projection property*, denoted ppp, if for every $u \in G$, $G = u^{\perp\perp} \oplus u^{\perp}$. (One also says, "$G$ is projectable.") Here $\oplus$ denotes internal direct sum.

We note that every conditionally σ-complete l-group has ppp and is archimedean [**8**, 11.2.3].

We now consider κ-coessential embeddings and the role of weak order units.

Proposition 2.5. *For any κ, suppose $G \leq H$ is a κ-coessential κ-complete embedding and H is laterally σ-complete and has ppp. If there is countable $A \subseteq G$ with $\kappa_G(A) = G$, then $\kappa_H(A) = H$ so that H has a weak unit.*

Lemma 2.6 [**6**, 2.3(c)]. *If an l-group H has* ppp *and A is a countable subset of H, then there is countable, pairwise disjoint B with $B^{\perp H} = A^{\perp H}$.*

Proof of Proposition 2.5. Since $\kappa_G(A) = G$, we have $G \subseteq \kappa_H(A)$ by Proposition 2.0. Now, since $G \leq H$ is κ-coessential, the only κ-ideal in H that contains G is H itself, so that $\kappa_H(A) = H$. Hence, $A^{\perp\perp H} = H$, and thus $A^{\perp H} = (0)$. Since H has ppp, there is countable pairwise disjoint B with $B^{\perp H} = (0)$ by Lemma 2.6. Then since H is laterally σ-complete, $(\vee B)$ exists and is a weak unit in H.

Corollary 2.7. *Suppose $G \leq H$ is a coessential embedding and H is laterally σ-complete with* ppp. *If there is countable $A \subseteq G$ with $ak_G(A) = G$, then $ak_H(A) = H$, so that H has a weak unit.*

Proof. Take $\kappa = \omega_0$ in Proposition 2.5.

Corollary 2.8. *Let $G \leq H$ be a coessential embedding. If there is*

$u \in G$ with $ak_G(u) = G$, then $ak_H(u) = H$, so that u is a weak unit in H.

Proof. Since $ak_G(u) = G$, we have $G \subseteq ak_H(u)$ by Proposition 2.0. Now $G \leq H$ is coessential, so the only archimedean kernel in H that contains G is H. Hence, $ak_H(u) = H$, and thus, $u^{\perp\perp} = H$.

Corollary 2.8 shows that, in particular, if u is a *strong unit* in G (an element u in G with the property that $(u)_G = G$) and $G \leq H$ is coessential, then u is a weak unit in H.

3. When $\beta^{\kappa}_{\mathcal{A}}(G)$ has weak order unit. In this section, we work in the category Arch. Let $G \leq H$ be a coessential embedding, and let H be laterally σ-complete with ppp. The next proposition provides an answer to the general question: When does H have a weak unit?

Proposition 3.0. *Let $G \leq H$ be a coessential embedding, and let H be laterally σ-complete with* ppp. *The following statements are equivalent*:

(1) *H has a weak unit.*

(2) *There is countable $A \subseteq G$ with $\kappa_H(A) = H$ for any uncountable cardinal number κ.*

(3) *There is countable $A \subseteq G$ with $A^{\perp\perp} = H$.*

If H is actually an epicompletion of G, then the following statement is equivalent to those above:

(4) *There is countable $A \subseteq G$ with $ak_H(A) = H$.*

Proof. To show that (1) implies (2) and (1) implies (4), we first prove the following lemma.

Lemma 3.1. *In* Arch, *$G \leq H$ is coessential if and only if for every $h \in H$, there is countable $A \subseteq G$ with $h \in ak_H(A)$.*

Proof. Suppose $G \leq H$ is coessential. Then $H = ak_H(G) =$

$\cup_{\alpha<\omega_1} G_\alpha$. We proceed by induction on α.

If $h \in G_0$, then there is $a \in G$ with $|h| \leq a$ and clearly $h \in ak_H(a)$.

Let $h \in G_\alpha$ and suppose the conclusion holds for all elements of G_β, for each $\beta < \alpha$. If α is a limit ordinal, then there is nothing to prove. If $\alpha = \beta + 1$, then there is $\{h_n : n \in N\} \subseteq G_\beta$ and $v \in H$ with $h_n \to h$ r.u. with regulator v. Since $h_n \in G_\beta$ for each n, there is countable $A_n \subseteq G$ with $h_n \in ak_H(A_n)$. Let $A = \cup_n A_n$. Then A is countable and $h_n \in ak_H(A_n) \subseteq ak_H(A)$ for all n. Since $h_n \to h$ r.u., then $h \in \{h_n : n \in N\}'^H \subseteq ak_H(\cup_n h_n)$, and since $\cup_n h_n \subseteq ak_H(A)$, then $ak_H(\cup_n h_n) \subseteq ak_H(A)$. Hence, $h \in ak_H(A)$. Invoking induction concludes the proof.

The converse is trivial.

(1) implies (2). Suppose u is a weak unit in H. Since $G \leq H$ is coessential, there is countable $A \subseteq G$ with $u \in ak_H(A) \subseteq \kappa_H(A)$ by Lemma 3.1. Then $\kappa_H(u) = u^{\perp\perp H} = H$, by Proposition 2.2, since u is a weak unit; whence, $\kappa_H(A) = H$.

(2) implies (3) and (4) implies (2) since $ak_H(A) \subseteq \kappa_H(A) \subseteq A^{\perp\perp H}$ for any uncountable cardinal κ.

To show that (3) implies (1), take countable $A \subseteq G$ with $A^{\perp\perp H} = H$. Since H has ppp, disjointify A, by Lemma 2.6. Then there is countable, pairwise disjoint $B \subseteq H$ with $A^{\perp H} = B^{\perp H} = (0)$. Since H is laterally σ-complete, $\vee B$ exists in H and is a weak unit.

Now suppose H is actually an epicompletion of G.

Then (1) implies (4). Let H be an Arch epicompletion of G with weak unit u. By Lemma 3.1, there is countable $A \subseteq G$ with $u \in ak_H(A)$ since the embedding of G in H is coessential. Since H has weak unit u, H is $\mathcal{W}$-epicomplete and, hence, $H \cong D(X)$ for some basically disconnected, compact, Hausdorff space X, with $u \mapsto 1$ [**4**, 3.9]. Hence H is an *f-ring* [**8**, 9.1.2], with identity a weak unit. By [**3**, 8.5.5], [**10**, p. 62], if H is an archimedean f-ring with identity u a weak unit, then $(u)'^H = ak_H(u) = H$. It then follows that $ak_H(A) = H$.

One cannot conclude that statement (4) is equivalent to the others if H is not epicomplete, since $ak_H(u)$ need not be all of H.

Upon replacing the generic epicompletion H in Proposition 3.0 by the specific one, $\beta^\kappa_{\mathcal{A}}(G)$, the criteria for having a weak unit can be refined considerably to involve only G.

Theorem 3.2. *Let G and κ be given. The following statements are equivalent*:

(1) *$\beta^{\kappa}_{\mathcal{A}}(G)$ has a weak unit.*

(2) *There is countable $A \subseteq G$ with $\kappa_G(A) = G$.*

(3) *If $G \leq H$ is a κ-coessential, κ-complete embedding and H is laterally σ-complete, with* ppp, *then H has a weak unit.*

(4) *If $G \leq H$ is a κ-coessential, κ-complete embedding and H is epicomplete, then H has a weak unit.*

(5) *If H is an epicompletion of G in which G is κ-completely embedded, then H has a weak unit.*

(6) *If $\phi : G \to H$ is a κ-complete, epic morphism with H epicomplete, then H has a weak unit.*

Proof. (2) implies (3) follows from Proposition 2.5.

(3) implies (4) is clear, since epicomplete H is always laterally σ-complete and has ppp, by Theorem 1.0 and [**8**, 11.2.3].

(4) implies (5) is obvious; (5) implies (1) and (6) implies (1) are also clear, since $\beta^{\kappa}_{\mathcal{A}}(G)$ is an epicompletion of G in which G is κ-completely embedded.

(1) implies (6). Suppose $\phi : G \to H$ is a κ-complete, epic morphism with H epicomplete. Then by the properties of $\beta^{\kappa}_{\mathcal{A}}(G)$, there is a surjection $\tau : \beta^{\kappa}_{\mathcal{A}}(G) \to H$ (Theorem 1.2) and since all morphisms out of epicomplete objects are σ-complete [**14**, 5.1], τ is σ-complete. Now, if e is a weak unit in $\beta^{\kappa}_{\mathcal{A}}(G)$, it follows by Corollary 2.3 that $\tau(e)$ is a weak unit in H.

It remains now only to show that (1) implies (2).

Suppose $\beta^{\kappa}_{\mathcal{A}}(G)$ has weak unit. Then, by Proposition 3.0, there is countable $A \subseteq G$ with $ak_{\beta^{\kappa}_{\mathcal{A}}(G)}(A) = \beta^{\kappa}_{\mathcal{A}}(G)$. Now let $\langle A \rangle$ denote the sub-l-group of G generated by A. We shall show that $\kappa_G(\langle A \rangle) = G$ (i.e., $\langle A \rangle \leq G$ is κ-coessential) from which it follows that $\kappa_G(A) = G$. Toward that end, let $\alpha : G \to H$ be a κ-complete morphism in Arch with $\alpha|_{\langle A \rangle} = 0$, and let $\varepsilon : H \hookrightarrow \beta^{\infty}_{\mathcal{A}}(H)$ be the embedding of H into the epicompletion $\beta_{\mathcal{A}}^{\infty}(H)$. Then $\varepsilon \circ \alpha : G \to \beta_{\mathcal{A}}^{\infty}(H)$ is a κ-complete morphism out of G to epicomplete $\beta_{\mathcal{A}}^{\infty}(H)$. By the universal mapping property of $\beta_{\mathcal{A}}^{\kappa}(G)$, there is $\tau : \beta_{\mathcal{A}}^{\kappa}(G) \to \beta_{\mathcal{A}}^{\infty}(H)$ so that $\tau|_G = \varepsilon \circ \alpha$. Now $\langle A \rangle \leq \beta_{\mathcal{A}}^{\kappa}(G)$ is coessential (i.e., $ak_{\beta_{\mathcal{A}}^{\kappa}(G)}(\langle A \rangle) = \beta_{\mathcal{A}}^{\kappa}(G)$ since $ak_{\beta_{\mathcal{A}}^{\kappa}(G)}(A) = \beta_{\mathcal{A}}^{\kappa}(G)$) and $\tau|_{\langle A \rangle} = (\varepsilon \circ \alpha)|_{\langle A \rangle} = 0$, so that $\tau = 0$. Hence, $\tau|_G = \varepsilon \circ \alpha = 0$ and since ε is one-to-one, we have that $\alpha = 0$. Therefore, $\kappa_G(\langle A \rangle) = G$; whence, $\kappa_G(A) = G$.

The proof of Theorem 3.2 is now complete. For the interested reader, we provide a proof of (1) implies (4).

(1) implies (4). Suppose $G \leq H$ is a κ-coessential, κ-complete embedding and H is epicomplete. Factor $G \leq H$ into $G \leq K \leq H$, so that $G \leq K$ is epic and $K \leq H$ is extremal monic [**14**, 2.1]. Since K is an extremal subobject of epicomplete H, K is epicomplete (Theorem 1.0) and since $G \leq H$ is κ-complete and $K \leq H$ is one-to-one, $G \leq K$ is κ-complete [**13**]. Hence, there is surjective $\tau : \beta_{\mathcal{A}}^{\kappa}(G) \to K$ by the mapping properties of $\beta_{\mathcal{A}}^{\kappa}(G)$. Since $\beta_{\mathcal{A}}^{\kappa}(G)$ has weak unit, say e, then $\tau(e)$ is a weak unit in K by [**14**, 5.1] and the remark after Corollary 2.3. Now $K \leq H$ is κ-coessential (as the second factor of the κ-coessential embedding $G \leq H$) and is σ-complete, since K is epicomplete [**14**, 5.1]. It then follows that $\tau(e)$ is a weak unit in H. First, since K is epicomplete, K is an archimedean f-ring, so by [**3**, 8.5.5], [**10**, p. 62] and by Proposition 2.0, $K = ak_K(\tau(e)) \subseteq ak_H(\tau(e)) \subseteq \kappa_H(\tau(e))$. Moreover, $K \leq H$ κ-coessential implies that $\kappa_H(\tau(e)) = H$; whence, $\tau(e)$ is a weak unit in H.

Corollary 3.3. *The following statements are equivalent about G:*

(1) *$\beta_{\mathcal{A}}^{\omega_0}(G)$ has a weak unit.*

(2) *There is countable $A \subseteq G$ with $ak_G(A) = G$.*

(3) *If H is an epicompletion of G, then H has a weak unit.*

Proof. Take $\kappa = \omega_0$ in Theorem 3.2.

Corollary 3.4. *Let $\kappa > \omega_0$. The following statements are equivalent about G.*

(1) *$\beta_{\mathcal{A}}^{\infty}(G)$ has a weak unit.*

(2) *$\beta^\kappa_{\mathcal{A}}(G)$ has a weak unit for every κ.*

(3) *There is countable $A \subseteq G$ with $A^{\perp\perp} = G$.*

Proof. By Theorem 3.2, for $\kappa = \infty$, $\beta^\infty_{\mathcal{A}}(G)$ has a weak unit precisely when there is countable $A \subseteq G$ with $A^{\perp\perp} = G$. Now this is precisely the condition that $\beta^\kappa_{\mathcal{A}}(G)$ has a weak unit for every $\kappa > \omega_0$, since any $A \subseteq G$ with $|A| < \kappa$ has $A^{\perp\perp} = \kappa_G(A)$ by Proposition 2.2.

However, for the case $\kappa = \omega_0$, this condition is necessary but *not* sufficient. We give an example of such a G in the next section.

Proposition 3.5. *Let κ be an infinite cardinal number or the symbol ∞, and let e be a weak unit in G. Then e is a weak unit in $\beta^\kappa_{\mathcal{A}}(G)$ precisely when $\kappa_G(e) = G$.*

Proof. If $\kappa_G(e) = G$, then $\kappa_{\mathcal{B}^\kappa_{\mathcal{A}}(G)}(e) = \beta^\kappa_{\mathcal{A}}(G)$, by Proposition 2.5; whence, e is a weak unit in $\beta^\kappa_{\mathcal{A}}(G)$.

Conversely, if e is a weak unit in $\beta^\kappa_{\mathcal{A}}(G)$, then $ak_{\beta^\kappa_{\mathcal{A}}(G)}(e) = \beta^\kappa_{\mathcal{A}}(G)$ by [**3**, 8.5.5], [**10**, p. 62], since $\beta^\kappa_{\mathcal{A}}(G)$ is an archimedean f-ring. It then follows from the proof of (1) implies (2) in Theorem 3.2 that $\kappa_G(e) = G$.

In [**14**] it is shown that the construction of $\beta^\kappa_{\mathcal{A}}(G)$ in Arch may also be applied to any G in the category $\mathcal{W}$ to yield canonical epicompletions, which we shall denote $\beta^\kappa_{\mathcal{W}}(G)$ for each κ. These $\mathcal{W}$-epicompletions have properties analogous to properties of their corresponding Arch epicompletions. In order to distinguish which category we are in, we shall write $\beta^\kappa_{\mathcal{W}}(G)$ for the $\mathcal{W}$-epicompletion of any G in $\mathcal{W}$.

Corollary 3.6. *Let G be an archimedean l-group with a weak unit e and let $\omega_1 \leq \kappa \leq \infty$. Then e is also a weak unit in $\beta^\kappa_{\mathcal{A}}(G)$ so that $\beta^\kappa_{\mathcal{A}}(G) \cong \beta^\kappa_{\mathcal{W}}(G)$ over G.*

Proof. Since $e^{\perp\perp G} = \kappa_G(e) = G$ (Proposition 2.2), we have that e is a weak unit in $\beta^\kappa_{\mathcal{A}}(G)$ by Proposition 3.5. The result now follows by the uniqueness of these objects.

Corollary 3.6 need not hold for $\kappa = \omega_0$, however, as is illustrated in the next section.

4. G with weak unit for which $\beta_{\mathcal{A}}^{\omega_0}(G)$ has no weak unit. We now present an example of an archimedean l-group with a weak unit for which $\beta_{\mathcal{A}}^{\omega_0}(G)$ has no weak unit. Hence, while the condition that there is countable $A \subseteq G$ such that $A^{\perp\perp} = G$ is necessary in ensuring that $\beta_{\mathcal{A}}^{\omega_0}(G)$ has a weak unit, it is not sufficient. As a consequence, we have a $G \in \mathcal{W}$ for which $\beta_{\mathcal{A}}^{\omega_0}(G)$ is not isomorphic to $\beta_{\mathcal{W}}^{\omega_0}(G)$ over G.

Example. For every $r \in [0,1]$, let $s_r(x) = 1/|x-r|$. Let G be the archimedean l-group in $D[0,1]$ generated by $\{s_r : r \in [0,1]\}$ and all the constant functions. Let the constant function 1 be the distinguished weak order unit in G. Note, for each $r \in [0,1]$, s_r is a continuous $[-\infty,\infty]$-valued function on $[0,1]$ which attains a maximum and minimum.

We note that, while $D[0,1]$ is not an l-group, it is a lattice. In generating the l-group G in $D[0,1]$, one need only be concerned with adding two functions $g, h \in G$. The sum of two functions is not difficult to compute when one notes that any function $g \in G$ is simply a finite expression, in all the operations, of the form

$$g = \bigwedge_i \bigvee_j \sum_k (\alpha_{ijk} s_{r_{ijk}} + \beta_{ijk}).$$

By Corollary 3.3, we show that $\beta_{\mathcal{A}}^{\omega_0}(G)$ has no weak unit by showing that whenever A is a countable subset of G, then $ak_G(A) \neq G$.

We first notice the following:

(1) For every $g \in G$, $g^{-1}(\infty)$ is finite since $s_r^{-1}(\infty)$ is finite, for any $r \in [0,1]$.

(2) For every $g \in G$, there is a neighborhood U_r of r and $k \in N^+$ such that $|g| \leq k s_r$ on U_r.

(3) If $g \in G$ and $g(r) \in R$, then there is a neighborhood V_r of r with $|g| \leq s_r$ on V_r.

Let A be a countable subset of G. Let I be the l-ideal generated by all s_t, where $t \in \cup\{a^{-1}(\infty) : a \in A\} \equiv A^{-1}(\infty)$. (If A consists only of constant functions, so that $A^{-1}(\infty)$ is empty, let I be the l-ideal generated by the constant function 1.)

Now $A^{-1}(\infty)$ is countable, since A is countable, and $a^{-1}(\infty)$ is finite for each $a \in A$. Then clearly $A \subseteq I$. For, each $a \in A$ is a finite combination of s_r's and constant functions. So if $a(r) = \infty$, for some $r \in [0,1]$, then s_r must appear in the expression of a and $s_r(r) = \infty$. All constant functions also belong to I, since each s_r is bound away from 0 and $ks_r \geq 1$ for some $k \in N$. If A consists only of constant functions, so that $A^{-1}(\infty)$ is empty, then I is generated by the constant function 1 and therefore contains A.

We show that if $r \notin A^{-1}(\infty)$, then $s_r \notin ak_G(I)$; whence, $s_r \notin ak_G(A)$, so that $ak_G(A) \neq G$.

Write $ak_G(I) = \cup_{\alpha<\omega_1} I_\alpha$. We proceed by transfinite induction on α to show that $r \notin A^{-1}(\infty)$ implies that $s_r \notin I_\alpha$. We use the following fact that is not difficult to show: $s_r \in I_\alpha$ if and only if there is some $g \in I_\alpha$ with $r \in g^{-1}(\infty)$.

Clearly, $s_r \notin I$. Suppose $s_r \notin I_\alpha$ for all $\alpha < \beta$. If β is a limit ordinal, then $s_r \notin I_\beta = \cup_{\alpha<\beta} I_\alpha$. If $\beta = \alpha + 1$ and $s_r \in I_{\alpha+1} \setminus I_\alpha$, then there is $\{f_n : n \in N\} \subseteq I_\alpha$ and $u \in G$ so that $f_n \to s_r$ r.u. with regulator u. Since $s_r \notin I_\alpha$, then for all $g \in I_\alpha$, $r \notin g^{-1}(\infty)$. Hence, $r \notin f_n^{-1}(\infty)$ for all n. Assume $f_n > 0$ for each $n \in N$. Now there is a neighborhood U_r of r and $k \in N^+$ with $u \leq ks_r$ on U_r by remark (2).

Let $\varepsilon = 1/2k$ and choose n so that $|f_n - s_r| \leq 1/2k \cdot u$. Now on U_r, $|f_n - s_r| \leq 1/2k \cdot u \leq 1/2k \cdot ks_r = 1/2s_r$. There is a neighborhood V_r of r so that $f_n \leq s_r$ on V_r by remark (3). Thus, on $T = U_r \cap V_r$, we have $s_r - f_n \leq 1/2s_r$; or $1/2s_r = s_r - 1/2s_r \leq f_n$ on T. However, this cannot occur, since $s_r(r) = \infty$ while $f_n(r) \in R$.

The proof then follows by invoking induction.

5. The partially ordered set $\mathcal{A}(G)/\sim$ of arch epicompletions of G. In this section we fix an archimedean l-group G and continue our study of Arch epicompletions of G. We first consider the class of all Arch epicompletions of G, equipped with a quasi-order. Then we define an equivalence relation on this class and consider the resulting partially ordered set. We then give results about maximal and minimal elements in this poset. Among other things, we show that this poset has a maximum element but no minimum element.

The material presented here closely models the work done by Ball and Hager in the category $\mathcal{W}$ in [**5**, Sections 8, 9].

The following result is used in what follows and is the reason that τ is a surjection if θ is epic in Theorem 1.2.

Proposition 5.0 [**11**, 1.14]. *In* Arch*, the homomorphic image of an epicomplete object is epicomplete. That is, the class of* Arch *epicomplete objects is closed under* Arch *surjections.*

Let G be a fixed archimedean l-group, and let $\mathcal{A}(G)$ denote the class of all Arch epicompletions $\phi : G \to E$ of G. We shall write $(\phi, E) \in \mathcal{A}(G)$. If (ϕ_1, E_1) and $(\phi_2, E_2) \in \mathcal{A}(G)$, we write $(\phi_1, E_1) \geq (\phi_2, E_2)$ if there is $h : E_1 \to E_2$ with $h \circ \phi_1 = \phi_2$. We note that such an h is unique (respectively, surjective) since ϕ_1 (respectively, ϕ_2) is an epimorphism. Since $\geq$ is both reflexive and transitive, $\geq$ is a quasi-order on $\mathcal{A}(G)$.

Define a relation $\sim$ on $\mathcal{A}(G)$ by declaring that $(\phi_1, E_1) \sim (\phi_2, E_2)$ if $(\phi_1, E_1) \geq (\phi_2, E_2)$ and $(\phi_2, E_2) \geq (\phi_1, E_1)$. Then $(\phi_1, E_1) \sim (\phi_2, E_2)$ means there is $h : E_1 \to E_2$ and $k : E_2 \to E_1$ with $h \circ \phi_1 = \phi_2$ and $k \circ \phi_2 = \phi_1$, respectively. It then follows that h is an isomorphism over G. Now $\sim$ is an equivalence relation on $\mathcal{A}(G)$ and $\mathcal{A}(G)/\sim$ is a set, since Arch is co-well-powered [**14**, 2.1], which is partially ordered by $[(\phi_1, E_1)] \geq [(\phi_2, E_2)]$ if $(\phi_1, E_1) \geq (\phi_2, E_2)$.

Henceforth, we will focus on the poset $\mathcal{A}(G)/\sim$ but shall suppress mention of the equivalence classes. Instead, we will think of the representatives of the equivalence classes as the elements of this poset.

An element in a partially ordered set is called *maximal* (respectively, *minimal*) if it is not strictly bounded above (respectively, below) by any other member of the set. An element which is larger (respectively, smaller) than all other elements in the partially ordered set is called the *maximum* (respectively, *minimum*) element of the poset.

Let $(\varepsilon, \beta^{\kappa}_{\mathcal{A}}(G))$ denote the unique (up to isomorphism over G) epicompletion of G in which G is κ-completely embedded and which lifts all κ-complete morphisms out of G to epicomplete objects in Arch.

Proposition 5.1. $(\varepsilon, \beta^{\omega_0}_{\mathcal{A}}(G))$ *is the maximum element in* $\mathcal{A}(G)/\sim$.

Proof. Let $(\phi, E) \in \mathcal{A}(G)$. Then there is $\tau : \beta_{\mathcal{A}}^{\omega_0}(G) \to E$ so that $\tau \circ \varepsilon = \phi$, by the universal mapping property of $\beta_{\mathcal{A}}^{\omega_0}(G)$. Thus, $(\varepsilon, \beta_{\mathcal{A}}^{\omega_0}(G)) \geq (\phi, E)$ for any $(\phi, E) \in \mathcal{A}(G)$.

Proposition 5.2. (1) *$(\varepsilon, \beta_{\mathcal{A}}^{\kappa}(G))$ is the minimum element in $\mathcal{A}(G)/\sim$ with the property of lifting κ-complete morphisms out of G to epicomplete objects.*

(2) *$(\varepsilon, \beta_{\mathcal{A}}^{\kappa}(G))$ is the maximum element in $\mathcal{A}(G)/\sim$ in which G is κ-completely embedded.*

Proof. (1) Let $(\phi, E) \in \mathcal{A}(G)$ have the property that it lifts κ-complete morphisms out of G to epicomplete objects. Then since G is κ-completely embedded in $\beta_{\mathcal{A}}^{\kappa}(G)$, there is $\tau : E \to \beta_{\mathcal{A}}^{\kappa}(G)$ with $\tau \circ \phi = \varepsilon$. Hence, $(\phi, E) \geq (\varepsilon, \beta_{\mathcal{A}}^{\kappa}(G))$ for all $(\phi, E) \in \mathcal{A}(G)$ with this property.

(2) Let $(\phi, E) \in \mathcal{A}(G)$ with ϕ κ-complete. Then there is $\tau : \beta_{\mathcal{A}}^{\kappa}(G) \to E$ with $\tau \circ \varepsilon = \phi$, by the universal mapping property of $\beta_{\mathcal{A}}^{\kappa}(G)$. Hence, $(\varepsilon, \beta_{\mathcal{A}}^{\kappa}(G)) \geq (\phi, E)$ for all $(\phi, E) \in \mathcal{A}(G)$ with ϕ κ-complete.

Let G be a sub-l-group of an archimedean l-group H. Then the embedding $G \leq H$ is called *essential* (or one says that G is *large* in H) if every nontrivial l-ideal of H intersects G nontrivially or, equivalently, if every l-homomorphism of H which is one-to-one on G is one-to-one on H.

In [**14**, 7.4] it is shown that, up to isomorphism over G, $\beta_{\mathcal{A}}^{\infty}(G)$ is the unique essential epicompletion of G.

Proposition 5.3. *$(\varepsilon, \beta_{\mathcal{A}}^{\infty}(G))$ is, up to isomorphism over G, the unique epicompletion of G in which G is completely embedded.*

Proof. Suppose $(\phi, E) \in \mathcal{A}(G)$ with ϕ complete. Then, by the universal mapping property of $\beta_{\mathcal{A}}^{\infty}(G)$, there is $\tau : \beta_{\mathcal{A}}^{\infty}(G) \to E$ with $\tau \circ \varepsilon = \phi$. Since ϕ is epic, τ is onto, and since $\tau \circ \varepsilon = \phi$ is one-to-one and ε is essential, τ is one-to-one. Hence, τ is an isomorphism over G.

Proposition 5.4. *Let (ϕ, E) be an* Arch *epicompletion of G. ϕ is essential if and only if (ϕ, E) is minimal in $\mathcal{A}(G)/\sim$.*

Proof. Let ϕ be essential and suppose $(\phi, E) \geq (\delta, D)$ for some $(\delta, D) \in \mathcal{A}(G)$. Then there is $\tau : E \to D$ so that $\tau \circ \phi = \delta$. Since δ is epic, τ is onto, and since $\tau \circ \phi = \delta$ is one-to-one and ϕ is essential, τ is one-to-one. Hence, τ is an isomorphism over G, so that $(\phi, E) \sim (\delta, D)$.

Conversely, suppose $\psi : E \to H$ in Arch has $\psi \circ \phi$ one-to-one. Let $\psi' : E \to \psi(E)$ be the range restriction of ψ. Then $(\psi' \circ \phi, \psi(E)) \in \mathcal{A}(G)$, by [**11**, 1.14] and $(\phi, E) \geq (\psi' \circ \phi, \psi(E))$, by definition. Since (ϕ, E) is minimal, we must have $(\psi' \circ \phi, \psi(E)) \sim (\phi, E)$ which implies that ψ' is an isomorphism over G since ϕ is epic. Hence, ψ is one-to-one.

Proposition 5.5. *Let $(\phi, E) \in \mathcal{A}(G)$. The following statements are equivalent*:

(1) *(ϕ, E) is minimal in $\mathcal{A}(G)/\sim$.*

(2) *ϕ is essential.*

(3) *ϕ is complete.*

(4) *There is an isomorphism $h : \beta^{\infty}_{\mathcal{A}}(G) \to E$ so that $h \circ \varepsilon = \phi$.*

Proof. (1) and (2) are equivalent by Proposition 5.4. (2) implies (3) since all essential embeddings are complete [**8**, 12.1.12]. (3) implies (4) since $(\varepsilon, \beta^{\infty}_{\mathcal{A}}(G))$ is the unique epicompletion of G in which G is completely embedded (Proposition 5.3). Finally, (4) implies (2) since ε and h are both essential.

While Proposition 5.5 shows that $(\varepsilon, \beta^{\infty}_{\mathcal{A}}(G))$ is minimal in $\mathcal{A}(G)/\sim$, it is usually not minimum. In [**5**, 9.11], Ball and Hager use results in $\mathcal{W}$ to show that if $G = C[0,1]$ with weak unit 1, then $\beta^{\infty}_{\mathcal{W}}(G)$ is not least in the poset of $\mathcal{W}$-epicompletions of G, equipped with the equivalence relation $\sim$. Now since 1 is a strong unit in $C[0,1]$, 1 is a weak unit in all Arch epicompletions of $C[0,1]$ by the remark after Corollary 2.8, so all Arch epicompletions of $C[0,1]$ are $\mathcal{W}$-epicompletions. The converse is also true, since every $\mathcal{W}$-epic morphism into an "algebra" is Arch epic [**3**, 8.5.2]. Hence, $\beta^{\infty}_{\mathcal{A}}(G)$ is isomorphic to $\beta^{\infty}_{\mathcal{W}}(G)$ over G, and it is not least in $\mathcal{A}(G)/\sim$.

6. Remarks. An element u in an l-group G is a *strong unit* if $(u)_G = G$. For an element $u \in G$, the property $ak_G(u) = G$ is stronger than u being a weak unit and weaker than u being a strong unit, since

$(u)_G \subseteq ak_G(u) \subseteq u^{\perp\perp G}$. We note that while it is always the case, by Proposition 2.2, that $\kappa_G(u) = u^{\perp\perp G}$ (for any uncountable cardinal number κ), it is possible to have $(u)_G \subset ak_G(u)$ and $ak_G(u) \subset u^{\perp\perp G}$.

Now κ-ideals have been studied to some degree (see [**7**]). For $\kappa = \infty$, there is a very nice description of $\kappa_G(A)$ [**8**, 11.1.6]. That is, if G is an archimedean l-group and $A \subseteq G$, then for $0 \leq b \in A^{\perp\perp}$, $b = \vee_{a\in A, n\in N}(b \wedge n|a|)$. When $\kappa = \omega_0$, $\kappa_G(A)$ refers to the archimedean kernel generated by A in G, and in this case and all cases $\kappa < \infty$, we have no such useful description involving suprema of elements of (A). Recall that the description we do have of $ak_G(A)$ (see page 4) involves transfinite induction, and calculation is difficult because it requires having a good understanding of relative uniform convergence. Hence, having an alternative description of $ak_G(A)$ or $\kappa_G(A)$ for uncountable κ, which is easier to work with, would be useful. In particular, it would be helpful in studying elements $u \in G$ with the property that $ak_G(u) = G$. These are called *near units* in [**15**]. We note that Arch surjections preserve near units by Corollary 2.1 for $\kappa = \omega_0$. Exploring what importance, if any, such elements hold in the theory of archimedean l-groups seems interesting.

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